EXTENSION OF THE Sp(2,C) GROUP FOR DESCRIPTION OF A THREE-BODY SYSTEM

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We propose a new approach to the three-body problem that is based on the extension of the $Sp(2, C)$ group, which is the universal covering group for the Lorentz group, to the $Sp(4, C)$ one. Angular momenta of the particles in the phase space of a system with an inner interaction are obtained. This result can be used to obtain eigenfunctions of angular momenta, and exact quantum mechanical solutions for the system defined by Dirac-like equations, e.g. a system of three zero-spin particles or Regge trajectories of N-baryons.

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1. INTRODUCTION

Description of a relativistic three-body system is an important issue in modern nuclear and particle physics. Statistical approaches or even many-body technics are not efficient for three-body problems, and a good treatment of the center of mass is necessary and internal coordinates must be employed. Various approaches to the solution of this problem exist. For example, the system can be described by means of relativistic quantum mechanics (using the Lorentz-invariant wave equations) [1–3] or quantum field theory (Faddeev equation for three particles) [4, 5]. An important research direction is connected with extensions of the Minkowski and spinor spaces [6, 7].

Using the method of extension of the $Sp(2, C)$ group in a minimal manner, we develop the model that describes the three-body system with an inner interaction. Expressions for positions, momenta, angular momenta and their eigenfunctions are found. Furthermore, this new method of spacetime extension may be used for description of interacting standart model particles, e.g., N-baryons.

2. SPACETIME EXTENSION

The following is the general overview of the extension of the two-dimensional spinor space and corresponding groups of symmetry. The proposed method of extension of two-dimensional spinor space has an important advantage over the method of the spacetime extension by addition of spatial variables only, as number of dimensions increases only quadratically, not exponentially. Therefore, a significantly smaller number of subsidiary conditions is required.

The Lorentz group $SO(1, 3)$ is covered by the $SL(2, C) \equiv Sp(2, C)$ group. There is an one-to-one correspondence between the $Sp(2, C)$ Hermitian spin-tensors of second rank and the Minkowski four-vectors that describe the space-time position of a relativistic particle. The $Sp(2, C)$ group can be extended to the $Sp(4, C)$ one for the description of few-particle systems. Similar to the $Sp(2, C)$ group case, there is a correspondence between $Sp(4, C)$ 4×4 Hermitian spin-tensor and a real 16-vector [8, 9]. We will choose the matrices of the basis as follows:

$$\mu^M \equiv \mu^{(a,m)} = \Sigma^a \otimes \sigma^m, \tag{1}$$

Values of $M = 1..16$ can be represented through 4×4 combinations of two indices ($a, m = 0..3$). The Σ^a and σ^m matrices can be explicitly expressed through 2×2 unitary matrix I and the Paupli matrices τ_1, τ_2, τ_3:

$$\begin{aligned}
\sigma^0 = \tilde{\sigma}^0 = \Sigma^2 = \tilde{\Sigma}^2 = I, \\
\sigma^1 = \tilde{\sigma}^1 = \Sigma^1 = \tilde{\Sigma}^1 = \tau_1, \\
\sigma^2 = -\tilde{\sigma}^2 = \Sigma^0 = -\tilde{\Sigma}^0 = \tau_2, \\
\sigma^3 = -\tilde{\sigma}^3 = \Sigma^3 = \tilde{\Sigma}^3 = \tau_3,
\end{aligned} \tag{2}$$

Now we obtain the metrics of 16-dimensional real space:

$$g^{MN} = \frac{1}{4} Tr(\mu^M \tilde{\mu}^N) = \hat{h}^{ab} h^{mn}, \tag{3}$$

where $h^{mn} = \text{diag}\,(1, -1, -1, -1)$ is the Minkowski metrics and $\hat{h}^{ab}$ is the Minkowski metrics with the opposite sign caused by the group extension. The discussed real space is a sum of four Minkowski metrics. The 16-dimensional momentum operator in the considered space can be represented as four relativistic four-momenta. Since ten of the 16 dimensions are spacelike, whereas six other are time-like and we need only one time for a physical interpretation, it is necessary to exclude five of six time-like components.

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Let us introduce the quadratic function of the momentum $P_M = i\partial_M$:

$$(P\tilde{P})_{\alpha\beta} = P_{\alpha\tilde{\alpha}}\tilde{P}^{\beta\tilde{\beta}} = \mu^M_{\alpha\tilde{\alpha}}P_M\mu^{N\beta\tilde{\beta}}P_N, \qquad (4)$$

where $\alpha, \bar{\alpha}, \beta, \bar{\beta} = 1..4$ are spinoral indices.

The matrix $P_{\alpha\beta}$ is antisymmetric under the interchange of the α and β indices. Let us consider the antisymmetric part, which has six linearly independent matrices in the four-dimensional space. One of them is proportional to the metric, and, consequently, is invariant under the $Sp(4, C)$. The remaining five matrices create a five-dimensional complex space. In such way the five-dimensional representation of the $Sp(4, C)$ group, i.e. the special orthogonal group $SO(5, C)$, is obtained. These groups are locally isomorphic.

We introduce linkage to the system by requiring five of complex quadratic combinations of momenta to be equal to zero. In this case they are invariant under the transformations of the $Sp(4, C)$ group. In the absence of interaction these momenta have the form of conventional derivatives, commute with each other and do not create secondary linkage. The system of subsidiary conditions, written in the form of ten real equations, is:

$$(s, p) = (s, q) = 0, \qquad \mathbf{r}s^0 - r^0\mathbf{s} = [\mathbf{p}, \mathbf{q}], \qquad (5)$$

$$(r, p) = (r, q) = 0, \qquad \mathbf{p}q^0 - p^0\mathbf{q} = [\mathbf{s}, \mathbf{r}], \qquad (6)$$

where $s_m = P_{3m}, p_m = P_{1m}, q_m = P_{2m}, r_m = P_{0m}$ are the components of the momentum vector $P_{a,m}$, round brackets denote the scalar products of four-vectors as $(a, b) = a_0b_0 - \mathbf{ab}$, and square brackets denote the vector products of three-vectors.

