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On Non-Gaussian Limiting Laws for Certain Statistics of Wigner Matrices

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This paper is a continuation of our papers [12–14] in which the limiting laws of fluctuations were found for the linear eigenvalue statistics $\mathrm{Tr}\,\varphi(M^{(n)})$ and for the normalized matrix elements $\sqrt{n}\varphi_{jj}(M^{(n)})$ of differentiable functions of real symmetric Wigner matrices $M^{(n)}$ as $n \to \infty$. Here we consider another spectral characteristic of Wigner matrices, $\xi_n^A[\varphi] = \mathrm{Tr}\,\varphi(M^{(n)})A^{(n)}$, where $\{A^{(n)}\}_{n=1}^{\infty}$ is a certain sequence of non-random matrices. We show first that if $M^{(n)}$ belongs to the Gaussian Orthogonal Ensemble, then $\xi_n^A[\varphi]$ satisfies the Central Limit Theorem. Then we consider Wigner matrices with i.i.d. entries possessing the entire characteristic function and find the limiting probability law for $\xi_n^A[\varphi]$, which in general is not Gaussian.

Key words: Wigner matrices, spectral characteristics, central limit theorem.

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1. Introduction

The asymptotic behavior of spectral characteristics of $n \times n$ random matrices $M^{(n)}$, as their size n tends to infinity, is of great interest in random matrix theory and its applications. One of the main questions under study is the validity of the Central Limit Theorem (CLT) for various spectral characteristics. In the last two decades there was obtained a number of results on the CLT for linear eigenvalue statistics $\mathrm{Tr}\,\varphi(M^{(n)})$ and other spectral characteristics (see [1, 3, 5, 7–9, 16, 17, 19–22] and references therein). It was found that in many cases the fluctuations of various spectral characteristics of eigenvalues of random matrix ensembles are asymptotically Gaussian (see [1, 3, 8, 9, 17, 19, 21, 22]). But the CLT is not always the case. For instance, it was shown in [16] that the CLT for linear eigenvalue statistics is not necessarily valid for so-called hermitian matrix models, for which in certain cases there appear non-Gaussian limiting laws.

Another example of the non-Gaussian limiting behavior is presented in works [14, 18, 19] dealing with the normalized individual matrix elements $\sqrt{n}\varphi_{jj}(M^{(n)})$ of functions of real symmetric Wigner random matrix. The particular case of the matrix elements $\sqrt{n}\varphi_{jj}(\widehat{M}^{(n)})$ with $\widehat{M}^{(n)}$ belonging to the Gaussian Orthogonal Ensemble (GOE) was considered earlier in [13], where it was proved that the centered $\sqrt{n}(\varphi_{jj}(\widehat{M}^{(n)}))$ satisfies the CLT. But in [14, 18, 19] it was shown that in general case of Wigner matrices the limiting law of fluctuations for $\sqrt{n}(\varphi_{jj}(M^{(n)}))$ is not Gaussian but the sum of the Gaussian law and probability law of the entries of $\sqrt{n}M^{(n)}$ modulo a certain rescaling, and to obtain the CLT, one has to impose a certain condition on the test function.

In particular, the fact that, in contrast to the linear statistics of eigenvalues, individual matrix elements in general do not satisfy the CLT shows the influence of eigenvectors and gives some information about asymptotic properties of eigenvectors. Indeed, in the case of the Gaussian random matrices (GOE, null Wishart) the eigenvectors are rotationally invariant and according to recent works [2, 6, 10] the eigenvectors of the non-Gaussian random matrices (Wigner, sample covariance) are similar in several aspects to the eigenvectors of the Gaussian random matrices. On the other hand, the results of [13] and [14, 18, 19] imply that there are asymptotic properties of eigenvectors of the non-Gaussian random matrices which are different from those for the Gaussian random matrices.

This paper continues the investigations of [12–14]. Here we consider the random variable

$$\xi_n^A[\varphi] = \operatorname{Tr}\varphi(M^{(n)})A^{(n)}, \tag{1.1}$$

where φ is a smooth enough test function and $\{A^{(n)}\}_{n=1}^{\infty}$ is a sequence of $n\times n$ non-random matrix satisfying

$$\text{(i)}\ \lim_{n\to\infty} n^{-1}\operatorname{Tr} A^{(n)T}A^{(n)} = 1, \tag{1.2}$$

$$\text{(ii)}\ \exists \lim_{n\to\infty} n^{-1}\operatorname{Tr} A^{(n)} = T_A. \tag{1.3}$$

Here are some examples of choices of $A^{(n)}$:

1. *Linear eigenvalue statistics.* If $A^{(n)} = I^{(n)}$, then $T_A = 1$ and

$$\xi_n^A[\varphi] = \operatorname{Tr}\varphi(M^{(n)}). \tag{1.4}$$

2. *Matrix elements.* If $A^{(n)}_{lm} = \sqrt{n}\delta_{jl}\delta_{jm}$, then $T_A = 0$ and

$$\xi_n^A[\varphi] = \sqrt{n}\varphi_{jj}(M^{(n)}). \tag{1.5}$$

3. *Bilinear forms.* If $A_{lm}^{(n)} = \sqrt{n}\eta_l\eta_m$, where

$$\eta^{(n)} = (\eta_1^{(n)}, \dots, \eta_n^{(n)})^T, \quad \lim_{n\to\infty}\sum_{l=1}^{n}(\eta_l^{(n)})^2 = 1, \tag{1.6}$$

then $T_A = 0$ and

$$\xi_n^A[\varphi] = \sqrt{n}(\varphi(M^{(n)})\eta^{(n)}, \eta^{(n)}). \tag{1.7}$$

Denote

$$\xi_n^{A\circ}[\varphi] = \xi_n^A[\varphi] - \mathbf{E}\{\xi_n^A[\varphi]\}. \tag{1.8}$$

Our main result is Theorem 5.1 below, where the limiting expression for the characteristic function of $\xi_n^{A\circ}[\varphi]$ is given and written via the cumulants of matrix entries and quantities depending on a sequence $\{A^{(n)}\}_{n=1}^{\infty}$. Let us note that the corresponding theorems for linear eigenvalue statistics (1.4) and matrix elements (1.5) of [12–14, 18] can be obtained from Theorem 5.1 as particular cases (however, under much stronger conditions).

The paper is organized as follows. Section 2 contains definitions, some known facts and technical means used throughout the paper. In Section 3, we consider the case of the GOE and prove the CLT for $\xi_n^A[\varphi]$ (see [13] for the analogous statements for matrix elements). Then we find the limiting variance (Section 4) and the limiting probability law (Section 5) for $\xi_n^{A\circ}[\varphi]$ for Wigner matrices. Section 6 contains auxiliary results. We confine ourselves to real symmetric matrices, although our results as well as the main ingredients of proofs remain valid in the hermitian case with natural modifications.

2. Definitions and Technical Means

To make the paper self-consistent, we present several definitions and technical facts that will be often used below. We start with the definition of the Wigner real symmetric matrix $M^{(n)}$ and put

$$M^{(n)} = n^{-1/2}W^{(n)}, \quad W^{(n)} = \{W_{jk}^{(n)} \in \mathbb{R},\ W_{jk}^{(n)} = W_{kj}^{(n)}\}_{j,k=1}^{n}, \tag{2.1}$$

where $\{W_{jk}^{(n)}\}_{1\le j\le k\le n}$ are independent random variables satisfying

$$\mathbf{E}\{W_{jk}^{(n)}\} = 0, \quad \mathbf{E}\{(W_{jk}^{(n)})^2\} = w^2(1+\delta_{jk}). \tag{2.2}$$

The case of the Gaussian random variables obeying (2.2) corresponds to the GOE:

$$\widehat{M}^{(n)} = n^{-1/2}\widehat{W}^{(n)}, \quad \widehat{W}^{(n)} = \{\ \widehat{W}_{jk} = \widehat{W}_{kj} \in \mathbb{R},\ \widehat{W}_{jk} \in \mathcal{N}(0, w^2(1+\delta_{jk}))\}_{j,k=1}^{n}. \tag{2.3}$$

Here for simplicity's sake we define the Wigner matrix so that the first two moments of its entries match those of the GOE. It can be shown that if $\mathbf{E}\{(W_{jj}^{(n)})^2\} = w^2 w_2$, then the corresponding expressions for the limiting variance and characteristic function have additional terms proportional to $(w_2 - 2)$ (see Remarks 4.5 and 5.2). In what follows, we will assume additional conditions on the distributions of $W_{jk}^{(n)}$, mostly in the form of the existence of certain moments of $W_{jk}^{(n)}$ whose order will depend on the problem under study.

