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Generalized Duality, Hamiltonian Formalism and New Brackets

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It is shown that any singular Lagrangian theory: 1) can be formulated without the use of constraints by introducing a Clairaut-type version of the Hamiltonian formalism; 2) leads to a special kind of nonabelian gauge theory which is similar to the Poisson gauge theory; 3) can be treated as the many-time classical dynamics. A generalization of the Legendre transform to the zero Hessian case is done by using the mixed (envelope/general) solution of the multidimensional Clairaut equation. The equations of motion are written in the Hamilton-like form by introducing new antisymmetric brackets. It is shown that any classical degenerate Lagrangian theory is equivalent to the many-time classical dynamics. Finally, the relation between the presented formalism and the Dirac approach to constrained systems is given.

Key words: Dirac constraints, nonabelian gauge theory, degenerate Lagrangian, Hessian, Legendre transform, multidimensional Clairaut equation, gauge freedom, Poisson bracket, many-time dynamics.

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1. Introduction

Nowadays, many fundamental physical models are based on gauge field theories [73, 17]. On the classical level, they are described by a singular (degenerate) Lagrangian, which makes the passage to the Hamiltonian description, which is important for quantization, highly nontrivial and complicated (see, e.g., [66, 59]).

A common way to deal with singular theories is the Dirac approach [19] based on extending the phase space and constraints. This treatment of constrained theories has been deeply reviewed, e.g., in lecture notes [74] and books [30, 38]. In spite of its general success, the Dirac approach has some problems [20, 57, 72] and is not directly applicable in some cases, e.g., for irregular constrained systems (with linearly dependent constraints)

[56, 6] or so-called "pathological examples" [50]. Therefore, it is worthwhile to reconsider basic ideas of the Hamiltonian formalism in general from another point of view [24, 23].

In the standard approach to nonsingular theories [64, 4], the transition from Lagrangian to Hamiltonian description is carried out by using the Legendre transform and then finding the Hamiltonian as an envelope solution of the corresponding Clairaut equation [42, 41]. The main idea of our formulation is the following [25]: for singular theories, instead of the Lagrange multiplier procedure developed by Dirac [19], we construct and solve the corresponding multidimensional Clairaut equation [41]. In this way, we state that the ordinary duality can be generalized to the Clairaut duality [25].

In this paper we use our previous works [24, 25] to construct a self-consistent analog of the canonical (Hamiltonian) formalism and present a general algorithm for describing a Lagrangian system (singular or not) as a set of first-order differential equations without introducing the Lagrange multipliers. From mathematical viewpoint, we extend to the singular dynamical systems the well-known construction of Hamiltonian as a solution of the Clairaut equation developed in [4] for unconstrained systems. To simplify matters, we consider the systems with a finite number of degrees of freedom, use the local coordinates and the clear language of differential equations together with the Clairaut equation theory [42, 41].

Using the fact that for a singular Lagrangian system the Hessian matrix is degenerate and therefore has the rank less than its size, we separate out the group of "physical" (or regular/non-degenerate) and "non-physical" (degenerate) dynamical variables such that the Hessian matrix of the former is non-degenerate. On the other hand, the Clairaut equation has two kinds of solutions: the general solution and the envelope one [41]. The key idea is to use the envelope solution for "physical" variables and the general solution for "non-physical" ones, and therefore the separation of variables is unavoidable. In this way we obtain a unique analog of the Hamiltonian (called the mixed Hamilton–Clairaut) function which (formally) coincides with the the Hamiltonian function derived by the geometric approach [11, 67] and by the generalized Legendre transformation [13]. Then, using the mixed Hamilton–Clairaut function, we pass from the second-order Lagrange equations of motion to a set of the first-order Hamilton-like equations. The next important step is to exclude the so-called degenerate "momenta" and introduce the "physical" Hamilton–Clairaut function (which corresponds to the total Dirac Hamiltonian), which allows us to present the equations of motion as a system of differential equations for "physical" coordinates and momenta together with a system of linear equations for unresolved ("non-physical") velocities. Different kinds of solutions of this system of linear equations lead to the classification of singular systems which reminds the classification of constraints but does not coincide with it: the former does not contain analogs of higher constraints because there are no corresponding degenerate "momenta" at all. Some formulations without (primary) constraints were given in [32, 18, 51], and without any constraints but for special (regularizable) kind of Lagrangians, in [47, 46].

The "shortened" approach can play an important role for quantization of such complicated constrained systems as gauge field theories [43] and gravity [58]. To illustrate the power and simplicity of our method, we consider such examples, as the Cawley Lagrangian [12], which leads to difficulties in the Dirac approach, and the relativistic particle. The above Hamilton-like form of the equations of motion is described in terms of the newly defined antisymmetric brackets. Quantization of brackets can be done by means of the standard methods (see, e.g., [37]) without using Dirac quantization [19].

While analyzing the equations of motion corresponding to "unresolved" velocities, we arrive effectively at a kind of nonabelian gauge theory in the "degenerate" coordinate subspace which is similar to the Poisson gauge theory [28]. But in our case partial derivatives and Poisson brackets "live" in different subspaces. We also outline that the Clairaut-type formulation is equivalent to the many-time classical dynamics developed in [21, 48], if "nondynamical" (degenerate) coordinates are treated as additional "times". Finally, in Appendix, after introduction of "non-dynamical" momenta corresponding Lagrange multipliers and respective constraints, we show that the Clairaut-type formulation presented here corresponds to the Dirac approach [19].

2. The Legendre–Fenchel and Legendre Transforms

We start with a brief description of the standard Legendre–Fenchel and Legendre transforms for the theory with nondegenerate Lagrangian [5, 61]. Let $L\left(q^A, v^A\right)$, $A = 1, \dots n$, be a Lagrangian given by a function of $2n$ variables (n generalized coordinates q^A and n velocities $v^A = \dot{q}^A = dq^A/dt$) on the configuration space $\mathsf{T}M$, where M is a smooth manifold. We use the indices in the arguments to distinguish different kinds of coordinates (similarly to [69]). For the same reason, we use the summation signs with explicit ranges. Also, we consider the time-independent case for simplicity and conciseness, which will not influence on the main procedure.

By the convex approach (see, e.g., [60, 5]), a Hamiltonian $H\left(q^A, p_A\right)$ is a dual function on the phase space T^*M (or convex conjugate [61]) to the Lagrangian (in the second set of variables p_A) constructed by using the *Legendre–Fenchel transform* $L \overset{\mathfrak{Leg}^{\mathrm{Fen}}}{\longmapsto} H^{\mathrm{Fen}}$ defined by [27, 60],

$$H^{\mathrm{Fen}}\left(q^A, p_A\right) = \sup_{v^A} G\left(q^A, v^A, p_A\right), \tag{2.1}$$

$$G\left(q^A, v^A, p_A\right) = \sum_{B=1}^{n} p_B v^B - L\left(q^A, v^A\right). \tag{2.2}$$

Note that this definition is very general and it can be applied to nonconvex [2] and nondifferentiable [70] functions $L\left(q^A, v^A\right)$, which can lead to numerous extended versions of Hamiltonian formalism (see, e.g., [15, 62, 40]). Also, a generalization of convex conjugacy can be achieved by substituting in (2.2) the form $p_A v^A$ by any function $\Psi\left(p_A, v^A\right)$ satisfying special conditions [33].

In the standard mechanics [36], one usually restricts to convex, smooth and differentiable Lagrangians (see, e.g., [5, 65]). Then the coordinates $q^A(t)$ are treated as fixed (passive with respect to the Legendre transform) parameters, and the velocities $v^A(t)$ are assumed to be independent functions of time.

According to our assumptions, the supremum (2.1) is attained by finding an extremum point $v^A = v^A_{extr}$ of the ("pre-Hamiltonian") function $G\left(q^A, v^A, p_A\right)$ which leads to the supremum condition

$$p_B = \left.\frac{\partial L\left(q^A, v^A\right)}{\partial v^B}\right|_{v^A = v^A_{extr}}. \tag{2.3}$$

It is commonly assumed (see, e.g., [5, 65, 36]) that the only way to get rid of dependence on the velocities v^A in the r.h.s. of (2.1) is to resolve (2.3) with respect to velocities and find its solution given by a set of functions

$$v^B_{extr} = V^B\left(q^A, p_A\right). \tag{2.4}$$

This can be done only in the class of nondegenerate Lagrangians $L\left(q^A, v^A\right) = L^{\text{nondeg}}\left(q^A, v^A\right)$ (in the second set of variables v^A), which is equivalent to the case

$$\det \left\| \frac{\partial^2 L^{\text{nondeg}}\left(q^A, v^A\right)}{\partial v^B \partial v^C} \right\| \neq 0. \tag{2.5}$$

Then, substituting v^A_{extr} to (2.1), we can obtain the standard Hamiltonian (see, e.g., [5, 36]),

$$\begin{aligned} H\left(q^A, p_A\right) &\overset{def}{=} G\left(q^A, v^A_{extr}, p_A\right) \\ &= \sum_{B=1}^{n} p_B V^B\left(q^A, p_A\right) - L^{\text{nondeg}}\left(q^A, V^A\left(q^A, p_A\right)\right). \end{aligned} \tag{2.6}$$

The passage from the nondegenerate Lagrangian $L^{\text{nondeg}}\left(q^A, v^A\right)$ to the Hamiltonian $H\left(q^A, p_A\right)$ is called the *Legendre transform* (of functions) which will be denoted by $L^{\text{nondeg}} \overset{\mathfrak{Leg}}{\longmapsto} H$.

By the geometric approach [68, 54, 1], the *Legendre transform* of the functions $L^{\text{nondeg}} \overset{\mathfrak{Leg}}{\longmapsto} H$ is tantamount to the *Legendre transformation* from the configuration space to the phase space $\mathsf{Leg} : \mathsf{T}M \to \mathsf{T}^*M$ (or between submanifolds in the presence of constraints [55, 8, 29]). Nevertheless, here we will use local coordinates and the language of differential equations associated with function transforms, in particular the Clairaut equation theory [42, 41].

