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On One Nonlinear Boundary-Value Problem in Kinetic Theory of Gases

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In the paper, the solvability of one nonlinear boundary-value problem arising in kinetic theory of gases is studied. We prove the existence of global solvability of a boundary-value problem in the Sobolev space $W^1_\infty(\mathbb{R}^+)$. The limit of the solution is found by using some a'priori estimations. For the case of power nonlinearity, the uniqueness of the solution in a certain class of functions is proved. Some examples illustrating the obtained results are given.

Key words: boundary-value problem, monotony, nonlinear integral equation, iteration, limit of solution.

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1. Introduction. Statement of the Problem

The paper is devoted to the following nonlinear boundary-value problem:

$$\pm s\frac{\partial \varphi^{\pm}(x,s)}{\partial x} + \varphi^{\pm}(x,s) = G(U(x)), \quad x>0,\, s>0, \tag{1}$$

$$\varphi^{+}(0,s) = G_1\left(\int_0^\infty Q(s,p)\varphi^{-}(0,p)dp\right), \tag{2}$$

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$$\varphi^-(x,s) = o\left(e^{\frac{x}{s}}\right), \quad x \to +\infty, \tag{3}$$

where

$$U(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-p^2}[\varphi^+(x,p) + \varphi^-(x,p)]dp. \tag{4}$$

The functions G and G_1 describe the nonlinear dependence in the right-hand side of integro-differential equation (1) and the nonlinear dependence of boundary condition (2), respectively. The function $Q(s,p)$ describes the general law of reflection and possesses the substochasticity property

$$Q(s,p) \geq 0,\ (s,p) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \int_0^\infty Q(s,p)dp \leq 1. \tag{5}$$

Boundary-value problem (1)–(4) can be derived from the Boltzmann equation within the framework of one model suggested in [5] and it has important applications in kinetic theory of gases (see [1–6] and references therein). By means of equations (1), (4) with boundary value conditions (2), (3), the flow of a gas with average mass velocity $U(x)$ in a half space $x > 0$ bounded by the plate wall $x = 0$ is described.

Problem (1)–(4) in a standard way can be reduced to the nonlinear integral equation

$$U(x) = \mu(x,U) + \int_0^\infty K(x-t)G(U(t))dt, \tag{6}$$

where

$$\mu(x,U) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{x}{s}} e^{-s^2} G_1 \left[\int_0^\infty Q(s,p)dp \int_0^\infty e^{-\frac{t}{p}} G(U(t))\frac{dt}{p}\right] ds, \tag{7}$$

$$K(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{|x|}{s}} e^{-s^2} \frac{ds}{s}. \tag{8}$$

In the linear case, where $G(x) \equiv x$, $\quad G_1(x) \equiv x$, the investigation of the problem (1)–(4) was carried out in a number of works (see [1, 6] and references therein).

In all the papers mentioned, the average mass velocity possesses asymptotics $O(x)$ when x tends to $+\infty$.

In the case of the linear law of reflection (i.e., where $G_1(x) \equiv x$), in [5], by imposing some natural conditions on the function G, it was shown that there exists qualitative difference between the solutions for the linear ($G(x) \equiv x$) and nonlinear cases. In the linear case, the solution has a linear growth away from the wall, while in the nonlinear case it has a bounded solution with the finite limit at infinity.

In the present paper, the question of solvability of nonlinear integral equation (6) is considered. Under certain conditions imposed on the functions G, G_1 (see below), the existence of a positive bounded solution of equation (6) is proved. The limit of the solution at infinity representing isothermal sliding coefficient is found. In the case of power nonlinearity (i.e., where $G(x) = x^\alpha$), the uniqueness of the solution in a certain class of functions is proved. Some examples of nonlinearity illustrating the obtained results are given.

2. Basic Results

Let $G_0(z)$ be a real measurable function defined on the set $(-\infty, +\infty)$ and satisfying the following conditions:

a) The numbers η and ξ are assumed to be the first positive roots of the equations $G_0(z) = z$ and $G_0(z) = 2z$, respectively, and besides $2\xi < \eta$,

b) $G_0 \in C[0, \eta]$, $G_0 \uparrow$ on the interval $[\xi, \eta]$.

Here are the examples of the above function:

$$1)\quad G_0(z) = z^p;\ 0 < p < 1,\ \xi = \left(\frac{1}{2}\right)^{\frac{1}{1-p}};\ \eta = 1, \tag{9}$$

$$2)\quad G_0(z) = e^{z-1},\ \xi \approx 0,2,\ \eta = 1. \tag{10}$$

Below, assuming that the initial function $G(z)$ is the local majorant for the function $G_0(z)$, η is a fixed point for the function $G_1(z)$, and imposing some natural conditions on the functions G and G_1, we will prove global solvability of equation (6) in the space of essentially bounded functions.