The next proposition presents certain facts on Gaussian random variables.

Proposition 2.1. *Let $\zeta = \{\zeta_l\}_{l=1}^p$ be the independent Gaussian random variables of zero mean, and $\Phi : \mathbb{R}^p \to \mathbb{C}$ be a differentiable function with polynomially bounded partial derivatives Φ_l', $l = 1, \dots, p$. Then we have*

$$\mathbf{E}\{\zeta_l \Phi(\zeta)\} = \mathbf{E}\{\zeta_l^2\}\mathbf{E}\{\Phi_l'(\zeta)\},\ l = 1, \dots, p, \tag{2.4}$$

$$\mathbf{Var}\{\Phi(\zeta)\} \le \sum_{l=1}^p \mathbf{E}\{\zeta_l^2\}\mathbf{E}\left\{|\Phi_l'(\zeta)|^2\right\}. \tag{2.5}$$

The first formula is a version of the integration by parts. The second one is a version of the Poincaré inequality (see, e.g., [4]). Formula (2.4) is a particular case of a more general formula. To write it, we recall some definitions. If a random variable ζ has a finite pth absolute moment, $p \ge 1$, then we have the expansions as $t \to 0$:

$$\mathbf{E}\{e^{it\zeta}\} = \sum_{j=0}^p \frac{\mu_j}{j!}(it)^j + o(t^p), \quad \log \mathbf{E}\{e^{it\zeta}\} = \sum_{j=0}^p \frac{\kappa_j}{j!}(it)^j + o(t^p), \tag{2.6}$$

where " log " denotes the principal branch of logarithm. The coefficients in the expansion of $\mathbf{E}\{e^{it\zeta}\}$ are the moments $\{\mu_j\}$ of ζ, and the coefficients in the expansion of $\log \mathbf{E}\{e^{it\zeta}\}$ are the cumulants $\{\kappa_j\}$ of ζ. In particular, if $\mu_1 = 0$, then

$$\kappa_1 = 0, \quad \kappa_2 = \mu_2 = \mathbf{Var}\{\zeta\}, \quad \kappa_3 = \mu_3, \quad \kappa_4 = \mu_4 - 3\mu_2^2, \ \dots \tag{2.7}$$

We have [9, 12, 17]:

Proposition 2.2. (i) *Let ζ be a random variable such that $\mathbf{E}\{|\zeta|^{p+2}\} < \infty$ for a certain non-negative integer p. Then for any function $\Phi : \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded partial derivatives $\Phi^{(l)}$, $l = 1, \dots, p+1$, we have*

$$\mathbf{E}\{\zeta\Phi(\zeta)\} = \sum_{l=0}^p \frac{\kappa_{l+1}}{l!}\mathbf{E}\{\Phi^{(l)}(\zeta)\} + \varepsilon_p, \tag{2.8}$$

where

$$|\varepsilon_p| \le C_p \mathbf{E}\{|\zeta|^{p+2}\} \sup_{t\in\mathbb{R}} |\Phi^{(p+1)}(t)|, \;\; C_p \le \frac{1+(3+2p)^{p+2}}{(p+1)!}. \tag{2.9}$$

(ii) If the characteristic function $\mathbf{E}\{e^{it|\zeta|}\}$ *is entire and* $\Phi \in C^\infty$, *then*

$$\mathbf{E}\{\zeta\Phi(\zeta)\} = \sum_{l=0}^{\infty} \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\zeta)\} \tag{2.10}$$

provided that for some $a > 0$

$$|\mathbf{E}\{\Phi^{(l)}(\zeta)\}| \le a^l, \tag{2.11}$$

and for some $R = ca$, $c > 1$,

$$\sum_{l=0}^{\infty} \frac{|\kappa_{l+1}| R^l}{l!} < \infty. \tag{2.12}$$

The next proposition presents simple facts of linear algebra.

Proposition 2.3. *Let* M *and* M' *be* $n \times n$ *matrices, and* $t \in \mathbb{R}$. *Then we have the following:*

(*i*) *the Duhamel formula*

$$e^{(M+M')t} = e^{Mt} + \int_0^t e^{M(t-s)} M' e^{(M+M')s} ds, \tag{2.13}$$

(*ii*) *if for a real symmetric* $n \times n$ *matrix* $M^{(n)}$ *we put*

$$U(t) = U^{(n)}(t) := e^{itM^{(n)}}, \; t \in \mathbb{R}, \tag{2.14}$$

then $U(t)$ *is a symmetric unitary matrix satisfying*

$$U(t_1)U(t_2) = U(t_1+t_2), \quad ||U(t)|| = 1, \quad \sum_{j=1}^{n} |U_{jk}(t)|^2 = 1, \tag{2.15}$$

(*iii*) *if* $D_{lm} = \partial/\partial M_{lm}$, *then*

$$D_{lm} U_{ab}(t) = i\beta_{lm} \left(U_{al} * U_{bm} + U_{bl} * U_{am}\right)(t), \tag{2.16}$$

where symbol "$*$" is a convolution sign, and

$$\beta_{lm} = (1+\delta_{lm})^{-1} = 1 - \delta_{lm}/2, \tag{2.17}$$

(iv) if $A^{(n)}$ is an $n \times n$ matrix, $C^{(n)} = A^{(n)} + A^{(n)T}$, and

$$\xi_n^A(t) = \mathrm{Tr} A^{(n)} U(t) \tag{2.18}$$

is a particular case of $\xi_n^A[\varphi]$ corresponding to $\varphi(\lambda) = e^{it\lambda}$, then

$$D_{lm}(A^{(n)}U)_{ab}(t) = i\beta_{lm}\left((A^{(n)}U)_{al} * U_{bm} + U_{bl} * (A^{(n)}U)_{am}\right)(t), \tag{2.19}$$

$$D_{lm}\xi_n^A(t) = i\beta_{lm}(U * C^{(n)}U)_{lm}(t), \tag{2.20}$$

$$D_{lm}^2\xi_n^A(t) = -\beta_{lm}^2\big(U_{ll} * (U * C^{(n)}U)_{mm} + U_{mm} * (U * C^{(n)}U)_{ll} + 2U_{lm} * (U * C^{(n)}U)_{lm}\big)(t), \tag{2.21}$$

$$D_{lm}(U * A^{(n)}U)_{jk}(t) = i\beta_{lm}\big(U_{jl} * (U * A^{(n)}U)_{mk} + U_{jm} * (U * A^{(n)}U)_{lk} + U_{lk} * (U * A^{(n)}U)_{jk} + U_{mk} * (U * A^{(n)}U)_{jl}\big)(t), \tag{2.22}$$

$$D_{lm}(U * A^{(n)}U)_{lm}(t) = i\beta_{lm}\big(U_{ll} * (U * A^{(n)}U)_{mm} + U_{mm} * (U * A^{(n)}U)_{ll} + 2U_{lm} * (U * A^{(n)}U)_{lm}\big)(t). \tag{2.23}$$

It follows from the above that if $A^{(n)}$ satisfies (1.2)–(1.3) and

$$C_A: \ \mathrm{Tr}\, A^{(n)T}A^{(n)} \le C_A n, \ \forall n \in \mathbb{N}, \tag{2.24}$$

then

$$|(A^{(n)}U^{(n)})_{lm}| \le (A^{(n)T}A^{(n)})_{ll}^{1/2} \le O(n^{1/2}), \tag{2.25}$$

$$|(U^{(n)}A^{(n)}U^{(n)})_{lm}| \le (\mathrm{Tr}\, A^{(n)T}A^{(n)})^{1/2} \le C_A n^{1/2}, \tag{2.26}$$

$$\sum_{l,m=1}^{n} |(U^{(n)}A^{(n)}U^{(n)})_{lm}|^2 = \mathrm{Tr}\, A^{(n)T}A^{(n)} = O(n), \quad n \to \infty, \tag{2.27}$$

$$|\xi_n^A(t)| \le (n\mathrm{Tr}\, A^{(n)T}A^{(n)})^{1/2} = O(n), \quad n \to \infty, \tag{2.28}$$

$$|D_{lm}^p\xi_n^A(t)| \le n^{1/2}c_p|t|^p, \quad c_p = C_A 2^{p+1}/p! \tag{2.29}$$

At last in the next proposition we will summarize some facts concerning the integral equations we need, which were proved in [13, 17] by using the generalized Fourier transform.