3. The Legendre–Clairaut Transform

The connection between the Legendre transform, convexity and the Clairaut equation has a long story [42, 64] (see also [4]). Here we present an alternative way by applying the supremum condition (2.3) and considering the related multidimensional Clairaut equation proposed in [24].

We differentiate (2.6) by the momenta p_A and use the supremum condition (2.3) to get

$$\frac{\partial H\left(q^A, p_A\right)}{\partial p_B} = V^B\left(q^A, p_A\right) + \sum_{C=1}^{n}\left(p_C - \left.\frac{\partial L\left(q^A, v^A\right)}{\partial v^C}\right|_{v^C = V^C\left(q^A, p_A\right)}\right)\frac{\partial V^C\left(q^A, p_A\right)}{\partial p_B} = V^B\left(q^A, p_A\right), \quad (3.1)$$

which can be called the *dual supremum condition* (indeed, this gives the first set of the Hamilton equations, see below). Relations (2.3), (2.6) and (3.1) represent a particular case of the Donkin theorem (see, e.g., [36]).

Then we substitute (3.1) in (2.6) and obtain

$$H\left(q^A, p_A\right) \equiv \sum_{B=1}^{n} p_B \frac{\partial H\left(q^A, p_A\right)}{\partial p_B} - L^{\mathrm{nondeg}}\left(q^A, \frac{\partial H\left(q^A, p_A\right)}{\partial p_C}\right), \quad (3.2)$$

which contains no manifest dependence on velocities at all. It is important that for nonsingular Lagrangians, relation (3.2) is an *identity*, which follows from (2.3), (2.6) and (3.1) by our construction. This relation can also be obtained if the geometric approach from [10] is used.

Now we make the main step: we consider equation (3.2) *by itself* (without referring to (2.3), (2.6) and (3.1)) as a *definition of a new transform* which is a solution of the nonlinear partial differential equation (the multidimensional Clairaut equation) [24, 25]

$$H^{\mathrm{Cl}}\left(q^A, \lambda_A\right) = \sum_{B=1}^{n} \lambda_B \frac{\partial H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_B} - L\left(q^A, \frac{\partial H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_A}\right) \quad (3.3)$$

in the formal independent variables λ_A (initially not connected with p_A defined by (2.3)) and $L\left(q^A, v_A\right)$, that is, *any* differentiable smooth function of $2n$ variables q^A, v_A, where the coordinates q^A play the role of external parameters. It is very important that in (3.3) we do not demand that nondegeneracy condition (2.5) be imposed on $L\left(q^A, v_A\right)$. We call the transform defined by (3.3), $L \overset{\mathfrak{Leg}^{\mathrm{Cl}}}{\longmapsto} H^{\mathrm{Cl}}$, a *Clairaut duality transform* (or the *Legendre–Clairaut transform*) and $H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)$, a *Hamilton–Clairaut function* [24, 25].

Note that relation (2.3), which is commonly treated as a definition of all dynamical momenta p_A, in our approach is the supremum condition for some of the independent variables of the Clairaut duality transform λ_A. In the differential equation language, λ_A are independent mathematical variables having no connection with any physical dynamics. Before solving the Clairaut equation (3.3) and applying supremum condition (2.3), which in our language is $\lambda_A = p_A = \partial L \diagup \partial v^A$, we must notice that the independent variables λ_A are not connected with the Lagrangian and therefore cannot be called momenta. The independent variables λ_A are used to find all possible solutions of the Clairaut Eq. (3.3) for nondegenerate and degenerate Lagrangians $L\left(q^A, v^A\right)$. Only those of λ_A which will be restricted by supremum condition (2.3) can be interpreted as momenta with the corresponding geometric description in terms of the cotangent space.

The difference between the Legendre–Clairaut transform and the Legendre transform is crucial for degenerate Lagrangian theories [24]. Specifically, multidimensional Clairaut Eq. (3.3) has solutions even for degenerate Lagrangians $L\left(q^A, v^A\right) = L^{\deg}\left(q^A, v^A\right)$ when the Hessian is zero,

$$\det \left\| \frac{\partial^2 L^{\deg}\left(q^A, v^A\right)}{\partial v^B \partial v^C} \right\| = 0. \tag{3.4}$$

In this case, $\mathfrak{Leg}^{\mathrm{Cl}}$, the Legendre–Clairaut transform of functions (3.3), is another along with the Legendre–Fenchel transform $\mathfrak{Leg}^{\mathrm{Fen}}$ counterpart to the ordinary Legendre transform (2.6) in the case of degenerate Lagrangians. The Clairaut equation (3.3) always has a solution which is independent of the properties of the Hessian as well as of solving the supremum condition (2.3) with respect to velocities.

To find the solutions of (3.3), we differentiate it by λ_C to obtain

$$\sum_{B=1}^{n} \left[\lambda_B - \left. \frac{\partial L\left(q^A, v^A\right)}{\partial v^B} \right|_{v^B = \frac{\partial H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_B}} \right] \frac{\partial^2 H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_B \partial \lambda_C} = 0. \tag{3.5}$$

Now we apply the ordinary method of solving the Clairaut equation (see Appendix A). There are two possible solutions of (3.5), the first, in which square brackets vanish (envelope solution), and the second, in which double derivative in velocity vanishes (general solution). The l.h.s. of (3.5) is a sum over B and it is quite conceivable that one may vanish for some B and the other vanishes for other B. The physical reason of choosing a particular solution is given in Sec. 4. Thus we have two solutions of the Clairaut equation:

1) The *envelope solution* defined by the first multiplier in (3.5) being zero, that is,

$$\lambda_B = p_B = \frac{\partial L\left(q^A, v^A\right)}{\partial v^B}, \tag{3.6}$$

which coincides with supremum condition (2.3), together with (3.1). In this way, we obtain the standard Hamiltonian (2.6),

$$H_{env}^{\mathrm{Cl}}\left(q^A,\lambda_A\right)|_{\lambda_A=p_A}=H\left(q^A,p_A\right). \tag{3.7}$$

Thus, in the nondegenerate case, the "envelope" Legendre–Clairaut transform $\mathfrak{Leg}_{env}^{\mathrm{Cl}}: L\to H_{env}^{\mathrm{Cl}}$ coincides with the ordinary Legendre transform constructed here.

2) A *general solution* defined by

$$\frac{\partial^2 H^{\mathrm{Cl}}\left(q^A,\lambda_A\right)}{\partial\lambda_B\partial\lambda_C}=0, \tag{3.8}$$

which gives $\dfrac{\partial H^{\mathrm{Cl}}\left(q^A,\lambda_A\right)}{\partial\lambda_B}=c^B$, where c^B are arbitrary smooth functions of q^A, and the latter are considered in (3.3) as parameters (passive variables). Then the general solution takes the form

$$H_{gen}^{\mathrm{Cl}}\left(q^A,\lambda_A,c^A\right)=\sum_{B=1}^{n}\lambda_B c^B-L\left(q^A,c^A\right), \tag{3.9}$$

which corresponds to a "general" Legendre–Clairaut transform $\mathfrak{Leg}_{gen}^{\mathrm{Cl}}: L\to H_{gen}^{\mathrm{Cl}}$. Note that the general solution $H_{gen}^{\mathrm{Cl}}\left(q^A,\lambda_A,c^A\right)$ is always linear in the variables λ_A and the latter are not actually the dynamical momenta p_A, because we do not have the envelope solution condition (3.6), and therefore now there is no supremum condition (2.3). The variables c^A are in fact unresolved velocities v^A in the case of the general solution.

Note that in the standard way, $\mathfrak{Leg}_{env}^{\mathrm{Cl}}$ can be also obtained by finding the envelope of the general solution [4], i.e., differentiating (3.9) by c^A,

$$\frac{\partial H_{gen}^{\mathrm{Cl}}\left(q^A,\lambda_A,c^A\right)}{\partial c^B}=\lambda_B-\frac{\partial L\left(q^A,c^A\right)}{\partial c^B}=0, \tag{3.10}$$

which coincides with (3.6) and (2.3). This means that $H_{gen}^{\mathrm{Cl}}\left(q^A,\lambda_A,c^A\right)|_{c^A=v^A}$ is the "pre-Hamiltonian" (2.2) needed to find the supremum in (2.1).

Let us consider the classical example of a one-dimensional oscillator.

E x a m p l e 3.1. Let $L(x,v)=mv^2/2-kx^2/2$ (m, k are constants), then the corresponding Clairaut equation (3.3) for $H=H^{\mathrm{Cl}}(x,\lambda)$ is

$$H=\lambda H_\lambda'-\frac{m}{2}\left(H_\lambda'\right)^2+\frac{kx^2}{2}, \tag{3.11}$$

where the prime denotes partial differentiation with respect to λ. The general solution of (3.11) is

$$H_{gen}^{\mathrm{Cl}}(x,\lambda,c)=\lambda c-\frac{mc^2}{2}+\frac{kx^2}{2}, \tag{3.12}$$

where c is an arbitrary function ("unresolved velocity" v). The envelope solution (with $\lambda = p$) can be found from the condition

$$\frac{\partial H^{\mathrm{Cl}}}{\partial c} = p - mc = 0 \Longrightarrow c_{extr} = \frac{p}{m},$$

which gives

$$H_{env}^{\mathrm{Cl}}(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2} \tag{3.13}$$

in the standard way.

E x a m p l e 3.2. Let $L(x,v) = x\exp kv$, then the corresponding Clairaut equation for $H = H^{\mathrm{Cl}}(x,\lambda)$ is

$$H = \lambda H'_\lambda - x\exp\left(kH'_\lambda\right). \tag{3.14}$$

The general solution is

$$H_{gen}^{\mathrm{Cl}}(x,\lambda) = \lambda c - x\exp kc, \tag{3.15}$$

where c is a smooth function of x.

