GRASSMANN DYNAMICS OF CLASSICAL SPIN IN NONABELIAN GAUGE FIELDS

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Using Grassmann variant of classical mechanics, we construct Lagrangian dynamics of classical spinning particle in (possibly non-abelian) gauge fields. Quantization of this model is briefly discussed.

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In [1] the Lagrangian theory of pseudoclassical particle moving in electromagnetic field was constructed and its quantization leading to Dirac equation was considered. In the present note, we generalize this theory onto the case of arbitrary gauge group. Quantization and the case of nonzero anomalous magnetic momentum are discussed in last two paragraphs.

Let x^μ be even "space-time" coordinated of our particle, ξ^μ odd spin variables, A^μ_a even gauge potential, and J_a generators of certain finite-dimensional anti-hermitian representation of our gauge group (e.g. spinor representation of $SO(3)$), so $[J_a, J_b] = if_{ab}{}^c J_c$. Let $q, \bar{q}$ be even coordinates of internal gauge degrees of freedom of our particle (their indexes will not be written explicitly). Define their covariant derivatives as:

$$\begin{aligned}\frac{\mathcal{D}q}{\mathcal{D}s} &= \frac{dq}{ds} + ieA^a_\mu \dot{x}^\mu J_a q,\\ \frac{\mathcal{D}\bar{q}}{\mathcal{D}s} &= \frac{d\bar{q}}{ds} - ieA^a_\mu \dot{x}^\mu \bar{q} J_a,\end{aligned} \tag{1}$$

where s is even coordinate of a worldline and overdot means derivative w.r.t. s. Note that $\dot{x}^\mu \dot{x}_\mu \neq 1$ in general, see [2] for discussion. Define gauge charge as $Q_a = \bar{q} J_a q$, so

$$\frac{\mathcal{D}Q_a}{\mathcal{D}s} = \frac{dQ_a}{ds} + eA^b_\mu \dot{x}^\mu f_{ab}{}^c Q_c. \tag{2}$$

Consider the following lagrangian

$$\begin{aligned}L = \frac{m}{2}\dot{x}^\mu \dot{x}_\mu - \frac{1}{4}\xi^\mu \dot{\xi}_\mu + \\ + \frac{i}{2}\left(-\frac{\mathcal{D}\bar{q}}{\mathcal{D}s}q + \bar{q}\frac{\mathcal{D}q}{\mathcal{D}s}\right) + \frac{\mu'}{2m}F^a_{\mu\nu}S^{\mu\nu}Q_a,\end{aligned} \tag{3}$$

where $\mu' = e$ is the magnetic moment of a particle (see last paragraph however), $S_{\mu\nu} = \frac{1}{2}\xi_\mu \xi_\nu$ is spin tensor and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + eA^b_\mu A^c_\nu f_{bc}{}^a \tag{4}$$

is gauge field tensor. Note that unfolding the brackets in r.h.s. of (3) we obtain usual interaction term $-eA^a_\mu \dot{x}^\mu Q_a$ due to (1). Varying the action $\int L\,ds$ w.r.t. q and $\bar{q}$ we obtain:

$$\begin{aligned}i\dot{q} &= \left(-eA^a_\mu \dot{x}^\mu + \frac{\mu'}{2m}F^a_{\mu\nu}S^{\mu\nu}\right)J_a q,\\ -i\dot{\bar{q}} &= \left(-eA^a_\mu \dot{x}^\mu + \frac{\mu'}{2m}F^a_{\mu\nu}S^{\mu\nu}\right)\bar{q} J_a,\end{aligned}$$

so using (2) we obtain equations of motion of gauge charge

$$\frac{\mathcal{D}Q_a}{\mathcal{D}s} = \frac{\mu'}{2m}F^b_{\mu\nu}S^{\mu\nu}f_{ab}{}^c Q_c. \tag{5}$$