Proposition 2.4. *Consider*

$$v(t) = \int_{-2w}^{2w} e^{it\lambda}\rho_{sc}(\lambda)d\lambda, \tag{2.30}$$

where ρ_{sc} is the density of the semicircle law

$$\rho_{sc}(\lambda) = (2\pi w^2)^{-1}\big((4w^2-\lambda^2)^{1/2}\mathbb{I}_{[-2w,2w]}\big)^{1/2}. \tag{2.31}$$

Then the unique differentiable solutions of the integral equations

$$F_1(t) + w^2\int_0^t dt_1 \int_0^{t_1} v(t_1-t_2)F_1(t_2)dt_2 = 1, \tag{2.32}$$

$$F_2(t) + w^2\int_0^t dt_1 \int_0^{t_1} v(t_1-t_2)F_2(t_2)dt_2 = \int_0^t R(t_1)dt_1, \tag{2.33}$$

$$F_3(t_1,t_2) + w^2\int_0^{t_1} dt_3 \int_0^{t_3} v(t_3-t_4)F_3(t_4,t_2)dt_4$$
$$= -w^2\int_0^{t_1} dt_3 \int_0^{t_2} v(t_2-t_4)v(t_3+t_4)dt_4, \tag{2.34}$$

$$F_4(t_1,t_2) + 2w^2\int_0^{t_1} dt_3 \int_0^{t_3} v(t_3-t_4)F_4(t_4,t_2)dt_4 = -2w^2t_2\int_0^{t_1} v(t_2+t_3)dt_3, \tag{2.35}$$

are given by

$$F_1(t) = v(t), \quad F_2(t) = \int_0^t v(t-t_1)R(t_1)dt_1, \quad F_3(t_1,t_2) = v(t_1+t_2) - v(t_1)v(t_2), \tag{2.36}$$

$$F_4(t_1,t_2) = \frac{1}{2\pi^2}\int_{-2w}^{2w}\int_{-2w}^{2w} \frac{\Delta e(t_1)\Delta e(t_2)}{(\lambda_1-\lambda_2)^2}\frac{4w^2-\lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}}d\lambda_1 d\lambda_2, \tag{2.37}$$

where we denote $\Delta e(t) = e^{it\lambda_2} - e^{it\lambda_1}$.

3. The Case of Gaussian Orthogonal Ensemble

In this section we consider $\xi_n^A[\varphi]$ corresponding to the GOE matrix $M^{(n)} = \widehat{M}^{(n)}$ of (2.3). We find the limiting variance of $\xi_n^A[\varphi]$ and prove the CLT for its fluctuations.

In view of the orthogonal invariance of the GOE probability measure, we have

$$\mathbf{E}\{U_{jk}(t)\} = \delta_{jk}\mathbf{E}\{v_n(t)\}, \tag{3.1}$$

so that

$$n^{-1}\mathbf{E}\{\xi_n^A(t)\} = \mathbf{E}\{v_n(t)\}n^{-1}\mathrm{Tr}\, A^{(n)},$$

where $\xi_n^A(t)$ is defined in (2.18), and

$$v_n(t) = n^{-1}\xi_n^I(t) = n^{-1}\mathrm{Tr}\, U(t). \tag{3.2}$$

Since for any bounded continuous φ

$$\lim_{n\to\infty} n^{-1}\mathbf{E}\{\mathrm{Tr}\,\varphi(M^{(n)})\} = \int_{-2w}^{2w} \varphi(\lambda)\rho_{sc}(\lambda)d\lambda,$$

where $M^{(n)}$ is the Wigner matrix and ρ_{sc} is the density of the semicircle law (2.31) (see, e.g., [15] and references therein), then we have

$$\lim_{n\to\infty} \mathbf{E}\{v_n(t)\} = v(t), \tag{3.3}$$

where v is defined in (2.30). Hence,

$$\lim_{n\to\infty} n^{-1}\mathbf{E}\{\xi_n^A(t)\} = T_A \cdot v(t), \tag{3.4}$$

where T_A is defined in (1.3). We also have:

Lemma 3.1. *Let $\widehat{M}^{(n)}$ be the GOE matrix* (2.3)*. Then for any test function $\varphi : \mathbb{R} \to \mathbb{C}$, whose Fourier transform*

$$F[\varphi](t) = \frac{1}{2\pi}\int e^{-it\lambda}\varphi(\lambda)d\lambda \tag{3.5}$$

satisfies the condition

$$\int (1+|t|)|F[\varphi](t)|dt < \infty, \tag{3.6}$$

we have the bound

$$\mathbf{Var}\{\xi_n^A[\varphi]\} := \mathbf{E}\{|\xi_n^{A\circ}[\varphi]|^2\} \le c\Big(\int (1+|t|)|F[\varphi](t)|dt\Big)^2. \tag{3.7}$$

P r o o f. It follows from Poincaré inequality (2.5) and (2.20) that

$$\mathbf{Var}\{\xi_n^A(t)\} \le \frac{w^2}{n} \sum_{1\le l\le m\le n} \beta_{lm}^{-1}\mathbf{E}\{|D_{lm}\xi_n^A(t)|^2\}$$
$$\le \frac{2w^2}{n}\sum_{l,m=1}^{n}\mathbf{E}\{|(U*A^{(n)}U)_{lm}(t)|^2\} = \frac{2w^2|t|^2}{n}\mathrm{Tr}\, AA^{(n)T},$$

so that

$$\mathbf{Var}\{\xi_n^A(t)\} \le 2C_A w^2|t|^2, \tag{3.8}$$

where C_A is defined in (2.24). Writing the Fourier inversion formula

$$\varphi(\lambda) = \int e^{i\lambda t}F[\varphi](t)dt \tag{3.9}$$

and using the spectral theorem for symmetric matrices, we obtain

$$\xi_n^A[\varphi] = \int \xi_n^A(t)F[\varphi](t)dt. \tag{3.10}$$

By (3.10) and the Schwarz inequality,

$$\mathbf{Var}\{\xi_n^A[\varphi]\} \le \left(\int \mathbf{Var}^{1/2}\{\xi_n^A(t)\}|F[\varphi](t)|dt\right)^2. \tag{3.11}$$

This, (3.6), and (3.8) yield (4.3). ■

We have two theorems:

Theorem 3.2. *Let $\widehat{M}^{(n)}$ be the GOE matrix* (2.3), *and $\varphi_{1,2}:\mathbb{R}\to\mathbb{R}$ be the test functions satisfying* (3.6). *Denote*

$$\mathbf{Cov}\{\xi_n^A[\varphi_1],\xi_n^A[\varphi_2]\} = \mathbf{E}\{\xi_n^{A\circ}[\varphi_1]\xi_n^A[\varphi_2]\}.$$

Then we have

$$C_{GOE}[\varphi_1,\varphi_2] := \lim_{n\to\infty}\mathbf{Cov}\{\xi_n^A[\varphi_1],\xi_n^A[\varphi_2]\}$$
$$= \frac{T_A^2}{2\pi^2}\int_{-2w}^{2w}\int_{-2w}^{2w}\frac{\Delta\varphi_1}{\Delta\lambda}\frac{\Delta\varphi_2}{\Delta\lambda}\frac{4w^2-\lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}}d\lambda_1 d\lambda_2$$
$$+ (T_{A(A+A^T)}/2 - T_A^2)\int_{-2w}^{2w}\int_{-2w}^{2w}\Delta\varphi_1\Delta\varphi_2\rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2)d\lambda_1 d\lambda_2, \tag{3.12}$$

where T_A *is defined in* (1.3),

$$\Delta\varphi = \varphi(\lambda_1) - \varphi(\lambda_2), \quad \Delta\lambda = \lambda_1 - \lambda_2, \tag{3.13}$$

and ρ_{sc} *is the density of the semicircle law* (2.31).

Theorem 3.3. *Let* $\widehat{M}^{(n)}$ *be the GOE matrix* (2.3)*, and* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfies* (3.6)*. Then the random variable* $\xi_n^{A\circ}[\varphi]$ *converges in distribution to the Gaussian random variable with zero mean and the variance given by*

$$\begin{aligned} V_{GOE}[\varphi] =& \frac{T_A^2}{2\pi^2} \int_{-2w}^{2w}\int_{-2w}^{2w} \Big(\frac{\Delta\varphi}{\Delta\lambda}\Big)^2 \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}} d\lambda_1 d\lambda_2 \\ &+ (T_{A(A+A^T)}/2 - T_A^2) \int_{-2w}^{2w}\int_{-2w}^{2w} \big(\Delta\varphi\big)^2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \end{aligned} \tag{3.14}$$

R e m a r k 3.4. Note that $V_{GOE}[\varphi]$ can be written in the form

$$V_{GOE}[\varphi] = T_A^2 \cdot V_{GOE}^{\mathcal{N}_n}[\varphi] + (T_{A(A+A^T)}/2 - T_A^2) \cdot V_{GOE}^{jj}[\varphi], \tag{3.15}$$

where

$$V_{GOE}^{\mathcal{N}}[\varphi] = \frac{1}{2\pi^2} \int_{-2w}^{2w}\int_{-2w}^{2w} \Big(\frac{\Delta\varphi}{\Delta\lambda}\Big)^2 \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}} d\lambda_1 d\lambda_2, \tag{3.16}$$

$$V_{GOE}^{jj}[\varphi] = \int_{-2w}^{2w}\int_{-2w}^{2w} \big(\Delta\varphi\big)^2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \tag{3.17}$$

are the limiting variances corresponding to the linear eigenvalue statistics (1.4) and matrix elements (1.5), respectively (compare with the results of [12] and [13]).