The envelope solution (with $\lambda = p$) can be found by differentiating the general solution (3.15),

$$\frac{\partial H^{\mathrm{Cl}}}{\partial c} = p - x\exp kc = 0 \Longrightarrow c_{extr} = \frac{1}{k}\ln\frac{p}{x},$$

which leads to

$$H_{env}^{\mathrm{Cl}}(x,p) = \frac{p}{k}\ln\frac{p}{x} - p. \tag{3.16}$$

4. The Mixed Legendre–Clairaut Transform

Now consider a singular Lagrangian $L\left(q^A, v^A\right) = L^{\deg}\left(q^A, v^A\right)$ for which the Hessian is zero (3.4). This means that the rank of the Hessian matrix $W_{AB} = \frac{\partial^2 L\left(q^A, v^A\right)}{\partial v^B \partial v^C}$ is $r < n$, and we suppose that r is constant. We rearrange the indices of W_{AB} in such a way that a nonsingular minor of rank r appears in the upper left corner [31]. Represent the index A as follows: if $A = 1,\ldots,r$, we replace A with i (the "regular" index), and if $A = r+1,\ldots,n$, we replace A with α (the "degenerate" index). Obviously, $\det W_{ij} \neq 0$, and $\operatorname{rank} W_{ij} = r$. Thus any set of the variables labelled by a single index splits as a disjoint union of two subsets. We call these subsets *regular* (having Latin indices) and *degenerate* (having Greek indices).

The standard Legendre transform $\mathfrak{Leg}$ is not applicable in the degenerate case because condition (2.5) is not valid [11, 67]. Therefore the supremum condition (2.3) cannot be resolved with respect to degenerate A, but it can be resolved only for regular

A because $\det W_{ij} \neq 0$. On the contrary, the Clairaut duality transform given by (3.3) is independent in spite of whether the Hessian is zero or not [24]. Thus we state the main idea of the formalism we present here: *the ordinary duality can be generalized to the Clairaut duality*, i.e., the standard Legendre transform $\mathfrak{Leg}$, given by (2.6), can be generalized to the singular Lagrangian theory using the Legendre-Clairaut transform $\mathfrak{Leg}^{\mathrm{Cl}}$ given by the multidimensional Clairaut equation (3.3).

To find its solutions, we differentiate (3.3) by λ_A and split the sum (3.5) in B as follows:

$$\sum_{i=1}^{r}\left[\lambda_i - \frac{\partial L\left(q^A, v^A\right)}{\partial v^i}\right]\frac{\partial^2 H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_i \partial \lambda_C} + \sum_{\alpha=r+1}^{n}\left[\lambda_\alpha - \frac{\partial L\left(q^A, v^A\right)}{\partial v^\alpha}\right]\frac{\partial^2 H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_\alpha \partial \lambda_C} = 0. \tag{4.1}$$

As $\det W_{ij} \neq 0$, we suggest to replace (4.1) by the conditions

$$\lambda_i = p_i = \frac{\partial L\left(q^A, v^A\right)}{\partial v^i}, \quad i = 1, \ldots, r, \tag{4.2}$$

$$\frac{\partial^2 H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_\alpha \partial \lambda_C} = 0, \quad \alpha = r+1, \ldots n. \tag{4.3}$$

In this way we obtain a *mixed envelope/general solution* of the Clairaut equation (cf. [24]). We resolve (4.2) by the regular velocities $v^i = V^i\left(q^A, p_i, c^\alpha\right)$ and write down a solution of (4.3) as

$$\frac{\partial H^{\mathrm{Cl}}\left(q^A, \lambda_A\right)}{\partial \lambda_\alpha} = c^\alpha, \tag{4.4}$$

where c^α are arbitrary variables corresponding to the unresolved velocities v^α. Finally we obtain a *mixed Hamilton–Clairaut* function

$$H^{\mathrm{Cl}}_{mix}\left(q^A, p_i, \lambda_\alpha, v^\alpha\right) = \sum_{i=1}^{r} p_i V^i\left(q^A, p_i, v^\alpha\right) + \sum_{\beta=r+1}^{n} \lambda_\beta v^\beta - L\left(q^A, V^i\left(q^A, p_i, v^\alpha\right), v^\alpha\right), \tag{4.5}$$

which is the desired "mixed" Legendre–Clairaut transform of the functions $L \overset{\mathfrak{Leg}^{\mathrm{Cl}}_{mix}}{\longmapsto} H^{\mathrm{Cl}}_{mix}$ written in coordinates.

Note that (4.5) was obtained formally as a mixed general/envelope solution of the Clairaut equation for the sought-for Hamilton–Clairaut function without any reference to the dynamics (this connection will be considered in the next section). Nevertheless, H^{Cl}_{mix} coincides with the corresponding functions derived from the "slow and careful

Legendre transformation" [69] and the "generalized Legendre transformation" [13], as well as from the implicit partial differential equation on the cotangent bundle [52, 16] in the local coordinates [75] and in the general geometric approach [53].

E x a m p l e 4.3. Let $L(x, y, v_x, v_y) = myv_x^2/2 + kxv_y$, then the corresponding Clairaut equation for $H = H^{\mathrm{Cl}}(x, y, \lambda_x, \lambda_y)$ is

$$H = \lambda_x H'_{\lambda_x} + \lambda_y H'_{\lambda_y} - \frac{my}{2}\left(H'_{\lambda_x}\right)^2 - kxH'_{\lambda_y}. \tag{4.6}$$

The general solution of (4.6) is

$$H_{gen}^{\mathrm{Cl}}(x, y, \lambda_x, \lambda_y, c_x, c_y) = \lambda_x c_x + \lambda_y c_y - \frac{myc_x^2}{2} - kxc_y,$$

where c_x, c_y are arbitrary functions of the passive variables x, y. Then we differentiate

$$\frac{\partial H_{gen}^{\mathrm{Cl}}}{\partial c_x} = p_x - myc_x = 0, \implies c_x^{extr} = \frac{p_x}{my},$$
$$\frac{\partial H_{gen}^{\mathrm{Cl}}}{\partial c_y} = \lambda_y - kx.$$

Finally, we solve the first equation with respect to c_x and treat $c_y \longmapsto v_y$ as an "unresolved velocity" . This way we obtain the mixed Hamiltonian–Clairaut function

$$H_{mix}^{\mathrm{Cl}}(x, y, p_x, \lambda_y, v_y) = \frac{p_x^2}{2my} + v_y(\lambda_y - kx). \tag{4.7}$$

This result can be compared with that obtained in the geometric approach based on the reduction of the Hamiltonian Morse family in [69].

5. Hamiltonian Formulation of Singular Lagrangian Systems

Let us use the mixed Hamilton–Clairaut function $H_{mix}^{\mathrm{Cl}}\left(q^A, p_i, \lambda_\alpha, v^\alpha\right)$ (4.5) to describe a singular Lagrangian theory in terms of the system of ordinary first-order differential equations. In our formulation we split a set of the standard Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L\left(q^A, v^A\right)}{\partial v^B} = \frac{\partial L\left(q^A, v^A\right)}{\partial q^B} \tag{5.1}$$

into two subsets according to the index B being either regular ($B = i = 1, \ldots, r$) or degenerate ($B = \alpha = r+1, \ldots n$). We use the designation of "physical" momenta (4.2) in the regular subset only such that the Lagrange equations become

$$\frac{dp_i}{dt} = \frac{\partial L\left(q^A, v^A\right)}{\partial q^i}, \tag{5.2}$$

$$\frac{dB_\alpha\left(q^A, p_i\right)}{dt} = \left.\frac{\partial L\left(q^A, v^A\right)}{\partial q^\alpha}\right|_{v^i = V^i\left(q^A, p_i, v^\alpha\right)}, \tag{5.3}$$

where

$$B_\alpha\left(q^A,p_i\right) \overset{def}{=} \left.\frac{\partial L\left(q^A,v^A\right)}{\partial v^\alpha}\right|_{v^i=V^i\left(q^A,p_i,v^\alpha\right)} \tag{5.4}$$

are given functions determining the dynamics of the singular Lagrangian system in the "degenerate" sector. The functions $B_\alpha\left(q^A,p_i\right)$ are independent of the unresolved velocities v^α since the $\operatorname{rank} W_{AB}=r$. One should also take into account that now

$$\frac{dq^i}{dt}=V^i\left(q^A,p_i,v^\alpha\right), \quad \frac{dq^\alpha}{dt}=v^\alpha. \tag{5.5}$$

Note that before imposing the Lagrange Eqs. (5.2), when solving the Clairaut Eq. (3.3), the arguments of $L\left(q^A,v^A\right)$ were treated as independent variables.