Moreover, by using special a'priori estimations, the limit of the solution at infinity will be found. In one important particular case, where $G(z) = z^p$, $0 < p < \frac{1}{2}$ and the function G_1 additionally satisfies the Lipschitz condition

$$|G_1(z_1) - G_1(z_2)| \le \alpha |z_1 - z_2|, \quad \alpha \in (0; 1], \quad z_1; z_2 \in \left[\left(\frac{1}{2}\right)^{\frac{1}{1-p}}, 1\right], \tag{11}$$

the uniqueness of the solutions in a certain class of functions will be proved.

The following results are true:

Theorem 1. *Let the functions $G(s)$ and $G_1(z)$ satisfy the following conditions:*

$$i_1)\quad G(z) \ge G_0(z), \quad z \in [\xi, \eta],\ G(\eta) = G_1(\eta) = \eta, \tag{12}$$

i_2) $G, G_1 \uparrow$ on the interval $[\xi,\eta]$ and $G_1(z) \geq 0,\ z \in [\xi,\eta],\ G; G_1 \in C[0;\eta]$ (13)

and the function $Q(s,p)$ satisfy condition (5).

Then equation (6) has a positive essentially bounded solution $U(x)$, and besides

$$\lim_{x\to\infty} U(x) = \eta. \tag{14}$$

Theorem 2. *Let $G(z) = z^p;\ p \in \left(0, \frac{1}{2}\right)$, and the function G_1 satisfy the conditions of Theorem 1 and condition (11). Then equation (6) has a unique solution in the following class of measurable functions:*

$$P = \{f(x) : \left(\frac{1}{2}\right)^{\frac{1}{1-p}} \leq f(x) \leq 1,\ x \in (0,+\infty)\}.$$

3. Proof of the Main Results

P r o o f of Theorem 1. With the help of equation (6), we consider the auxiliary Hammerstein type nonlinear integral equation

$$\varphi(x) = \int_0^\infty K(x-t)G_0(\varphi(t))dt, \quad x > 0, \tag{15}$$

with respect to an unknown measurable real function $\varphi(x)$, where kernel K is given by formula (8).

From (8), it follows that

$$K(-x) = K(x), \quad x \geq 0 \text{ and } \int_{-\infty}^{+\infty} K(x)dx = 1. \tag{16}$$

In [4], not only the existence of positive solution $\varphi(x)$ for equations (15), (16) was proved, but also the following properties were established:

$$\lim_{x\to\infty} \varphi(x) = \eta; \quad \varphi(x) \geq \xi, \quad x \geq 0. \tag{17}$$

Let us consider the iteration for basic equation (6) taking into account (7),

$$\begin{aligned} U_{n+1}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} G_1 \left(\int_0^\infty Q(s,p)\frac{dp}{p} \int_0^\infty e^{-\frac{t}{p}} G(U_n(t))dt \right) ds \\ + \int_0^\infty K(x-t)G(U_n(t))dt, \end{aligned} \tag{18}$$

$$U_0(x) = \varphi(x), \quad n = 0; 1; 2; \ldots, \; x \geq 0. \tag{19}$$

Due to monotony of the functions G and G_1 on the interval $[\xi, \eta]$, it is easy to check that

a) $U_n(x) \uparrow$ in n,

b) the functions $U_n(x)$ are measurable on the set $\mathbb{R}^+$; $n = 0; 1; 2, \ldots$.

Below we prove that

$$U_n(x) \leq \eta, \quad n = 0; 1; 2, \ldots. \tag{20}$$

In fact, in the case $n = 0$, inequality (20) is obvious because of $U_0(x) = \varphi(x)$, $\varphi(x) \uparrow$ on $\mathbb{R}^+$ and $\lim\limits_{x\to\infty} \varphi(x) = \eta$.

We assume that inequality (20) takes place for some $n \in \mathbb{N}$. Then in view of (12) and (5), from (18) we get

$$\begin{aligned} U_{n+1}(x) &\leq \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} G_1\left(\eta \int_0^\infty Q(s,p)dp\right) ds + \eta \int_0^\infty K(x-t)dt \\ &\leq \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} G_1(\eta) ds + \eta \int_{-\infty}^x K(y)dy \\ &= \eta \left(\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} ds + \int_{-\infty}^x K(y)dy \right) \equiv J. \end{aligned}$$

It is easy to verify that

$$\int_x^\infty K(y)dy = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} ds,$$

therefore $J = \eta$, and hence $U_{n+1} \leq \eta$.

Thus, the sequence of measurable functions $\{U_n(x)\}_{n=0}^\infty$ has a pointwise limit as $n \to +\infty$. By B. Levi's theorem, the function $U(x) = \lim\limits_{n\to\infty} U_n(x)$ satisfies equation (6). From (18)–(20), it also follows that

$$\varphi(x) \leq U(x) \leq \eta, \quad x \in \mathbb{R}^+. \tag{21}$$

As $\lim\limits_{x\to\infty} \varphi(x) = \eta$, then in view of (21), we immediately get

$$\lim_{x\to\infty} U(x) = \eta. \tag{22}$$

The theorem is proved.