Besides, for the limiting variance $V_{GOE}^{(M\eta,\eta)}[\varphi]$ corresponding to the bilinear form (1.7), we have

$$V_{GOE}^{(M\eta,\eta)}[\varphi] = V_{GOE}^{jj}[\varphi] = \int_{-2w}^{2w}\int_{-2w}^{2w} \big(\Delta\varphi\big)^2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \tag{3.18}$$

P r o o f of Theorem 3.2. Here we follow the scheme proposed in [17], Section 3.2. Since $\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\}$ is linear in $\varphi_{1,2}$, it suffices to consider the real valued $\varphi_{1,2}$. Writing the Fourier inversion formula (3.9) and using the linearity of $\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\}$ in $\varphi_{1,2}$ and the spectral theorem for symmetric matrices, we obtain

$$\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = \int\int \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} F[\varphi_1](t_1) F[\varphi_2](t_2) dt_1 dt_2 \tag{3.19}$$

with $\xi_n^A(t)$ of (2.18). Similarly to (3.8), with the help of Poincaré inequality (2.5) it can be shown that $\mathbf{Var}\{\xi_n^{A\prime}(t)\} \le ct^2$, where $\xi_n^{A\prime}(t) = i\mathrm{Tr}A^{(n)}\widehat{M}e^{it_1\widehat{M}}$. This, (3.8), and the Schwarz inequality imply the bounds

$$\left|\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}\right| \le c|t_1||t_2|, \ \left|\partial\ \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}/\partial t_i\right| \le c|t_1||t_2|, \ i = 1, 2. \tag{3.20}$$

Hence, in view of (3.6), the integrand in (3.19) admits an integrable and n-independent upper bound, and by the dominated convergence theorem it suffices to prove the pointwise in $t_{1,2}$ convergence of $\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}$ to a certain limit as $n \to \infty$, implying (3.12). It also follows from (3.20) that there exists a convergent subsequence $\{\mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^A(t_2)\}\}_{j=1}^{\infty}$. We will show that every such a subsequence has the same limit leading through (3.19) to (3.12). Evidently, we can confine ourselves to $t_{1,2} \ge 0$. Consider

$$\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} = \mathbf{E}\{\xi_n^A(t_1)\xi_n^{B\circ}(t_2)\}, \tag{3.21}$$

putting in appropriate moment $A^{(n)} = B^{(n)}$. Here $\xi_n^{A,B}(t_1)$ correspond to $A^{(n)}$, $B^{(n)}$ satisfying (1.2), (1.3). By using Duhamel formula (2.13), we can write

$$\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} = i\int_0^{t_1} \sum_{l,m=1}^{n} \mathbf{E}\{\widehat{M}_{lm}(A^{(n)}U)_{lm}(t_3)\xi_n^{B\circ}(t_2)\}dt_3.$$

Applying differentiation formula (2.4) with (2.2) written in the form

$$\mathbf{E}\{(W_{lm}^{(n)})^2\} = w^2\beta_{lm}^{-1} \tag{3.22}$$

(see (2.17)), and then (2.19), (2.20), we obtain

$$\begin{aligned}
&\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} \\
&= iw^2\int_0^{t_1} \frac{1}{n}\sum_{l,m=1}^{n} \beta_{lm}^{-1}\mathbf{E}\{D_{lm}[(A^{(n)}U)_{lm}(t_3)\xi_n^{B\circ}(t_2)]\}dt_3 \\
&= -\frac{w^2}{n}\int_0^{t_1} dt_3 \int_0^{t_3} \mathbf{E}\{[\xi_n^I(t_3 - t_4)\xi_n^A(t_4) + \xi_n^A(t_3)]\xi_n^{B\circ}(t_2)\}dt_4 \\
&-\frac{w^2}{n}\int_0^{t_1} dt_3 \int_0^{t_2} \mathbf{E}\{\mathrm{Tr}A^{(n)}U(t_3 + t_4)(B^{(n)} + B^{(n)T})U(t_2 - t_4)\}dt_4.
\end{aligned} \tag{3.23}$$

Putting

$$H_n^{A,C}(t_1,t_2) = \mathrm{Tr} A^{(n)} U(t_1) C^{(n)} U(t_2), \quad C^{(n)} = B^{(n)} + B^{(n)T}, \tag{3.24}$$

$$\xi_n^A = \xi_n^{A\circ} + \overline{\xi}_n^A, \quad \overline{\xi}_n^A = \mathbf{E}\{\xi_n^A\}, \tag{3.25}$$

from (3.23) we get

$$\begin{aligned}
\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} = & - w^2 \int_0^{t_1} dt_3 \int_0^{t_3} \overline{v}_n(t_3 - t_4) \mathbf{E}\{\xi_n^A(t_4) \xi_n^{B\circ}(t_2)\} dt_4 \\
& - \frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_3} \overline{\xi}_n^A(t_3 - t_4) \mathbf{E}\{\xi_n^I(t_4) \xi_n^{B\circ}(t_2)\} dt_4 \\
& - \frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_2} \mathbf{E}\{H_n^{A,C}(t_3 + t_4, t_2 - t_4)\} dt_4 + r_n(t_1, t_2),
\end{aligned} \tag{3.26}$$

where

$$r_n(t_1,t_2) = -w^2 \int_0^{t_1} \mathbf{E}\{[(v_n^\circ * \xi_n^{A\circ})(t_3) + t_3 n^{-1} \xi_n^A(t_3)] \xi_n^{B\circ}(t_2)\} dt_3.$$

By using Poincaré inequality (2.5) it can be shown that $\mathbf{Var}\left\{v_n^\circ \xi_n^{A\circ}\right\} = O(n^{-2})$, $n \to \infty$, which, together with (3.8), yields

$$r_n(t_1,t_2) = O(n^{-1}), \quad n \to \infty. \tag{3.27}$$

Consider the convergent subsequences

$$\{\mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^B(t_2)\}\}_{j=1}^\infty, \quad \{H_{n_j}^{A,C}(t_1,t_2)\}_{j=1}^\infty,$$

and denote

$$C^{A,B}(t_1,t_2) := \lim_{n_j \to \infty} \mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^B(t_2)\}, \; H^{A,C}(t_1,t_2) := \lim_{n_j \to \infty} \mathbf{E}\{H_{n_j}^{A,C}(t_1,t_2)\}. \tag{3.28}$$

It follows from (3.3), (3.4), and (3.26), (3.27) that $C^{A,B}(t_1,t_2)$ satisfies the equation

$$\begin{aligned}
C^{A,B}(t_1,t_2) \; + w^2 \int_0^{t_1} (v * C^{A,B}(\cdot, t_2))(t_3) dt_3 = & -w^2 T_A \int_0^{t_1} (v * C^{I,B}(\cdot, t_2))(t_3) dt_3 \\
- w^2 \int_0^{t_1} dt_3 \int_0^{t_2} H^{A,C}(t_3 + t_4, t_2 - t_4)\} dt_4.
\end{aligned} \tag{3.29}$$

In particular, putting here $A^{(n)} = I$, we get

$$C^{I,B}(t_1,t_2) + 2w^2 \int_0^{t_1} (v * C^{I,B}(\cdot,t_2))(t_3)dt_3 = -w^2 T_C t_2 \int_0^{t_1} v(t_2+t_3)dt_3, \quad (3.30)$$

so that by (2.37)

$$C^{I,B}(t_1,t_2) = \frac{T_B}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \frac{\Delta e(t_1)\Delta e(t_2)}{(\lambda_1-\lambda_2)^2} \frac{4w^2-\lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}} d\lambda_1 d\lambda_2. \quad (3.31)$$