A passage to an analog of the Hamiltonian formalism can be done by the standard procedure: consider the full differential of both sides of (4.5) and use supremum condition (4.2) which gives (note that in the previous sections the Lagrange equations of motion (5.1) were not used)

$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial p_i}=V^i\left(q^A,p_i,v^\alpha\right),$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial \lambda_\alpha}=v^\alpha,$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial q^i}=-\left.\frac{\partial L\left(q^A,v^A\right)}{\partial q^i}\right|_{v^i=V^i\left(q^A,p_i,v^\alpha\right)}+\sum_{\beta=r+1}^{n}\left[\lambda_\beta-B_\beta\left(q^A,p_i\right)\right]\frac{\partial v^\beta}{\partial q^i},$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial q^\alpha}=-\left.\frac{\partial L\left(q^A,v^A\right)}{\partial q^\alpha}\right|_{v^i=V^i\left(q^A,p_i,v^\alpha\right)}+\sum_{\beta=r+1}^{n}\left[\lambda_\beta-B_\beta\left(q^A,p_i\right)\right]\frac{\partial v^\beta}{\partial q^\alpha}.$$

Applying of (5.2) yields the system of equations which gives a Hamiltonian–Clairaut description of a singular Lagrangian system

$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial p_i}=\frac{dq^i}{dt}, \tag{5.6}$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial \lambda_\alpha}=\frac{dq^\alpha}{dt}, \tag{5.7}$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial q^i}=-\frac{dp_i}{dt}+\sum_{\beta=r+1}^{n}\left[\lambda_\beta-B_\beta\left(q^A,p_i\right)\right]\frac{\partial v^\beta}{\partial q^i}, \tag{5.8}$$
$$\frac{\partial H^{\mathrm{Cl}}_{mix}}{\partial q^\alpha}=\frac{dB_\alpha\left(q^A,p_i\right)}{dt}+\sum_{\beta=r+1}^{n}\left[\lambda_\beta-B_\beta\left(q^A,p_i\right)\right]\frac{\partial v^\beta}{\partial q^\alpha}. \tag{5.9}$$

The system (5.6)–(5.9) has two disadvantages: the first, it contains the "nondynamical momenta" λ_α; the second, it has derivatives of unresolved velocities v^α. We observe that we can get rid of these difficulties if we reformulate (5.6)–(5.9) by introducing a "physical" Hamiltonian

$$H_{phys}\left(q^A,p_i\right)=H_{mix}^{\mathrm{Cl}}\left(q^A,p_i,\lambda_\alpha,v^\alpha\right)-\sum_{\beta=r+1}^{n}\left[\lambda_\beta-B_\beta\left(q^A,p_i\right)\right]v^\beta \tag{5.10}$$

which does not depend on the variables λ_α ("nondynamical momenta") at all by the construction

$$\frac{\partial H_{phys}}{\partial\lambda_\alpha}=0 \tag{5.11}$$

(cf. (4.4) and (4.5)). Then the "physical" Hamiltonian (5.10) can be written in the form

$$\begin{aligned}H_{phys}\left(q^A,p_i\right)&=\sum_{i=1}^{r}p_iV^i\left(q^A,p_i,v^\alpha\right)\\&+\sum_{\alpha=r+1}^{n}B_\alpha\left(q^A,p_i\right)v^\alpha-L\left(q^A,V^i\left(q^A,p_i,v^\alpha\right),v^\alpha\right).\end{aligned} \tag{5.12}$$

Using (4.2), we can show that the r.h.s. of (5.12) does not depend on λ_α, and for the degenerate velocities v^α one has

$$\frac{\partial H_{phys}}{\partial v^\alpha}=0, \tag{5.13}$$

which justifies the term "physical". Therefore, the time evolution of the singular Lagrangian system (5.1) is determined by $(n-r+1)$ functions $H_{phys}\equiv H_{phys}\left(q^A,p_i\right)$ and $B_\alpha\equiv B_\alpha\left(q^A,p_i\right)$. Writing $\left(q^A,p_i\right)=\left(q^\alpha|q^i,p_i\right)\in R^{n-r}\times Sp\left(r,r\right)\equiv M_{phys}$, where R^{n-r} is a real space of the dimension $(n-r)$, and $Sp\left(r,r\right)$ is a symplectic space of the dimension (r,r), we observe that $H_{phys}:R^{n-r}\times Sp\left(r,r\right)\to R$ and $B_\alpha:R^{n-r}\times Sp\left(r,r\right)\to R^{n-r}$.

Then we use (5.6)–(5.9) to deduce the main result of our Clairaut-type formulation that the sought-for system of ordinary first-order differential equations (*the Hamilton–Clairaut system*) which describes any singular Lagrangian classical system (satisfying the second-order Lagrange Eqs. (5.1)), has the form

$$\frac{dq^i}{dt}=\left\{q^i,H_{phys}\right\}_{phys}-\sum_{\beta=r+1}^{n}\left\{q^i,B_\beta\right\}_{phys}\frac{dq^\beta}{dt},\ \ i=1,\ldots r, \tag{5.14}$$

$$\frac{dp_i}{dt}=\left\{p_i,H_{phys}\right\}_{phys}-\sum_{\beta=r+1}^{n}\left\{p_i,B_\beta\right\}_{phys}\frac{dq^\beta}{dt},\ \ i=1,\ldots r, \tag{5.15}$$

$$\begin{aligned}&\sum_{\beta=r+1}^{n}\left[\frac{\partial B_\beta}{\partial q^\alpha}-\frac{\partial B_\alpha}{\partial q^\beta}+\left\{B_\alpha,B_\beta\right\}_{phys}\right]\frac{dq^\beta}{dt}\\&=\frac{\partial H_{phys}}{\partial q^\alpha}+\left\{B_\alpha,H_{phys}\right\}_{phys},\qquad \alpha=r+1,\ldots,n,\end{aligned} \tag{5.16}$$

where

$$\{X, Y\}_{phys} = \sum_{i=1}^{n-r} \left(\frac{\partial X}{\partial q^i} \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i} \frac{\partial X}{\partial p_i} \right) \tag{5.17}$$

is the "physical" Poisson bracket (in regular variables q^i, p_i) for the functions X and Y on M_{phys}.

The system (5.14)–(5.16) is equivalent to the Lagrange equations of motion (5.1) by the construction. Thus, the Clairaut-type formulation (5.14)–(5.16) is valid for any Lagrangian theory without additional conditions, as opposite to other approaches (see, e.g., [57, 72]).

E x a m p l e 5.4. (Cawley [12]) Let $L = \dot{x}\dot{y} + zy^2/2$, then the equations of motion are

$$\ddot{x} = yz, \quad \ddot{y} = 0, \quad y^2 = 0. \tag{5.18}$$

Because the Hessian has rank 2, and the velocity $\dot{z}$ does not enter into the Lagrangian, the only degenerate velocity is $\dot{z}$ $(\alpha = z)$, the regular momenta are $p_x = \dot{y}$, $p_y = \dot{x}$ $(i = x, y)$. Thus, we have

$$H_{phys} = p_x p_y - \frac{1}{2} z y^2, \quad B_z = 0.$$

The equations of motion (5.14)–(5.15) are

$$\dot{p}_x = 0, \quad \dot{p}_y = yz, \tag{5.19}$$

and condition (5.16) gives

$$\frac{\partial H_{phys}}{\partial z} = -\frac{1}{2} y^2 = 0. \tag{5.20}$$

Observe that (5.19) and (5.20) coincide with the initial Lagrange equations of motion (5.18).

Since the number of equations $r + r + n - r = n + r$ coincides with the number of the sought-for variables $n_{q^i} = r$, $n_{p_i} = r$, $n_{q^\alpha} = n - r$, we deduce that there are no constraints in (5.14)–(5.16) at all. In particular, the system (5.16) has $(n - r)$ equations, which exactly coincides with the number of the sought-for "unresolved" velocities $v^\alpha = \frac{dq^\alpha}{dt}$. Therefore, (5.16) is a standard system of linear algebraic equations with respect to v^α, but not constraints (when there are more sought-for variables than equations).

E x a m p l e 5.5. ([7, 71]) Let us consider a classical particle on R^3 with the regular Lagrangian $\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/2$ subject to the nonholonomic constraint $\dot{z} = y\dot{x}$. To apply the Clairaut equation method, we introduce an extra coordinate u. Then this system is equivalent to the singular Lagrangian system on R^4 described by

$$L = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} + u\left(\dot{z} - y\dot{x}\right). \tag{5.21}$$

The Lagrange equations of motion are straightforward (cf. [7])

$$\ddot{x} - \dot{u}y - \dot{y}u = 0, \quad \ddot{y} + u\dot{x} = 0, \quad \ddot{z} + \dot{u} = 0, \quad \dot{z} - y\dot{x} = 0. \tag{5.22}$$

The Hessian of (5.21) is zero, and thus the system is singular. The rank of the Hessian matrix $\operatorname{diag}(1,1,1,0)$ being 3, we have 3 regular and 1 degenerate variables. First, we should find the "physical" Hamiltonian using the Clairaut equation formalism and then pass from the second-order Eqs. (5.22) to the first-order equations similar to (5.14)–(5.16). Let us consider multidimensional Clairaut Eq. (3.3) for the Hamilton–Clairaut function $H \equiv H^{Cl}(x, y, z, u, \lambda_x, \lambda_y, \lambda_z, \lambda_u)$,

$$\begin{aligned} H &= \lambda_x H'_{\lambda_x} + \lambda_y H'_{\lambda_y} + \lambda_z H'_{\lambda_z} + \lambda_u H'_{\lambda_u} \\ &\quad - \frac{1}{2}\left(H'_{\lambda_x}\right)^2 - \frac{1}{2}\left(H'_{\lambda_y}\right)^2 - \frac{1}{2}\left(H'_{\lambda_z}\right)^2 - uH'_{\lambda_z} + yuH'_{\lambda_x}. \end{aligned} \tag{5.23}$$

The general solution of (5.23) is

$$H_{gen} = \lambda_x c_x + \lambda_y c_y + \lambda_z c_z + \lambda_u c_u - \frac{c_x^2 + c_y^2 + c_z^2}{2} - uc_z + yuc_x, \tag{5.24}$$

where initially c_x, c_y, c_z, c_u are arbitrary functions of the passive (with respect to the Clairaut Eq. (5.23)) variables x, y, z, u. To find supremum conditions (3.10), we write the derivatives

$$\frac{\partial H_{gen}}{\partial c_x} = \lambda_x - c_x + yu = 0, \tag{5.25}$$

$$\frac{\partial H_{gen}}{\partial c_y} = \lambda_y - c_y = 0, \tag{5.26}$$

$$\frac{\partial H_{gen}}{\partial c_z} = \lambda_z - c_z - u = 0, \tag{5.27}$$

$$\frac{\partial H_{gen}}{\partial c_u} = \lambda_u. \tag{5.28}$$

Observe that only 3 first conditions here can be resolved with respect to c_i $(i = x, y, z)$, and therefore these λ_i correspond to the "physical" momenta (4.2), that is, $\lambda_i = p_i = \partial L \diagup \partial v_i$ $(i = x, y, z)$. Thus, the extremum values of c_i are

$$c_x^{extr} = p_x + yu, \quad c_y^{extr} = p_y, \quad c_z^{extr} = p_z - u, \tag{5.29}$$

while c_u becomes the "unresolved" velocity $c_u = v_u$. In this way, inserting (5.29) into (5.24), for the mixed Hamilton–Clairaut function (4.5) we have

$$H_{mix}^{Cl} = \frac{p_x^2 + p_y^2 + p_z^2}{2} + \lambda_u v_u + u\left(yp_x - p_z\right) + u^2\frac{1 + y^2}{2}. \tag{5.30}$$