P r o o f of Theorem 2. We assume the opposite. Let equation (6) have two solutions from P. We denote their difference by $\Delta U = U^1 - U^2$; $U^j \in P,\ j = 1, 2$. Then from (6), taking into account (11) and $G(z) = z^p$, $p \in \left(0, \frac{1}{2}\right)$, we have

$$\Delta U(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}}$$

$$\times \left[G_1\left(\int_0^\infty Q(s,p) \int_0^\infty e^{-\frac{t}{p}} G(U^1(t)) \frac{dtdp}{p} \right) - G_1\left(\int_0^\infty Q(s,p) \int_0^\infty e^{-\frac{t}{p}} G(U^2(t)) \frac{dtdp}{p} \right) \right] ds$$

$$+ \int_0^\infty K(x-t) \left[G(U^1(t)) - G(U^2(t)) \right] dt. \tag{23}$$

By the Lagrange theorem, it is easy to verity that

$$\text{if } \left(\frac{1}{2}\right)^{\frac{1}{1-p}} \le x_1, \quad x_2 \in 1, \quad \text{then } |x_1^p - x_2^p| \le 2p\,|x_1 - x_2|. \tag{24}$$

Using (24) and (5), from (23) we obtain

$$|\Delta U(x)| \le \frac{\alpha}{\sqrt{\pi}} \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} \int_0^\infty Q(s,p) \int_0^\infty e^{-\frac{t}{p}} |G(U^1(t)) - G(U^2(t))| \frac{1}{p}\, dtdpds$$

$$+ \int_0^\infty K(x-t) |G(U^1(t)) - G(U^2(t))| dt$$

$$\le \left[\frac{\alpha}{\sqrt{\pi}} 2p \int_0^\infty e^{-s^2} e^{-\frac{x}{s}} \int_0^\infty Q(s,p) dpds + 2p \int_{-\infty}^x K(y) dy \right] \sup_{t \ge 0} |U^1(t) - U^2(t)|$$

$$\le \left[2p\alpha \int_x^\infty K(y) dy + 2p \int_{-\infty}^x K(y) dy \right] \sup_{t \ge 0} |\Delta U(t)| \le 2p \sup_{t \ge 0} |\Delta U(t)|.$$

Hence,

$$(1 - 2p) \sup_{t \ge 0} |\Delta U(t)| \le 0. \tag{25}$$

As $p \in \left(0, \frac{1}{2}\right)$, then due to (25) we obtain that $\Delta U(t) = 0$ almost everywhere on $(0; +\infty)$, therefore $U^1(t) = U^2(t)$ almost everywhere on $(0; +\infty)$. The theorem is proved.

E x a m p l e s. As the function $G(z)$ can be chosen, for the examples of the functions G_0 see (9) and (10). However, we also give an example different from G_0,

$$G(z) = G_0(z) + \frac{\eta}{\pi}\frac{\sin^2 G_0(z)\pi}{\eta}.$$

Here are the examples of the functions $G_1(z)$:

$$\text{a)}\quad G_1(z) = e^{z-1};\ \eta = 1;\ \alpha = 1,$$

$$\text{b)}\quad G_1(z) = \eta - \beta\tilde{G}(\eta - z);\ \beta \in \left(0,\ \min\left(1; \frac{1}{\max\limits_{\xi\le z\le\eta} \tilde{G}'(z)}\right)\right],$$

where $\tilde{G}(\eta) = \eta$, $\tilde{G} \uparrow$ on $[\xi, \eta]$, $\max\limits_{\xi\le z\le\eta} \tilde{G}'(z) < +\infty$, $\tilde{G}(z) \ge 0$, $z \in [\xi, \eta]$.

For example, if $\tilde{G}(z) = z^2$, then $\eta = 1$ and $G_1(z) = 1 - \beta(1 - z)^2$, $\beta \in (0, \frac{1}{2}]$.

R e m a r k. It should be noted that the solution of initial boundary value problem (1)–(4) belongs to the space $W^1_\infty(\mathbb{R}^+)$ in x.

Thus, from (1)–(3), we get

$$\varphi^+(x, s) = C(s)e^{-\frac{x}{s}} + \int\limits_0^x e^{-\frac{(x-t)}{s}} G(U(t))\frac{dt}{s}, \tag{26}$$

$$\varphi^-(x, s) = \int\limits_x^\infty e^{-\frac{(t-x)}{s}} G(U(t))\frac{dt}{s}, \tag{27}$$

where

$$C(s) = G_1\left(\int\limits_0^\infty Q(s, p)\frac{dp}{p}\int\limits_0^\infty e^{-\frac{t}{p}} G(U(t))dt\right).$$

As $U \in L_\infty(0, +\infty)$, then from (26), (27) it follows that for each fixed $s \in (0, +\infty)$,

$$\varphi^\pm(x, s) \in W^1_\infty(\mathbb{R}^+).$$

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