Now let us calculate the second term in the r.h.s. of (3.29). By (3.1), we have

$$\begin{aligned} \mathbf{E}\{H_n^{A,C}(t_1,t_2)\} &= T_{AC}\overline{v}_n(t_1)\overline{v}_n(t_2) + F_n(t_1,t_2), \quad (3.32) \\ F_n(t_1,t_2) &= n^{-1}\sum_{j,l=1}^{n} \mathbf{E}\{(UC^{(n)})_{jl}(t_1)(U^\circ A^{(n)})_{lj}(t_2)\}. \end{aligned}$$

Repeating the steps leading from (3.21) to (3.29) and using consequently (2.13), the differentiation formulas (2.4) and (2.19), (2.20), and applying (3.8) to estimate the vanishing terms, one can easily get

$$\begin{aligned} &F_n(t_1,t_2) + w^2 \int_0^{t_1} (\overline{v}_n * F_n(\cdot,t_2))(t_3)dt_3 \\ &\quad = -w^2 \int_0^{t_1} dt_3 \int_0^{t_2} n^{-1}\overline{\xi}_n^A(t_3+t_4)\cdot n^{-1}\overline{\xi}_n^C(t_2-t_4)dt_4 + O(n^{-1}),\ n\to\infty. \end{aligned}$$

This, (3.3), and (3.4) yield for $F = \lim_{n_j\to\infty} F_{n_j}$:

$$\begin{aligned} &F(t_1,t_2) + w^2 \int_0^{t_1} (v * F(\cdot,t_2))(t_3)dt_3 \\ &\quad = -w^2 T_A T_C \int_0^{t_1} dt_3 \int_0^{t_2} v(t_3+t_4)v(t_2-t_4)dt_4. \end{aligned}$$

Hence, by (2.36), we get $F(t_1,t_2) = T_A T_C\big(v(t_1+t_2) - v(t_1)v(t_2)\big)$. This, (3.3), and (3.32) yield for $H^{A,C}$ of (3.28):

$$H^{A,C}(t_1,t_2) = T_A T_C v(t_1+t_2) + \big(T_{AC} - T_A T_C\big)v(t_1)v(t_2). \quad (3.33)$$

Putting (3.31) and (3.33) in (3.29), we obtain the equation

$$C^{A,B}(t_1,t_2)+w^2\int_0^{t_1}(v*C^{A,B}(\cdot,t_2))(t_3)dt_3$$
$$=-w^2\int_0^{t_1}\Big[T_A(v*C^{I,B}(\cdot,t_2))(t_3)+2T_AT_Bt_2v(t_3+t_2)$$
$$+\big(T_{A(B+B^T)}-2T_AT_B\big)\int_0^{t_2}v(t_3+t_4)v(t_2-t_4)dt_4\Big]dt_3,$$

solving which with the help of Lemma 2.4, we finally get

$$C^{A,B}(t_1,t_2)=\frac{T_AT_B}{2\pi^2}\int_{-2w}^{2w}\int_{-2w}^{2w}\frac{\Delta e(t_1)\Delta e(t_2)}{(\lambda_1-\lambda_2)^2}\frac{4w^2-\lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}}d\lambda_1 d\lambda_2$$
$$+\big(T_{A(B+B^T)}/2-T_AT_B\big)\int_{-2w}^{2w}\int_{-2w}^{2w}\Delta e(t_1)\Delta e(t_2)\rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2)d\lambda_1 d\lambda_2.$$

Putting this expression with $T_B=T_A$ in (3.19), we obtain (3.12) and hence the theorem. ■

P r o o f of Theorem 3.3. The detailed proofs of the CLTs for linear eigenvalue statistics (1.4) and for matrix elements (1.5) are given in [11, 12] and [13], respectively (see also [17], Section 3.2). The proof of Theorem 3.3 follows the same scheme, and here we only outline its main steps. According to this scheme it suffices to show that if

$$Z_n^A(x)=\mathbf{E}\{e^{ix\xi_n^{A\circ}[\varphi]}\},\tag{3.34}$$

then for any converging subsequences $\{Z_{n_i}^A\}_{i\ge1}$ and $\{Z_{n_i}^{A}{}'\}_{i\ge1}$ there exists $Z^A(x)$ such that

$$\lim_{i\to\infty}Z_{n_i}^A(x)=Z^A(x),\tag{3.35}$$

and

$$\lim_{i\to\infty}Z_{n_i}^{A}{}'(x)=-xV_{GOE}[\varphi]Z^A(x).\tag{3.36}$$

We obtain (3.35), (3.36), and hence the theorem for a class of test functions satisfying the condition

$$\int(1+|t|^2)|F[\varphi](t)|dt<\infty\tag{3.37}$$

(cf (3.6)). Then the theorem can be extended to the class of the functions satisfying (3.6) by using a standard approximation procedure (see, e.g., [17], Section 3.2). We denote

$$e_n(x) = \exp\{ix\xi_n^{A\circ}[\varphi]\}, \tag{3.38}$$

and according to (3.9) and (3.34), write

$$Z_n^{A\prime}(x) = i\mathbf{E}\left\{\xi_n^{A\circ}[\varphi]e^{ix\xi_n^{A\circ}[\varphi]}\right\} = i\int F[\varphi](t)Y_n^A(x,t)dt, \tag{3.39}$$

where

$$Y_n^A(x,t) = \mathbf{E}\left\{\xi_n^A(t)e_n^\circ(x)\right\}, \tag{3.40}$$

and $\xi_n^A(t)$ is defined in (2.18). It follows from the Schwarz inequality and (3.8) that

$$|Y_n^A(x,t)| \le c|t|.$$

This and (3.37) yield that the sequence $Z_n^{A\prime}$ is uniformly bounded. Hence, there is a convergent subsequence $Z_{n_i}^A$, and by the dominated convergence theorem to find its limit as $n \to \infty$ it suffices to find the pointwise limit of the corresponding subsequence $Y_{n_i}^A$. With the help of Poincaré inequality (2.5) and (3.37) it also can be shown that the sequences $\{\partial Y_n^A/\partial x\}$ and $\{\partial Y_n^A/\partial t\}$ are uniformly bounded in $(t,x) \in K \subset \mathbb{R}_+^2$, $n \in \mathbf{N}$, for any bounded K, so that the sequence $\{Y_n^A\}$ is equicontinuous on any finite set of $\mathbb{R}_+^2$ and contains convergent subsequences. Hence, for any converging subsequence $\{Z_{n_i}^A\}$ (see (3.35)) there is a converging subsequence $\{Y_{n_i'}^A\}$ and a continuous function Y^A (which obviously depends on $\{Z_{n_i}^A\}$) such that

$$\lim_{n_i'\to\infty} Y_{n_i'}^A = Y^A, \quad \lim_{n_i'\to\infty} Z_{n_i'}^A = Z^A. \tag{3.41}$$

We will show now that Y^A satisfies certain integral equation leading through (3.39) to (3.35), (3.36). Applying consequently (2.13) and differentiation formula (2.4) with (3.22), we get

$$Y_n^A(x,t) = \frac{iw^2}{n}\int_0^t \sum_{j,k=1}^n \beta_{jk}^{-1}\mathbf{E}\{D_{jk}\big((UA^{(n)})_{kj}(t_1)e_n^\circ(x)\big)\}dt_1,$$

where $D_{jk} = \partial/\partial M_{jk}$. It follows from (2.20) that

$$D_{jk}e_n(x) = -\beta_{jk}xe_n(x)\int (U * C^{(n)}U)_{jk}(\theta)F[\varphi](\theta)d\theta. \tag{3.42}$$

This, (2.19), (3.24), (3.25), and the relation $e_n = e_n^\circ + Z_n^A$ yield

$$Y_n^A(x,t) = -w^2 \int_0^t \left[\overline{v}_n * Y_n^A(x,\cdot) + \overline{\xi}_n * Y_n^I(x,\cdot)\right](t_1)dt_1$$

$$-iw^2 x Z_n^A(x) \int_0^t dt_1 \int F[\varphi](\theta)d\theta \int_0^\theta n^{-1}\mathbf{E}\left\{H_n^{A,C}(\theta-\theta_1,\theta_1+t_1)\right\}d\theta_1 + O(n^{-1})$$

as $n\to\infty$, where the vanishing term can be estimated with the help of (2.5).