Now we calculate function (5.4) and the "physical" Hamiltonian (5.10),

$$B_u = \frac{\partial L}{\partial \dot{u}} = 0, \tag{5.31}$$

$$H_{phys} = \frac{p_x^2 + p_y^2 + p_z^2}{2} + u\left(yp_x - p_z\right) + u^2 \frac{1+y^2}{2} \tag{5.32}$$

not depending on λ_u and v_u. Using (5.14)–(5.16), we obtain the Hamilton–Clairaut system

$$\dot{x} = p_x + yu, \quad \dot{y} = p_y, \quad \dot{z} = p_z - u, \tag{5.33}$$

$$\dot{p}_x = 0, \quad \dot{p}_y = -u\left(p_x + yu\right), \quad \dot{p}_z = 0, \tag{5.34}$$

$$yp_x - p_z + u\left(1 + y^2\right) = 0, \tag{5.35}$$

which coincides with the system of Lagrange equations of motion (5.22) by the construction. It is remarkable that the "degenerate"variable u is determined by the algebraic equation (5.35),

$$u = \frac{p_z - yp_x}{1 + y^2}, \tag{5.36}$$

and therefore the singular system (5.21) has no "gauge" degrees of freedom.

In general, if a dynamical system is nonsingular, it has no "degenerate" variables at all because the rank r of the Hessian is full ($r = n$). The distinguishing property of any singular system ($r < n$) is clear and simple in our Clairaut-type approach: it contains an additional system of the linear algebraic equations (5.16) for the "unresolved" velocities v^α (not constraints), which can be analyzed and solved by the standard linear algebra methods. Indeed, the linear algebraic system (5.16) gives a full classification of singular Lagrangian theories presented in the next section.

E x a m p l e 5.6. The classical relativistic particle is described by

$$L = -mR, \quad R = \sqrt{\dot{x}_0^2 - \sum_{i=x,y,z} \dot{x}_i^2}, \tag{5.37}$$

where a dot denotes a derivative with respect to the proper time. Because the rank of the Hessian is 3, we will treat the velocities $\dot{x}_i$ as regular variables and the velocity $\dot{x}_0$ as a degenerate variable. Then for the regular canonical momenta we have $p_i = \partial L \big/ \partial \dot{x}_i = m\dot{x}_i \big/ R$, which can be resolved with respect to the regular velocities,

$$\dot{x}_i = \dot{x}_0 \frac{p_i}{E}, \quad E = \sqrt{m^2 + \sum_{i=x,y,z} p_i^2}. \tag{5.38}$$

Using (5.4) and (5.12), we obtain

$$H_{phys} = 0, \quad B_{x_0} = \frac{\partial L}{\partial \dot{x}_0} = -m\frac{\dot{x}_0}{R} = -E. \tag{5.39}$$

The "physical sense" of $(-B_{x_0})$ is just the energy (5.38), while the "physical" Hamiltonian is zero. Equations of motion (5.14)–(5.15) are

$$\dot{x}_i = \dot{x}_0\frac{p_i}{E}, \quad \dot{p}_i = \frac{\partial B_{x_0}}{\partial x_i}\dot{x}_0 = 0,$$

which coincide with the Lagrange equations following from (5.37). Note that the velocity $\dot{x}_0$ is arbitrary here, and therefore we have one "gauge" degree of freedom.

6. Nonabelian Gauge Theory Interpretation

We observe that (5.16) can be written in a more compact form using the gauge theory notation. Let us introduce a "q^α-long derivative"

$$D_\alpha X = \frac{\partial X}{\partial q^\alpha} + \{B_\alpha, X\}_{phys}, \tag{6.1}$$

where $X = X\left(q^A, p_i\right)$ is a smooth scalar function on M_{phys}. We also notice that a multiplier in (5.16) to be called a "q^α-field strength" $F_{\alpha\beta} \equiv F_{\alpha\beta}\left(q^A, p_i\right)$ of the "q^α-gauge fields" B_α on M_{phys} defined by

$$F_{\alpha\beta} = \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{B_\alpha, B_\beta\}_{phys}. \tag{6.2}$$

Then the linear system of equations (5.16) for unresolved velocities can be written in a compact form

$$\sum_{\beta=r+1}^{n} F_{\alpha\beta}\frac{dq^\beta}{dt} = D_\alpha H_{phys}, \quad \alpha = r+1, \ldots, n. \tag{6.3}$$

The "q^α-field strength" $F_{\alpha\beta}$ is nonabelian due to the presence of the "physical" Poisson bracket in r.h.s. of (6.2). It is important to observe that in distinct of the ordinary Yang–Mills theory, the partial derivatives of B_α in (6.2) are defined in the q^α-subspace R^{n-r}, while the "noncompactivity" (the third term) is due to the Poisson bracket (5.17) in another symplectic subspace $Sp\left(r, r\right)$.

Note that the "q^α-long derivative" satisfies the Leibniz rule,

$$D_\alpha \{B_\beta, B_\gamma\}_{phys} = \{D_\alpha B_\beta, B_\gamma\}_{phys} + \{B_\beta, D_\alpha B_\gamma\}_{phys},$$

which is valid while acting on "q^α-gauge fields" B_α. The commutator of the "q^α-long derivatives" is now equal to the Poisson bracket with the "q^α-field strength"

$$\left(D_\alpha D_\beta - D_\beta D_\alpha\right) X = \left\{F_{\alpha\beta}, X\right\}_{phys}. \tag{6.4}$$

It follows from (6.4) that

$$D_\alpha D_\beta F_{\alpha\beta} = 0. \tag{6.5}$$

Let us introduce the B_α-transformation

$$\boldsymbol{\delta}_{B_\alpha} X = \left\{B_\alpha, X\right\}_{phys} \tag{6.6}$$

which satisfies

$$\left(\boldsymbol{\delta}_{B_\alpha}\boldsymbol{\delta}_{B_\beta} - \boldsymbol{\delta}_{B_\beta}\boldsymbol{\delta}_{B_\alpha}\right) B_\gamma = \boldsymbol{\delta}_{\left\{B_\alpha, B_\beta\right\}_{phys}} B_\gamma, \tag{6.7}$$

$$\boldsymbol{\delta}_{B_\alpha} F_{\beta\gamma}\left(q^A, p_i\right) = \left(D_\gamma D_\beta - D_\beta D_\gamma\right) B_\alpha, \tag{6.8}$$

$$\boldsymbol{\delta}_{B_\alpha}\left\{B_\beta, B_\gamma\right\}_{phys} = \left\{\boldsymbol{\delta}_{B_\alpha} B_\beta, B_\gamma\right\}_{phys} + \left\{B_\beta, \boldsymbol{\delta}_{B_\alpha} B_\gamma\right\}_{phys}. \tag{6.9}$$

This means that the "q^α-long derivative" D_α (6.1) is in fact a "q^α-covariant derivative" with respect to the B_α-transformation (6.6). Indeed, observe that D_α transforms as fields (6.6), which proves that it is really covariant (note the cyclic permutations in both sides)

$$\begin{aligned}&\boldsymbol{\delta}_{B_\alpha} D_\beta B_\gamma + \boldsymbol{\delta}_{B_\gamma} D_\alpha B_\beta + \boldsymbol{\delta}_{B_\beta} D_\gamma B_\alpha \\ &= \left\{B_\alpha, D_\beta B_\gamma\right\}_{phys} + \left\{B_\gamma, D_\alpha B_\beta\right\}_{phys} + \left\{B_\beta, D_\gamma B_\alpha\right\}_{phys}.\end{aligned} \tag{6.10}$$

The "q^α-Maxwell" equations of motion for the "q^α-field strength" are

$$D_\alpha F_{\alpha\beta} = J_\beta, \tag{6.11}$$

$$D_\alpha F_{\beta\gamma} + D_\gamma F_{\alpha\beta} + D_\beta F_{\gamma\alpha} = 0, \tag{6.12}$$

where $J_\alpha \equiv J_\alpha\left(q^A, p_i\right)$ is a "q^α-current" in M_{phys} which is a function of the initial Lagrangian (2.2) and its derivatives up to the third order. Due to (6.5), the "q^α-current" J_α is conserved

$$D_\alpha J_\alpha = 0. \tag{6.13}$$

Thus, a singular Lagrangian system leads effectively to a special kind of the nonabelian gauge theory in the direct product space $M_{phys} = R^{n-r} \times Sp\left(r, r\right)$. Here the "non-commutativity" (the third term in (6.2)) appears not due to a Lie algebra (as in the Yang-Mills theory), but "classically", due to the Poisson bracket in the symplectic subspace $Sp\left(r, r\right)$. The corresponding manifold can be interpreted locally as a special kind of the degenerate Poisson manifold (see, e.g., [9]).

The analogous Poisson type of "nonabelianity" (6.2) appears in the $N \to \infty$ limit of the Yang-Mills theory, and it is called the "Poisson gauge theory" [28]. In the $SU\left(\infty\right)$

Yang–Mills theory the group indices become the surface coordinates [26], which is connected with the Schild string [76]. The related algebra generalizations are called the continuum graded Lie algebras [63] (see also [45]). Here, because of the direct product structure of the space M_{phys}, the similar construction appears in (6.2) (in another initial context), while the "long derivative" (6.1), the "gauge transformations" (6.6) and the analog of the Maxwell equations (6.11)–(6.12) differ from the "Poisson gauge theory" [28].