Equations (6) are a consequence of (5), therefore we obtained five independent conditions (5), and it can be seen from their form that r_m is the pseudovector. The 16-momentum has the maximum amount of independent components when s^m is time-like. In that case p_m and q_m are space-like, and in the frame where $\mathbf{s} = 0$, we have $p_0 = q_0 = 0$, and $\mathbf{p}$, $\mathbf{q}$ are independent. Thus we can treat this frame as a center-of-momentum one. Then $\mathbf{p}$ and $\mathbf{q}$ are thought to be three-momenta of the relative motion, that have the corresponding $\mathbf{x}$ and $\mathbf{y}$ coordinates.

3. ANGULAR MOMENTA

The next step is to include an inner interaction in the system. We introduce the interaction that leads to the violation of the $Sp(4, C)$ symmetry, but relativistic invariance remains. The subsidiary conditions are primary constraints [10], and they can create secondary constraints. We demand that there is no secondary linkage. This strict requirement completely defines the form of interaction potentials up to gauge transformations. In the center-of-momentum frame instead of two sets of inner coordinates and corresponding momenta we obtain two independent coordinate spaces Q_a, Q'_a and two corresponding momenta spaces were obtained.:

$$\begin{aligned} a) \quad \mathbf{P}(\lambda) = \mathbf{p} - \lambda\mathbf{y}, \quad \mathbf{Q}(\lambda) = \mathbf{q} + \lambda\mathbf{x}, \\ b) \quad \mathbf{P}(-\lambda) = \mathbf{p} + \lambda\mathbf{y}, \quad \mathbf{Q}(-\lambda) = \mathbf{q} - \lambda\mathbf{x}. \end{aligned} \qquad (7)$$

The commutation relations between these vectors are

$$\begin{aligned} [Q_a(\lambda), P_b(\lambda)] = 2i\lambda\delta_{ab}, \\ [Q'_a(\lambda), P'_b(\lambda)] = 2i\lambda\delta_{ab}, \end{aligned} \qquad (8)$$

where $\mathbf{Q}'(\lambda) = \mathbf{P}(-\lambda)$ and $\mathbf{P}'(\lambda) = \mathbf{Q}(-\lambda)$; δ_{ab} is the Kronecker delta ($a, b = 1, 2, 3$ are Cartesian indices). The pairs $\mathbf{P}, \mathbf{Q}$ and $\mathbf{P}', \mathbf{Q}'$ commute with each other.

Using the two proposed sets of coordinates and momenta, $\mathbf{P}(\lambda), \mathbf{Q}(\lambda)$ and $\mathbf{P}'(\lambda), \mathbf{Q}'(\lambda)$, we can introduce two angular momentum operators:

$$\begin{aligned} \mathbf{M} = \mathbf{M}(\lambda) = \frac{1}{2\lambda}[\mathbf{Q}(\lambda), \mathbf{P}(\lambda)], \\ \mathbf{N} = \mathbf{M}(-\lambda) = \frac{1}{2(-\lambda)}[\mathbf{Q}'(-\lambda), \mathbf{P}'(\lambda)]. \end{aligned} \qquad (9)$$

$\mathbf{M}$ and $\mathbf{N}$ obey the usual commutation relations and they are independent.

4. CONCLUSIONS

In the present work, we have investigated the quantum system with nine degrees of freedom using a method based on the extension of the $Sp(2, C)$ group. After the introduction of inner interaction in a minimal manner, we redefined the coordinates and momenta of the system, that resulted in two independent angular momenta, which eigenfunctions will be found elsewhere. This new result lays the groundwork for determining the stationary states of a quantum three-body system with an inner interaction defined by Dirac-like equations. The theory can be applied, for example, to analysis of the system of three spinless particles or to find Regge trajectories of N-baryons.

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РАСШИРЕНИЕ ГРУППЫ Sp(2,C) ДЛЯ ОПИСАНИЯ ТРЁХЧАСТИЧНОЙ СИСТЕМЫ

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Предлагается новый подход к описанию трёхчастичной системы, основанный на расширении группы $Sp(2,C)$, которая является универсальной накрывающей группы Лоренца, до группы $Sp(4,C)$. В фазовом пространстве системы с внутренним взаимодействием получены угловые моменты частиц. На основе этого результата будут получены собственные функции углового момента, с помощью которых можно найти точное квантово-механическое решение системы, определённой уравнениями типа уравнений Дирака, например, системы трёх бесспиновых частиц, или определить траектории Редже барионов.

РОЗШИРЕННЯ ГРУПИ Sp(2,C) ДЛЯ ОПИСУ СИСТЕМИ ТРЬОХ ТІЛ

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Пропонується новий підхід до опису тричастинкової системи, що базується на розширенні групи $Sp(2,C)$, яка є універсальною накриваючою групи Лоренця, до групи $Sp(4,C)$. У фазовому просторі системи зі внутрішньою взаємодією отримані кутові моменти частинок. На основі цього результату будуть знайдені власні функції кутового моменту, за допомогою яких можна отримати точний квантово-механічний розв'язок системи, яка визначається рівняннями типа рівнянь Дірака, наприклад, системи трьох безспінових частинок, або визначити траєкторії Редже баріонів.