This, (3.4), and (3.33) lead to the pair of equations with respect to $Y^A = \lim_{n_j\to\infty} Y_{n_j}^A$ and Y^I:

$$Y^A(x,t) + w^2 \int_0^t (v * Y^A(x,\cdot))(t_1)dt_1 = -w^2 T_A \int_0^t (v * Y^I(x,\cdot))(t_1)dt_1$$

$$-iw^2 x Z^A(x) \int_0^t dt_1 \int F[\varphi](\theta)d\theta \int_0^\theta H(\theta-\theta_1,\theta_1+t_1)d\theta_1,$$

$$Y^I(x,t) + 2w^2 \int_0^t (v * Y^I(x,\cdot))(t_1)dt_1$$

$$= -2iw^2 x Z^A(x) T_A \int_0^t dt_1 \int F[\varphi](\theta)\theta v(\theta+t_1)d\theta.$$

Comparing these equations and (3.29), (3.30), one can see that

$$Y^A(x,t) = ixZ^A(x)\int C^{A,A}(t,\theta)F[\varphi](\theta)d\theta,$$

where $C^{A,A}$ is given by (3.21) with $A=B$. This and (3.39) yield

$$\lim_{i\to\infty} Z_{n_i}^{A\prime}(x) = -xZ^A(x)\int\int C^{A,A}(t,\theta)F[\varphi](t)F[\varphi](\theta)dtd\theta = -xZ^A(x)V_{GOE}[\varphi]$$

(see (3.14) and (3.19), and thus lead to (3.35), (3.36) and complete the proof of the theorem. ■

4. Covariance of $\xi_n^A[\varphi]$ in Wigner Case

We show first that if $M^{(n)}$ is the Wigner matrix (2.1)–(2.2) with uniformly bounded eighth moments of its entries, and the test-function φ is essentially of class $\mathbf{C}^4$, then the variance of $\xi_n^A[\varphi]$ is of the order $O(1)$ as $n\to\infty$. We have:

Lemma 4.1. *Let $M^{(n)} = n^{-1/2}W^{(n)}$ be the real symmetric Wigner matrix (2.1), (2.2). Assume that the third moments of its entries, $\mu_3 = \mathbf{E}\{(W_{jk}^{(n)})^3\}$, do not depend on j, k, and n, and the eighth moments are uniformly bounded*

$$w_8 := \sup_{n\in\mathbb{N}} \max_{1\le j,k\le n} \mathbf{E}\{(W_{jk}^{(n)})^8\} < \infty. \tag{4.1}$$

Then for any test function $\varphi : \mathbb{R} \to \mathbb{C}$, whose Fourier transform (3.5) *satisfies the condition*

$$\int (1+|t|)^4 |F[\varphi](t)| dt < \infty, \tag{4.2}$$

we have the bound

$$\mathbf{Var}\{\xi_n^A[\varphi]\} := \mathbf{E}\{|\xi_n^{A\circ}[\varphi]|^2\} \le c\Big(\int (1+|t|)^4 |F[\varphi](t)| dt\Big)^2. \tag{4.3}$$

The proof of (4.3) follows from (3.11), (4.2), and the bound (see (6.7))

$$\mathbf{Var}\{\xi_n^A(t)\} \le c(1+|t|)^8. \tag{4.4}$$

Theorem 4.2. *Let $M^{(n)} = n^{-1/2}W^{(n)}$ be the real symmetric Wigner matrix (2.1), (2.2), whose third and fourth moments, $\mu_l = \mathbf{E}\{(W_{jk}^{(n)})^l\}$, $l = 3, 4$, do not depend on j, k, and the eighth moments are uniformly bounded* (*see* (4.1)). *Let $\{A^{(n)}\}_{n=1}^\infty$ satisfy* (1.2), (1.3), *$C^{(n)} = A^{(n)} + A^{(n)T}$, and there exist*

$$K_A^{(1)} = \lim_{n\to\infty} n^{-3/2} \sum_{l,m=1}^n A_{ll}^{(n)} C_{lm}^{(n)} \tag{4.5}$$

$$K_A^{(2)} = T_A \lim_{n\to\infty} n^{-3/2} \sum_{l,m=1}^n C_{lm}^{(n)}, \tag{4.6}$$

$$K_A^{(3)} = \lim_{n\to\infty} n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \Big(A_{mm}^{(n)} - n^{-1}\mathrm{Tr}\, A^{(n)}\Big). \tag{4.7}$$

Then for any $\varphi_{1,2} : \mathbb{R} \to \mathbb{R}$ satisfying (4.2), *we have*

$$\lim_{n\to\infty} \mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = C_{GOE}[\varphi_1,\varphi_2] + C_{\kappa_3}[\varphi_1,\varphi_2] + C_{\kappa_4}[\varphi_1,\varphi_2], \tag{4.8}$$

where $C_{GOE}[\varphi_1, \varphi_2]$ *is defined in* (3.12),

$$C_{\kappa_3}[\varphi_1, \varphi_2] = \frac{\kappa_3}{w^6} \int_{-2w}^{2w} \int_{-2w}^{2w} \lambda_1 \Big(K_A^{(1)} (\lambda_2^2 - w^2) + K_A^{(2)} \Big(\frac{2w^4}{4w^2 - \lambda_2^2} - \lambda_2^2 \Big) \Big)$$

$$\times (\varphi_1(\lambda_1)\varphi_2(\lambda_2) + \varphi_1(\lambda_2)\varphi_2(\lambda_1)) \prod_{j=1}^{2} \rho_{sc}(\lambda_j) d\lambda_j, \tag{4.9}$$

$$C_{\kappa_4}[\varphi_1, \varphi_2] = \frac{\kappa_4}{w^8} \Big[K_A^{(3)} \prod_{j=1}^{2} \int_{-2w}^{2w} \varphi_j(\lambda)(w^2 - \lambda^2)\rho_{sc}(\lambda) d\lambda$$

$$+ \frac{T_A^2}{2\pi^2} \prod_{j=1}^{2} \int_{-2w}^{2w} \varphi_j(\lambda) \frac{2w^2 - \lambda^2}{\sqrt{4w^2 - \lambda^2}} d\lambda \Big], \tag{4.10}$$

$\kappa_3 = \mu_3$, *and* $\kappa_4 = \mu_4 - 3w^4$ *are the third and the fourth cumulants of the off-diagonal entries* (*see* (2.7)). *In particular,*

$$V_W[\varphi] := \lim_{n \to \infty} \mathbf{Var}\{\xi_n^A[\varphi]\} = V_{GOE}[\varphi] + C_{\kappa_3}[\varphi, \varphi] + C_{\kappa_4}[\varphi, \varphi] \tag{4.11}$$

with $V_{GOE}[\varphi]$ *of* (3.14).

R e m a r k 4.3. Note that for the limiting variances $V_W^{\mathcal{N}}[\varphi]$ and $V_W^{jj}[\varphi]$ of linear eigenvalue statistics (1.4) and matrix elements (1.5) we get

$$V_W^{\mathcal{N}}[\varphi] = V_{GOE}^{\mathcal{N}}[\varphi] + \frac{\kappa_4}{2\pi^2 w^8} \Big| \int_{-2w}^{2w} \varphi(\lambda) \frac{2w^2 - \lambda^2}{\sqrt{4w^2 - \lambda^2}} d\lambda \Big|^2, \tag{4.12}$$

$$V_W^{jj}[\varphi] = V_{GOE}^{jj}[\varphi] + \frac{\kappa_4}{w^8} \Big| \int_{-2w}^{2w} \varphi(\lambda)(w^2 - \lambda^2)\rho_{sc}(\lambda) d\lambda \Big|^2, \tag{4.13}$$

respectively (see (3.16), (3.17)). This coincides with the results of [12] and [14].