7. Classification, Gauge Freedom and New Brackets

Next we can classify singular Lagrangian theories as follows:

1. *Gaugeless theory*. The rank of the skew-symmetric matrix $F_{\alpha\beta}$ is "full", i.e., the $\operatorname{rank} F_{\alpha\beta} = n - r$ is constant, and therefore the matrix $F_{\alpha\beta}$ is invertible, and all the (degenerate) velocities v^{α} can be found from the system of the linear equations (5.16) (and (6.3)) in a purely algebraic way.

2. *Gauge theory*. The skew-symmetric matrix $F_{\alpha\beta}$ is singular. If the $\operatorname{rank} F_{\alpha\beta} = r_F < n - r$, then a singular Lagrangian theory has $n - r - r_F$ gauge degrees of freedom. We can take them arbitrary, which corresponds to the presence of some symmetries in the theory. Note that the rank r_F is even due to the skew-symmetry of $F_{\alpha\beta}$.

In the first case (gaugeless theory) one can resolve (6.3) as follows:

$$v^{\beta} = \sum_{\alpha=r+1}^{n} \bar{F}^{\beta\alpha} D_{\alpha} H_{phys}, \tag{7.1}$$

where $\bar{F}^{\alpha\beta}$is the inverse matrix to $F_{\alpha\beta}$, i.e.,

$$\sum_{\beta=r+1}^{n} F_{\alpha\beta} \bar{F}^{\beta\gamma} = \sum_{\beta=r+1}^{n} \bar{F}^{\gamma\beta} F_{\beta\alpha} = \delta_{\alpha}^{\gamma}. \tag{7.2}$$

Substitute (7.1) in (5.14)–(5.15) to present the system of equations for a gaugeless degenerate Lagrangian theory in the Hamilton-like form

$$\frac{dq^i}{dt} = \left\{ q^i, H_{phys} \right\}_{nongauge}, \tag{7.3}$$

$$\frac{dp_i}{dt} = \left\{ p_i, H_{phys} \right\}_{nongauge}, \tag{7.4}$$

where the "nongauge" bracket is defined by

$$\{X, Y\}_{nongauge} = \{X, Y\}_{phys} - \sum_{\alpha=r+1}^{n} \sum_{\beta=r+1}^{n} D_{\alpha} X \cdot \bar{F}^{\alpha\beta} \cdot D_{\beta} Y. \tag{7.5}$$

Then the time evolution of any function of the dynamical variables $X = X\left(q^A, p_i\right)$ is also determined by the bracket (7.5) as follows:

$$\frac{dX}{dt} = \{X, H_{phys}\}_{nongauge}\,. \tag{7.6}$$

The meaning of the new nongauge bracket (7.5) (which appears naturally in the Clairaut-type formulation [25]) is the same as the meaning of the ordinary Poisson bracket in the unconstrained Hamiltonian dynamics: it governs the dynamics by the set of first-order differential equations in the Hamilton-like form (7.3)–(7.4) and is responsible for the time evolution of any dynamical variable (7.6). Also, the second term in the new bracket (7.5) has a complicated coordinate dependence and is analogous to that of the Dirac bracket [19]. In the extended phase space both brackets coincide (see Appendix B). On the other hand, the appearance of the second term in (7.5) can be treated as a deformation of the Poisson bracket, which can lead to another kind of the generalized symplectic geometry [39].

In the second case (gauge theory), with the singular matrix $F_{\alpha\beta}$ of rank r_F, we rearrange its rows and columns to obtain a nonsingular $r_F \times r_F$ submatrix in the left upper corner. Thus, the first r_F equations of the system of linear (under also rearranged v^β) Eqs. (6.3) are independent. Then we express the indices α and β as the pairs $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, where α_1 and β_1 denote the first r_F rows and columns, while α_2 and β_2 denote the rest of $n - r - r_F$ rows and columns. Correspondingly, we decompose the system (6.3),

$$\sum_{\beta_1=r+1}^{r+r_F} F_{\alpha_1\beta_1} v^{\beta_1} + \sum_{\beta_2=r+r_F+1}^{n} F_{\alpha_1\beta_2} v^{\beta_2} = D_{\alpha_1} H_{phys}, \tag{7.7}$$

$$\sum_{\beta_1=r+1}^{r+r_F} F_{\alpha_2\beta_1} v^{\beta_1} + \sum_{\beta_2=r+r_F+1}^{n} F_{\alpha_2\beta_2} v^{\beta_2} = D_{\alpha_2} H_{phys}. \tag{7.8}$$

The matrix $F_{\alpha_1\beta_1}$ being nonsingular by the construction, we can find the first r_F velocities

$$v^{\beta_1} = \sum_{\alpha_1=r+1}^{r+r_F} \bar{F}^{\beta_1\alpha_1} D_{\alpha_1} H_{phys} - \sum_{\alpha_1=r+1}^{r+r_F} \bar{F}^{\beta_1\alpha_1} F_{\alpha_1\beta_2} v^{\beta_2}, \tag{7.9}$$

where $\bar{F}^{\beta_1\alpha_1}$ is the inverse of the nonsingular $r_F \times r_F$ submatrix $F_{\alpha_1\beta_1}$ satisfying (7.2).

Then, since the $\operatorname{rank} F_{\alpha\beta} = r_F$, the last $n - r - r_F$ equations (7.8) are the linear combinations of the first r_F independent equations (7.7), which gives

$$F_{\alpha_2\beta_1} = \sum_{\alpha_1=r+1}^{r+r_F} \lambda_{\alpha_2}^{\alpha_1} F_{\alpha_1\beta_1}, \tag{7.10}$$

$$F_{\alpha_2\beta_2} = \sum_{\alpha_1=r+1}^{r+r_F} \lambda_{\alpha_2}^{\alpha_1} F_{\alpha_1\beta_2}, \tag{7.11}$$

$$D_{\alpha_2} H_{phys} = \sum_{\alpha_1=r+1}^{r+r_F} \lambda_{\alpha_2}^{\alpha_1} D_{\alpha_1} H_{phys}, \tag{7.12}$$

where $\lambda_{\alpha_2}^{\alpha_1} = \lambda_{\alpha_2}^{\alpha_1}\left(q^A, p_i\right)$ are some $r_F \times (n - r - r_F)$ smooth functions. Using relation (7.10) and invertibility of $F_{\alpha_1\beta_1}$, we eliminate the functions $\lambda_{\alpha_2}^{\alpha_1}$ by

$$\lambda_{\alpha_2}^{\alpha_1} = \sum_{\alpha_1=r+1}^{r+r_F} \sum_{\beta_1=r+1}^{r+r_F} F_{\alpha_2\beta_1} \bar{F}^{\beta_1\alpha_1}. \tag{7.13}$$

This indicates that the gauge theory is fully determined by the first r_F rows of the (rearranged) matrix $F_{\alpha\beta}$ and the first r_F (rearranged) derivatives $D_{\alpha_1} H_{phys}$ only.

Next, we can make the unresolved $n - r - r_F$ velocities vanish

$$v^{\beta_2} = 0 \tag{7.14}$$

by some "gauge fixing" condition. Then (7.9) becomes

$$v^{\beta_1} = \sum_{\alpha_1=r+1}^{r+r_F} \bar{F}^{\beta_1\alpha_1} D_{\alpha_1} H_{phys}. \tag{7.15}$$

By analogy with (7.3)–(7.4), in the gauge case we can also write the system of equations for a singular Lagrangian theory in the Hamilton-like form. Now we introduce another new (gauge) bracket

$$\{X, Y\}_{gauge} = \{X, Y\}_{phys} - \sum_{\alpha_1=r+1}^{r+r_F} \sum_{\beta_1=r+1}^{r+r_F} D_{\alpha_1} X \cdot \bar{F}^{\alpha_1\beta_1} \cdot D_{\beta_1} Y. \tag{7.16}$$

Then substituting (7.14)–(7.15) into (5.14)–(5.15) and using (7.16), we obtain

$$\frac{dq^i}{dt} = \left\{q^i, H_{phys}\right\}_{gauge}, \tag{7.17}$$

$$\frac{dp_i}{dt} = \left\{p_i, H_{phys}\right\}_{gauge}. \tag{7.18}$$

Thus the gauge bracket (7.16) governs the time evolution in the gauge case

$$\frac{dX}{dt} = \{X, H_{phys}\}_{gauge}. \tag{7.19}$$

Note that the brackets (7.5) and (7.16) are antisymmetric and satisfy the Jacobi identity. Therefore the standard quantization scheme is applicable here (see, e.g., [37]). The difference is that only the canonical (regular) dynamic variables $\left(q^i, p_i\right)$ can be quantized, while the degenerate coordinates can be treated as some continuous parameters.

It is worthwhile to consider the *limit case*, when $r_F = 0$, i.e.,

$$F_{\alpha\beta} = 0 \tag{7.20}$$

identically, which can mean that $B_\alpha = 0$, so the Lagrangian can be independent of the degenerate velocities v^α. It follows from (5.16) that

$$D_\alpha H_{phys} = \frac{\partial H_{phys}}{\partial q^\alpha} = 0, \tag{7.21}$$

which leads to *the "independence" statement*: the "physical" Hamiltonian H_{phys} does not depend on the degenerate coordinates q^α iff the Lagrangian does not depend on the velocities v^α. In the limit case, both brackets (7.5) and (7.16) coincide with the Poisson bracket in the reduced "physical" phase space $\{\ ,\ \}_{nongauge,gauge} = \{\ ,\ \}_{phys}$.