Varying w.r.t. ξ^μ we obtain spin equations of motion

$$m\dot{S}^{\mu\nu} = \mu' F^{a\rho[\nu}S_\rho{}^{\mu]}Q_a. \tag{6}$$

Finally, varying w.r.t. x^μ we have to consider Q_a as **geodesically constant**: $\mathcal{D}Q_a/\mathcal{D}s = 0$ but not $\dot{Q}_a = 0$ **contrary to ordinary variational calculus**. Then using (2), (4) we obtain:

$$m\ddot{x}^\mu = eF^{a\mu\nu}\dot{x}_\nu Q_a + \frac{\mu'}{2m}(\mathcal{D}^\mu F^{\rho\sigma})^a S_{\rho\sigma}Q_a, \tag{7}$$

where $\mathcal{D}^\mu$ is ordinary covariant derivative w.r.t. A^a_μ, so $d(F^a_{\mu\nu}Q_a)/ds = (\mathcal{D}_\rho F_{\mu\nu})^a Q_a \dot{x}^\rho + F^a_{\mu\nu}\mathcal{D}Q_a/\mathcal{D}s$. Eqs. (5)-(7) are just the ones obtained by Heinz [3] by classicalizing the ordinary QCD hamiltonian (cf. also [4]).

Due to nonabelian Bianchi identity

$$(\mathcal{D}^{[\mu}F^{\rho\sigma]})^a = 0$$

we see that the following quantities are conserved:

$$\begin{aligned}C_1 &= \xi_\mu \dot{x}^\mu,\\ C_2 &= \frac{m}{2}\dot{x}^\mu \dot{x}_\mu + \frac{\mu'}{2m}F^a_{\mu\nu}S^{\mu\nu}Q_a,\\ C_3 &= \bar{q}q.\end{aligned} \tag{8}$$

Since orbits of coadjoint representation are distinguished by C_3 values, we see that different types of gauge charges arise in the sense of [5].

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To quantize our theory, we turn $q, \bar{q}$ into bosonic creation-destruction operators, then different values of occupation number operator $\hat{C}_3$ correspond to different representations of gauge group, ξ^μ turn into γ-matrices, and canonical momentum $p_\mu = \eta_{\mu\nu}\dot{x}^\nu - eA^a_\mu Q_a$ [6] turns into $-i\partial_\mu$. Thus $\hat{C}_1$ becomes Dirac operator (cf. [1] in the abelian case) and $\hat{C}_2$ becomes Hamiltonian. If coadjoint representation of gauge group is chosen, then $\hat{Q}_a$ turn into Gell-Mann matrices of ordinary QCD.

If $\mu' \neq e$, we obtain theory with anomalous magnetic momentum. Considering C_1 as a Lagrangian constraint, we see that all the above considerations go without substantial change, so we obtain BMT-type [7] equations (cf. [2] for abelian case):

$$m(\dot{x}^\rho \dot{x}_\rho)\dot{S}^{\mu\nu} = \mu'(\dot{x}^\kappa \dot{x}_\kappa)F^{a\rho[\nu}S_\rho{}^{\mu]}Q_a + \\ +(\mu' - e)F^{a\rho\sigma}\dot{x}_\rho S_\sigma{}^{[\mu}\dot{x}^{\nu]}Q_a,$$

and some additional terms in the r.h.s. of eq. (7) arise. For the case of $U(1)$ gauge group, the quantized version of C_2 for an arbitrary value of μ' was considered in [8].

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ГРАССМАНОВА ДИНАМИКА КЛАССИЧЕСКОГО СПИНА В НЕАБЕЛЕВЫХ КАЛИБРОВОЧНЫХ ПОЛЯХ

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На основе грассманова варианта классической механики построена лагранжева динамика классической частицы со спином в калибровочных полях (в т.ч. неабелевых). Кратко рассмотрено квантование предложенной модели.

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На основі грасманова варіанта класичної механіки побудовано лагранжеву динаміку класичної частинки зі спіном у калібрувальних полях (в т.ч. неабелевих). Коротко розглянуто квантування запропонованої моделі.