R e m a r k 4.4. In the case of bilinear forms (1.6), (1.7), $T_A = 0$ and the coefficients $K_A^{(j)}$, $j = 1, 2, 3$ of (4.5)–(4.7) are

$$K_A^{(1)} = 2 \lim_{n \to \infty} n^{-1/2} \sum_{m=1}^{n} \eta_m^{(n)} \sum_{l=1}^{n} (\eta_l^{(n)})^3, \quad K_A^{(2)} = 0, \quad K_A^{(3)} = \lim_{n \to \infty} \sum_{m=1}^{n} (\eta_m^{(n)})^4.$$

In particular, if for all $m = 1, \dots, n$ $\eta_m^{(n)} = O(n^{-1/2})$, $n \to \infty$, then $K_A^{(j)} = 0$, $j = 1, 2, 3$, and for the limiting variance (see (3.18)) we get

$$V_W^{(M\eta,\eta)}[\varphi] = V_{GOE}^{(M\eta,\eta)}[\varphi] = \int_{-2w}^{2w}\int_{-2w}^{2w} (\Delta\varphi)^2 \rho_{sc}(\lambda_1)\rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \quad (4.14)$$

R e m a r k 4.5. We choose here the Wigner matrix so that its first two moments match the first two moments of the GOE matrix (see (2.2)). It allows to use the known properties of the GOE and lies at the basis of interpolation procedure widely used in the proof of Lemma 6.1 below. In fact, this condition is a pure technical one, and we can replace condition (2.2) with a more general one and consider the Wigner matrix $\widetilde{M} = n^{-1/2}\widetilde{W}$ satisfying

$$\begin{aligned} &\mathbf{E}\{\widetilde{W}_{jk}^{(n)}\} = 0, \quad 1 \le j \le k \le n, \\ &\mathbf{E}\{(\widetilde{W}_{jk}^{(n)})^2\} = w^2,\ j \ne k, \quad \mathbf{E}\{(\widetilde{W}_{jj}^{(n)})^2\} = w_2 w^2,\ w_2 > 0. \end{aligned} \quad (4.15)$$

In this case there arise additional terms in (4.8) and (4.11) proportional to $w_2 - 2$. In particular, for the corresponding limiting variance we have

$$\begin{aligned} V_{\widetilde{W}}^{w_2}[\varphi] =& V_W[\varphi] + (w_2 - 2)w^{-2}\Big(K_A^{(3)}\Big(\int_{-2w}^{2w} \varphi(\mu)\mu\rho_{sc}(\mu)d\mu\Big)^2 \\ &+ T_A^2\Big(\frac{1}{2\pi}\int_{-2w}^{2w} \frac{\varphi(\mu)\mu}{\sqrt{4w^2 - \mu^2}}d\mu\Big)^2\Big), \end{aligned} \quad (4.16)$$

where $V_W[\varphi]$ is given by (4.11).

P r o o f of Theorem 4.2. It follows from (4.2) and (4.4) that the integrand in (3.19) admits an integrable and n-independent upper bound. Thus, by the dominated convergence theorem it suffices to prove the pointwise in $t_{1,2}$ convergence of $\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}$ to a certain limit as $n \to \infty$ implying (4.8). To do this we use the known result for the GOE matrix (see Theorem 3.2) and the interpolating procedure proposed in [9].

Let $\widehat{M}^{(n)} = n^{-1/2}\widehat{W}^{(n)}$ be the GOE matrix (2.3) independent of $M^{(n)}$, and

$$\widehat{U}(t) = \widehat{U}^{(n)}(t) := e^{it\widehat{M}^{(n)}}, \quad \widehat{\xi}_n^A(t) = \operatorname{Tr} A^{(n)}\widehat{U}(t). \quad (4.17)$$

Consider the "interpolating" random matrix

$$M^{(n)}(s) = s^{1/2}M^{(n)} + (1-s)^{1/2}\widehat{M}^{(n)}, \quad 0 \le s \le 1, \quad (4.18)$$

viewed as the matrix defined on the product of probability spaces of the matrices $W^{(n)}$ and $\widehat{W}^{(n)}$ (cf (2.18)). We denote again by $\mathbf{E}\{\dots\}$ the corresponding expectation in the product space. Since $M^{(n)}(1) = M^{(n)}$, $M^{(n)}(0) = \widehat{M}^{(n)}$, then putting

$$U(t,s) = U^{(n)}(t,s) := e^{itM^{(n)}(s)}, \quad \xi_n^A(t,s) = \mathrm{Tr}\, A^{(n)} U(t,s), \tag{4.19}$$

we can write

$$\begin{aligned} C_n^{\Delta}(t_1,t_2) :&= \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} - \mathbf{Cov}\{\widehat{\xi}_n^A(t_1), \widehat{\xi}_n^A(t_2)\} \\ &= \int_0^1 \frac{\partial}{\partial s} \mathbf{E}\{\xi_n^A(t_1,s)\xi_n^{A\circ}(t_2,s)\} ds = c_n^{\Delta}(t_1,t_2) + c_n^{\Delta}(t_2,t_1), \end{aligned} \tag{4.20}$$

$$\begin{aligned} c_n^{\Delta}(t_1,t_2) :&= \int_0^1 \mathbf{E}\Big\{\frac{\partial}{\partial s}(\xi_n^A(t_1,s)) \cdot \xi_n^{A\circ}(t_2,s)\Big\} ds \\ &= \frac{i}{2}\int_0^1 \Big(\frac{1}{\sqrt{ns}} \sum_{l,m=1}^n \mathbf{E}\Big\{W_{lm}^{(n)}\Phi_{lm}\Big\} - \frac{1}{\sqrt{n(1-s)}} \sum_{l,m=1}^n \mathbf{E}\Big\{\widehat{W}_{lm}\Phi_{lm}\Big\}\Big) ds, \end{aligned} \tag{4.21}$$

where

$$\Phi_{lm} = \Phi_{lm}(t_1,t_2,s) = (U * A^{(n)} U)_{ml}(t_1,s)\xi_n^{A\circ}(t_2,s). \tag{4.22}$$

A simple algebra based on (2.15)–(2.26) allows to obtain

$$|D_{lm}^q \Phi_{lm}| \le C_q (1 + |t_1| + |t_2|)^{q+1} n^{3/2}, \tag{4.23}$$

with C_q depending only on $q \in \mathbb{N}$. Besides, since $\partial/\partial W_{lm}^{(n)} = \sqrt{s/n} D_{lm}(s)$, $D_{lm}(s) = \partial/\partial M_{lm}^{(n)}(s)$, then every derivative with respect to $W_{lm}^{(n)}$ gives the factor $n^{-1/2}$. Therefore, applying differentiation formula (2.8) with $\zeta = W_{lm}^{(n)}$, $p = 6$, and $\Phi = \Phi_{lm}$ to every term of the first sum and differentiation formula (2.4) to every term of the second sum in the r.h.s. of (4.21), we obtain (see also (2.10))

$$c_n^{\Delta}(t_1,t_2) = \frac{i}{2}\int_0^1 \Big[\sum_{j=2}^6 s^{(j-1)/2} T_j^{(n)} + \varepsilon_6\Big] ds, \tag{4.24}$$

where

$$T_j^{(n)} = \frac{1}{j! n^{(j+1)/2}} \sum_{l,m=1}^n \kappa_{j+1,lm} \mathbf{E}\big\{D_{lm}^j \Phi_{lm}\big\}, \quad j = 2,\dots,6, \tag{4.25}$$

and by (2.9) and (4.23),

$$|\varepsilon_6| \le \frac{C_6 w_8}{n^4} \sum_{l,m=1}^{n} \sup_{M\in\mathcal{S}_n} |D_{lm}^7 \Phi_{lm}| \le c(1+|t_1|+|t_2|)^8 n^{-1/2}. \tag{4.26}$$

Now it follows from Lemma 4.6 below that

$$\int\!\!\int \Big[\frac{i}{2}\int_0^1 s^{1/2} \lim_{n\to\infty} (T_2^{(n)}(t_1,t_2)+T_2^{(n)}(t_2,t_1))ds\Big] \prod_{j=1}^{2} F[\varphi_j](t_j)dt_j = C_{\kappa_3}[\varphi_1,\varphi_2], \tag{4.27}$$

$$\int\!\!\int \Big[\frac{i}{2}\int_0^1 s \lim_{n\to\infty} (T_3^{(n)}(t_1,t_2)+T_3^{(n)}(t_2,t_1))ds\Big] \prod_{j=1}^{2} F[\varphi_j](t_j)dt_j = C_{\kappa_4}[\varphi_1,\varphi_2], \tag{4.28}$$

and

$$\lim_{n\to\infty} T_j^{(n)} = 0, \quad j = 4,5,6, \tag{4.29}$$

with $C_{\kappa_3}[\varphi_1,\varphi_2]$, $C_{\kappa_4}[\varphi_1,\varphi_2]$ of (4.9), (4.10). This, (4.24), (4.20), (3.19), and (3.12) lead to (4.8)–(4.10) and complete the proof. ■

Lemma 4.6. *Under the conditions of Theorem 4.2, the statements* (4.27)–(4.29) *are valid.*

P r o o f. Consider $T_2^{(n)}$ of (4.25). Note that by (2.7), $\kappa_{3,lm} = \mu_3 = \kappa_3$, and we have

$$\begin{aligned} T_2^{(n)}(t_1,t_2,s) =& \frac{\kappa_3}{2n^{3/2}} \sum_{l,m=1}^{n} \mathbf{E}\{\xi_n^{A\circ}(t_2,s) D_{lm}^2 (U * A^{(n)}U)_{ml}(t_1,s) \\ &+ 2D_{lm}(U * A^{(n)}U)_{ml}(t_1,s) D_{lm}\xi_n^A(t_2,s) \\ &+ (U * A^{(n)}U)_{ml}(t_1,s) D_{lm}^2 \xi_n^A(t_2,s)\} =: \kappa_3[T_{21}^{(n)} + T_{22}^{(n)} + T_{23}^{(n)}]. \end{aligned} \tag{4.30}$$