E x a m p l e 7.7. (Christ–Lee model [14]). The Lagrangian of $SU(2)$ Yang–Mills theory in $0+1$ dimensions in our notation is

$$L(x_i, y_\alpha, v_i) = \frac{1}{2}\sum_{i=1,2,3}\left(v_i - \sum_{j,\alpha=1,2,3}\varepsilon_{ij\alpha}x_j y_\alpha\right)^2 - U\left(x^2\right), \tag{7.22}$$

where $i, \alpha = 1,2,3$, $x^2 = \sum_i x_i^2$, $v_i = \dot{x}_i$ and ε_{ijk} is the Levi–Civita symbol. Because (7.22) is independent of the degenerate velocities $\dot{y}_\alpha$, all $B_\alpha \overset{(5.4)}{=} 0$, and therefore $F_{\alpha\beta} \overset{(6.2)}{=} 0$, we have the limit gauge case of the above classification. The corresponding Clairaut Eq. (3.3) for $H = H^{\mathrm{Cl}}(x_i, y_\alpha, \lambda_i, \lambda_\alpha)$ has the form

$$H = \sum_{i=1,2,3}\lambda_i H'_{\lambda_i} + \sum_{\alpha=1,2,3}\lambda_\alpha H'_{\lambda_\alpha} - \frac{1}{2}\sum_{i=1,2,3}\left(H'_{\lambda_i} - \sum_{j,\alpha=1,2,3}\varepsilon_{ij\alpha}x_j y_\alpha\right)^2 + U\left(x^2\right). \tag{7.23}$$

We show manifestly how to obtain the envelope solution for regular variables and the general solution for degenerate variables. The general solution is

$$H_{gen} = \sum_{i=1,2,3}\lambda_i c_i + \sum_{\alpha=1,2,3}\lambda_\alpha c_\alpha - \frac{1}{2}\sum_{i=1,2,3}\left(c_i - \sum_{j,\alpha=1,2,3}\varepsilon_{ij\alpha}x_j y_\alpha\right)^2 + U\left(x^2\right), \tag{7.24}$$

where c_i, c_α are arbitrary functions of coordinates. Recall that q^A are passive variables under the Legendre transform. We differentiate (7.24) by c_i, c_α,

$$\frac{\partial H_{gen}}{\partial c_i} = \lambda_i - \left(c_i - \sum_{j,\alpha=1,2,3} \varepsilon_{ij\alpha} x_j y_\alpha \right), \tag{7.25}$$

$$\frac{\partial H_{gen}}{\partial c_\alpha} = \lambda_\alpha, \tag{7.26}$$

and observe that only (7.25) can be resolved with respect to c_i, and therefore can lead to the envelope solution, while other c_α cannot be resolved, and therefore we consider only the general solution of the Clairaut equation. So we can exclude half of the constants using (7.25) (with the substitution $\lambda_i \overset{(4.2)}{\to} p_i$) and get the mixed solution (4.5) to the Clairaut Eq. (7.23),

$$H_{mix}^{\mathrm{Cl}} \left(x_i, y_\alpha, p_i, \lambda_\alpha, c_\alpha \right) = \frac{1}{2} \sum_{i=1,2,3} p_i^2 + \sum_{i,j,\alpha=1,2,3} \varepsilon_{ij\alpha} p_i x_j y_\alpha + \sum_{\alpha=1,2,3} \lambda_\alpha c_\alpha + U \left(x^2 \right). \tag{7.27}$$

Using (5.10), we obtain the "physical" Hamiltonian

$$H_{phys} \left(x_i, y_\alpha, p_i \right) = \frac{1}{2} \sum_{i=1,2,3} p_i^2 + \sum_{i,j,\alpha=1,2,3} \varepsilon_{ij\alpha} p_i x_j y_\alpha + U \left(x^2 \right). \tag{7.28}$$

On the other hand, the Hessian of (7.22) has the rank 3, and we choose x_i, v_i and y_α to be regular and degenerate variables, respectively. The degenerate velocities $v_\alpha = \dot{y}_\alpha$ cannot be defined from (5.16) at all, they are arbitrary, and the first integrals (5.16), (7.21) of the system (5.14)–(5.15) become (also in accordance to the independence statement)

$$\frac{\partial H_{phys} \left(x_i, y_\alpha, p_i \right)}{\partial y_\alpha} = \sum_{i,j=1,2,3} \varepsilon_{ij\alpha} p_i x_j = 0. \tag{7.29}$$

The preservation in time (7.19) of (7.29) is fulfilled identically due to the antisymmetry properties of the Levi–Civita symbols. It is clear that only 2 equations from 3 of (7.29) are independent, so we choose $p_1 x_2 = p_2 x_1$, $p_1 x_3 = p_3 x_1$ and insert them into (7.28) to get

$$\tilde{H}_{phys} = \frac{1}{2} p_1^2 \frac{x^2}{x_1^2} + U \left(x^2 \right). \tag{7.30}$$

The transformation $\tilde{p} = p_1 \sqrt{x^2} \big/ x_1$, $\tilde{x} = \sqrt{x^2}$ gives the well-known result [14, 34],

$$\tilde{H}_{phys} = \frac{1}{2} \tilde{p}^2 + U \left(\tilde{x} \right). \tag{7.31}$$

8. Singular Lagrangian Systems and Many-time Dynamics

The many-time classical dynamics and its connection with constrained systems were studied in [48, 44] as a generalization of some relativistic two-particle models [22]. We consider this connection from a different viewpoint, that is, in the Clairaut-type approach [25]. Recall that the Hamiltonian–Clairaut dynamics (5.14)–(5.16) of a Lagrangian singular system (5.1) is governed by the "physical" Hamiltonian function H_{phys} and $(n-r)$ "q^{α}-gauge fields" B_{α} defined on the direct product space $R^{n-r} \times Sp\,(r,r)$. Let us treat the degenerate coordinates $q^{\alpha} \in R^{n-r}$ as $(n-r)$ additional "time" variables together with $(n-r)$ corresponding "Hamiltonians" $-B_{\alpha}\left(q^{\alpha}|q^{i},p_{i}\right)$, $\alpha = r+1,\ldots,n$ (see (6.1)). Indeed, let us introduce $(n-r+1)$ generalized "times" t^{μ} and the corresponding "many-time Hamiltonians" $\mathsf{H}_{\mu}\left(\mathsf{t}^{\mu}|q^{i},p_{i}\right)$, $\mu = 0,\ldots n-r$ defined by

$$\mathsf{t}^{0} = t, \quad \mathsf{H}_{0}\left(\mathsf{t}^{\alpha}|q^{i},p_{i}\right) = H_{phys}\left(q^{\alpha}|q^{i},p_{i}\right), \quad \mu = 0, \tag{8.1}$$

$$\mathsf{t}^{\mu} = q^{\mu}, \quad \mathsf{H}_{\mu}\left(\mathsf{t}^{\mu}|q^{i},p_{i}\right) = -B_{r+\mu}\left(q^{r+\mu}|q^{i},p_{i}\right), \quad \mu = 1,\ldots,n-r. \tag{8.2}$$

Then Eqs. (5.14)–(5.15) can be presented in the differential form

$$dq^{i} = \sum_{\mu=0}^{n-r}\left\{q^{i},\mathsf{H}_{\mu}\right\}_{phys} d\mathsf{t}^{\mu}, \tag{8.3}$$

$$dp_{i} = \sum_{\mu=0}^{n-r}\left\{p_{i},\mathsf{H}_{\mu}\right\}_{phys} d\mathsf{t}^{\mu}, \tag{8.4}$$

where $\{\ ,\ \}_{phys}$ is defined in (5.17). The linear algebraic system of Eqs. (5.16) for the degenerate velocities then becomes

$$\sum_{\mu=0}^{n-r} \mathsf{G}_{\mu\nu} d\mathsf{t}^{\mu} = 0, \tag{8.5}$$

where

$$\mathsf{G}_{\mu\nu} = \frac{\partial \mathsf{H}_{\mu}}{\partial \mathsf{t}^{\nu}} - \frac{\partial \mathsf{H}_{\nu}}{\partial \mathsf{t}^{\mu}} + \left\{\mathsf{H}_{\mu},\mathsf{H}_{\nu}\right\}_{phys}. \tag{8.6}$$

It follows from (8.5) that the one-form $\omega = p_{i}dq^{i} - \mathsf{H}_{\mu}d\mathsf{t}^{\mu}$ is closed,

$$d\omega = \frac{1}{2}\sum_{\mu=0}^{n-r}\sum_{\nu=0}^{n-r} \mathsf{G}_{\mu\nu} d\mathsf{t}^{\mu} \wedge d\mathsf{t}^{\nu} = 0, \tag{8.7}$$

which agrees with the action principle for multi-time classical dynamics [21]. The corresponding set of the Hamilton–Jacobi equations for action $S\left(q^{\alpha}|q^{i},p_{i}\right) \longmapsto \mathsf{S}\left(\mathsf{t}^{\mu}|q^{i},p_{i}\right)$ is

$$\frac{\partial \mathsf{S}}{\partial \mathsf{t}^{\mu}} + \mathsf{H}_{\mu}\left(\mathsf{t}^{\mu}|q^{i},\frac{\partial \mathsf{S}}{\partial q^{i}}\right) = 0. \tag{8.8}$$

Therefore, we have come to the conclusion that any singular Lagrangian theory (in the Clairaut-type formulation [24, 25]) is equivalent to the many-time classical dynamics [21, 48]: the equations of motion are (8.3)–(8.4), which coincide with (5.14)–(5.15), and the integrability condition is (8.5), which coincides with the system of the linear algebraic equations for unresolved velocities (5.16) by the construction.

9. Conclusions

We have described the Hamilton-like evolution of singular Lagrangian systems by using $n-r+1$ functions $H_{phys}\left(q^{\alpha}|q^{i},p_{i}\right)$ and $B_{\alpha}\left(q^{\alpha}|q^{i},p_{i}\right)$ on the direct product space $R^{n-r}\times \mathrm{Sp}\left(r,r\right)$. To do this, we used the generalized Legendre–Clairaut transform, that is, we solved the corresponding multidimensional Clairaut equation without introducing the Lagrange multipliers. All variables are set as regular or degenerate according to the rank of the Hessian matrix of Lagrangian. We consider the reduced "physical" phase space formed by the regular coordinates q^{i} and momenta p_{i} only, while the degenerate coordinates q^{α} play a role of parameters. There are two reasons why the degenerate momenta λ_{α} corresponding to q^{α} need not be considered in the Clairaut-type formulation:

1) *the mathematical reason*: there is no possibility to find the degenerate velocities v^{α}, as can be done for the regular velocities v^{i} in (4.2), and "pre-Hamiltonian" (2.2) has no extremum in degenerate directions;

2) *the physical reason*: momentum is a measure of motion; however, there is *no dynamics* in "degenerate" directions and hence no reason to introduce the corresponding "physical" momenta at all.

Note that some possibilities to avoid constraints were considered in a different context in [18, 58] and for special forms of the Lagrangian in [32].

The Hamilton-like form of the equations of motion (7.3)–(7.4) is achieved by introducing new brackets (7.5) and (7.16) which are responsible for time evolution. They are antisymmetric and satisfy the Jacobi identity. Therefore we can quantize the brackets using the standard methods [37], but only for the regular variables, while the degenerate variables can be considered as some continuous parameters.

In the "nonphysical" coordinate subspace, we formulate some kind of nonabelian gauge theory such that "nonabelianity" appears due to the Poisson bracket in the physical phase space (6.2). This makes it similar to the Poisson gauge theory [28], but do not coincide with the latter.

Finally, we show that, in general, a singular Lagrangian system in the Clairaut-type formulation [24, 25] is equivalent to the many-time classical dynamics.

A. Multidimensional Clairaut Equation

The multidimensional Clairaut equation for a function $y = y(x_i)$ of n variables x_i is [41, 3]

$$y = \sum_{j=1}^{n} x_j y'_{x_j} - f(y'_{x_i}), \tag{A.1}$$

where prime denotes a partial differentiation by subscript and f is a smooth function of n arguments. To find and classify the solutions of (A.1), we have to find first derivatives y'_{x_i} in some way and then substitute them back into (A.1). We differentiate (A.1) by x_j and obtain n equations

$$\sum_{i=1}^{n} y''_{x_i x_j}(x_i - f'_{y'_{x_i}}) = 0. \tag{A.2}$$

The classification follows from the ways the factors in (A.2) can be set to zero. Here, for our physical applications, it is sufficient to suppose that the ranks of Hessians of y and f are

$$\operatorname{rank} y''_{x_i x_j} = \operatorname{rank} f''_{y'_{x_i} y'_{x_j}} = r. \tag{A.3}$$

This means that in each equation from (A.2) either the first or the second multiplier is zero, but it is not necessary to vanish both of them. The first multiplier can be set to zero without any additional assumptions. So we have

1) *The general solution.* It is defined by the condition

$$y''_{x_i x_j} = 0. \tag{A.4}$$

After one integration we can find $y'_{x_i} = c_i$ and substitute them into (A.1) to obtain

$$y_{gen} = \sum_{j=1}^{n} x_j c_j - f(c_i), \tag{A.5}$$

where c_i are n constants.

All second multipliers in (A.2) can be zero for $i = 1, \dots, n$, but this will give a solution if they can be resolved for y'_{x_i}. It is possible if the rank of Hessians f is full, i.e., $r = n$. In this case we obtain

2) *The envelope solution.* It is defined by

$$x_i = f'_{y'_{x_i}}. \tag{A.6}$$

We resolve (A.6) for the derivatives as $y'_{x_i} = C_i(x_j)$ and get

$$y_{env} = \sum_{i=1}^{n} x_i C_i(x_j) - f(C_i(x_j)), \tag{A.7}$$

where $C_i(x_j)$ are n smooth functions of n arguments.

In the intermediate case, we can use the envelope solution (A.7) for the initial s variables and the general solution (A.5) for the rest of $n-s$ variables to obtain

3) *The s-mixed solution*

$$y_{mix}^{(s)}=\sum_{j=1}^{s}x_jC_j(x_j)+\sum_{j=s+1}^{n}x_jc_j-f(C_1(x_j),\ldots,C_s(x_j),c_{s+1},\ldots,c_n). \quad \text{(A.8)}$$

If the rank r of the Hessians f is not full and a nonsingular minor of the rank r is in the upper left corner, then we can resolve the first r relations (A.6) only, and thus $s\leq r$. In our physical applications we use the limited case $s=r$.

E x a m p l e A.1. Let $f(z_i)=z_1^2+z_2^2+z_3$. Then the Clairaut equation for $y=y(x_1,x_{2,}x_3)$ has the form

$$y=x_1y'_{x_1}+x_2y'_{x_2}+x_3y'_{x_3}-\left(y'_{x_1}\right)^2-\left(y'_{x_2}\right)^2-y'_{x_3}, \quad \text{(A.9)}$$

and we have $n=3$ and $r=2$. The general solution can be found from (A.4) by integrating one time and using (A.5),

$$y_{gen}=c_1(x_1-c_1)+c_2(x_2-c_2)+c_3(x_3-1), \quad \text{(A.10)}$$

where c_i are constants.

Since $r=2$, we can resolve only 2 relations from (A.6) by $y'_{x_1}=\frac{x_1}{2}$, $y'_{x_2}=\frac{x_2}{2}$. So there is no envelope solution (for all variables), but we have several mixed solutions corresponding to $s=1,2$:

$$y_{mix}^{(1)}=\begin{cases}\dfrac{x_1^2}{4}+c_2(x_2-c_2)+c_3(x_3-1), \\ c_1(x_1-c_1)+\dfrac{x_2^2}{4}+c_3(x_3-1),\end{cases} \quad \text{(A.11)}$$

$$y_{mix}^{(2)}=\frac{x_1^2}{4}+\frac{x_2^2}{4}+c_3(x_3-1). \quad \text{(A.12)}$$

The case $f(z_i)=z_1^2+z_2^2$ can be obtained from the above formulas by putting $x_3=c_3=0$ and $y_{mix}^{(2)}$ becomes the envelope solution $y_{env}=\frac{x_1^2}{4}+\frac{x_2^2}{4}$.

B. Correspondence with the Dirac Approach

The constraints appear due to additional variables introduced into the theory of *additional* dynamical variables (because the Hamilton-like form of the equations of motion can be achieved *without* them in the presented approach), that is, momenta which correspond to the "degenerate" velocities. The relationship between the Clairaut-type formulation and the Dirac approach can be clarified by treating the variables λ_α in the general solution of the Clairaut equation as the "physical" degenerate momenta p_α using for them the same expression in terms of the Lagrangian as in (4.2),

$$\lambda_\alpha = p_\alpha = \frac{\partial L\left(q^A, v^A\right)}{\partial v^\alpha}. \tag{B.1}$$

Then we obtain the primary Dirac constraints (in the resolved form and our notation (5.4))

$$\mathbf{\Phi}_\alpha\left(q^A, p_A\right) = p_\alpha - B_\alpha = 0, \quad \alpha = r+1, \dots n, \tag{B.2}$$

which are defined now on the full phase space $\mathsf{T}^* M$. Using (5.10) and (5.12), we can arrive at the complete Hamiltonian of the first-order formulation [31] (corresponding to the total Dirac Hamiltonian [19]),

$$\begin{aligned} H_T\left(q^A, p_A, v^\alpha\right) &= H_{mix}^{\mathrm{Cl}}\left(q^A, p_i, \lambda_\alpha, v^\alpha\right)\Big|_{\lambda_\alpha = p_\alpha} \\ &= H_{phys}\left(q^A, p_i\right) + \sum_{\alpha=r+1}^{n} v^\alpha \mathbf{\Phi}_\alpha\left(q^A, p_A\right), \end{aligned} \tag{B.3}$$

which is equal to the mixed Hamilton–Clairaut function (4.5) with the substitution (B.1) and (B.2) being used. Then the Hamilton–Clairaut system of Eqs. (5.14)–(5.15) coincides with the Hamilton system in the first-order formulation [31],

$$\dot{q}^A = \left\{q^A, H_T\right\}_{full}, \quad \dot{p}_A = \left\{p_A, H_T\right\}_{full}, \quad \mathbf{\Phi}_\alpha = 0, \tag{B.4}$$

and (5.16) gives the second stage equations of the Dirac approach

$$\left\{\mathbf{\Phi}_\alpha, H_T\right\}_{full} = \left\{\mathbf{\Phi}_\alpha, H_{phys}\right\}_{full} + \sum_{\beta=r+1}^{n} \left\{\mathbf{\Phi}_\alpha, \mathbf{\Phi}_\beta\right\}_{full} v^\beta = 0, \tag{B.5}$$

where

$$\left\{X, Y\right\}_{full} = \sum_{A=1}^{n} \left(\frac{\partial X}{\partial q^A}\frac{\partial Y}{\partial p_A} - \frac{\partial Y}{\partial q^A}\frac{\partial X}{\partial p_A}\right) \tag{B.6}$$

is the (*full*) Poisson bracket on the whole phase space $\mathsf{T}^* M$. Notice that

$$F_{\alpha\beta} = \left\{\mathbf{\Phi}_\alpha, \mathbf{\Phi}_\beta\right\}_{full}, \tag{B.7}$$

$$D_\alpha H_{phys} = \left\{\mathbf{\Phi}_\alpha, H_{phys}\right\}_{full}. \tag{B.8}$$

It is important that the introduced new brackets (7.5) and (7.16) become the Dirac bracket [19]. Moreover, our cases 2) and 1) of Section 7 work as counterparts of the first and the second class constraints in the Dirac classification [19], respectively. The limit case with zero "q^α-field strength" $F_{\alpha\beta} = 0$ (7.20) (see (B.7)) corresponds to the Abelian constraints [35, 49].

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