Consider $T_{21}^{(n)}$. It follows from (2.16) and (2.22) that $D_{lm}^2(U * A^{(n)}U)_{ml}$ of $T_{21}^{(n)}$ gives the terms of the form

$$T_{21}^{1(n)} = n^{-3/2} \sum_{l,m=1}^{n} U_{lm} U_{lm} (UA^{(n)}U)_{lm}, \tag{4.31}$$

$$T_{21}^{2(n)} = n^{-3/2} \sum_{l,m=1}^{n} U_{lm} U_{ll} (UA^{(n)}U)_{mm}, \tag{4.32}$$

$$T_{21}^{3(n)} = n^{-3/2} \sum_{l,m=1}^{n} U_{ll} U_{mm} (UA^{(n)}U)_{lm}. \tag{4.33}$$

Here for shortness we omit the sign of conjugation " $*$ " and the arguments of U. Besides, we replace β_{lm} with 1 (this, in view of (2.27), gives error terms of the order $O(n^{-1/2})$, $n \to \infty$). It follows from the Schwarz inequality, (2.15), and (2.26) that $T_{21}^{1(n)} = O(n^{-1/2})$, $n \to \infty$, and from (2.15), (2.27) that

$$\begin{aligned} T_{21}^{2(n)} &\le n^{-3/2} ||U|| \cdot ||(U_{11}, \dots, U_{nn})^T|| \cdot ||((UA^{(n)}U)_{11}, \dots, (UA^{(n)}U)_{nn})^T|| \\ &= O(n^{-1/2}), \quad n \to \infty. \end{aligned}$$

This and (4.4) yield

$$\left|\mathbf{E}\{(T_{21}^{1(n)} + T_{21}^{2(n)})\xi_n^{A\circ}\}\right| \le cn^{-1/2} \mathbf{Var}\{\xi_n^A\}^{1/2} = O(n^{-1/2}), \quad n \to \infty. \tag{4.34}$$

We also have

$$T_{21}^{3(n)} = O(1), \quad n \to \infty. \tag{4.35}$$

Let us show that

$$\mathbf{E}\{T_{21}^{3(n)} \xi_n^{A\circ}\} = O(n^{-1/2}), \quad n \to \infty. \tag{4.36}$$

For this purpose consider

$$R_n = n^{-3/2} \sum_{l,m=1}^{n} \mathbf{E}\{U_{ll}(t_1) U_{mm}(t_2) (UA^{(n)}U)_{lm} \xi_n^{A\circ}\}.$$

Putting here $U_{jj} = \mathbf{E}\{U_{jj}\} + U_{jj}^{\circ}$ and using (6.7), we get

$$\begin{aligned} R_n =& v(t_1)v(t_2) n^{-3/2} \sum_{l,m=1}^{n} \mathbf{E}\{(UA^{(n)}U)_{lm} \xi_n^{A\circ}\} \\ &+ v(t_1) n^{-3/2} \sum_{l,m=1}^{n} \mathbf{E}\{U_{mm}^{\circ}(t_2)(UA^{(n)}U)_{lm} \xi_n^{A\circ}\} \\ &+ n^{-3/2} \sum_{l,m=1}^{n} \mathbf{E}\{U_{ll}^{\circ}(t_1) U_{mm}(t_2)(UA^{(n)}U)_{lm} \xi_n^{A\circ}\} + o(1), \quad n \to \infty. \end{aligned} \tag{4.37}$$

It follows from the Schwarz inequality, (6.7) and (6.8) that the first term in the r.h.s. of (4.37) is of the order $O(n^{-1/2})$, $n \to \infty$. In view of (1.2) and (2.15), we also have

$$n^{-3/2}\Big|\sum_{l,m=1}^{n} U^{\circ}_{mm}(t_2)(UA^{(n)}U)_{lm}\Big| \le n^{-1}||UAU||\cdot||(U^{\circ}_{11},\dots,U^{\circ}_{nn})^T||$$

$$\le n^{-1/2}\big(\sum_{m=1}^{n}|U^{\circ}_{mm}(t_2)|^2\big)^{1/2}.$$

Hence, by the Schwarz inequality and (6.7),

$$\Big|n^{-3/2}\sum_{l,m=1}^{n}\mathbf{E}\{U^{\circ}_{mm}(t_2)(UA^{(n)}U)_{lm}\xi_n^{A\circ}\}\Big|$$

$$\le n^{-1/2}\Big(\sum_{m=1}^{n}\mathbf{Var}\{U_{mm}(t_2)\}\Big)^{1/2}\mathbf{Var}\{\xi_n^A\}^{1/2} = O(n^{-1/2}),\ n\to\infty.$$

Thus, the second and the third terms in the r.h.s. of (4.37) are of the order $O(n^{-1/2})$, $n\to\infty$, and we get (4.36). Now (4.34)–(4.36) yield for $T_{21}^{(n)}$ of (4.30):

$$T_{21}^{(n)} = O(n^{-1/2}),\quad n\to\infty. \tag{4.38}$$

Applying (2.20)–(2.23) to calculate $T_{22}^{(n)}$ and $T_{23}^{(n)}$ of (4.30), we get the terms of the form

$$n^{-3/2}\sum_{l,m=1}^{n}U_{lm}(UA^{(n)}U)_{lm}(UA^{(n)}U)_{lm},$$

$$n^{-3/2}\sum_{l,m=1}^{n}U_{ll}(UA^{(n)}U)_{mm}(UA^{(n)}U)_{lm}, \tag{4.39}$$

where, as it follows from the Schwarz inequality and (2.27), the first term is of the order $O(n^{-1/2})$, and the second one is of the order $O(1)$, $n\to\infty$. Hence, we are left with

$$T_{22}^{(n)}+T_{23}^{(n)} = -\frac{1}{n^{3/2}}\sum_{l,m=1}^{n}\mathbf{E}\{2(U_{ll}*(U*A^{(n)}U))_{mm}(t_1)(U*C^{(n)}U)_{lm}(t_2)$$
$$+((U*C^{(n)}U))_{lm}(t_1)(U_{mm}*(U*C^{(n)}U)_{ll})(t_2)/2\}+O(n^{-1/2}), \tag{4.40}$$

$n\to\infty$. Now it follows from (4.30), (4.38), (4.40), and (6.13) that

$$\lim_{n\to\infty}T_2^{(n)}(t_1,t_2) = \kappa_3\lim_{n\to\infty}(T_{22}^{(n)}+T_{23}^{(n)})(t_1,t_2) = -\kappa_3\big[2T_2(t_1,t_2)+T_2(t_2,t_1)\big], \tag{4.41}$$

$$T_2(t_1,t_2) = \big[(K_A^{(1)}-K_A^{(2)})(v*v*v)(t_1)+K_A^{(2)}(v*tv)(t_1)\big]\cdot(v*v)(t_2) \tag{4.42}$$

with $K_A^{(1)}$, $K_A^{(2)}$ of (4.5)–(4.6) and v of (2.30). We also have

$$(v*v)(t) = -iw^{-2}\int_{-2w}^{2w} e^{i\mu t}\mu\rho_{sc}(\mu)d\mu, \tag{4.43}$$

$$(v*tv)(t) = w^{-2}\int_{-2w}^{2w} e^{i\mu t}\Big[1-\frac{2w^2}{4w^2-\mu^2}\Big]\rho_{sc}(\mu)d\mu, \tag{4.44}$$

$$(v*v*v)(t) = w^{-4}\int_{-2w}^{2w} e^{it\mu}(w^2-\mu^2)\rho_{sc}(\mu)d\mu. \tag{4.45}$$

Putting (4.43)–(4.45) in (4.42) and plugging the result in the l.h.s. of the r.h.s. of (4.27), after some calculations, we get (4.27). Consider now $T_3^{(n)}$ of (4.25),

$$T_3^{(n)} = \frac{1}{6n^2}\sum_{l,m=1}^{n}\kappa_{4,lm}\mathbf{E}\big\{D_{lm}^3\big((U*A^{(n)}U)_{ml}(t_1,s)\xi_n^{A\circ}(t_2,s)\big)\big\}, \tag{4.46}$$

where, in view of (2.7), $\kappa_{4,lm} = \kappa_4 - 9\delta_{lm}w^4$. It follows from (2.27) and (4.4) that in (4.47) we can replace $\kappa_{4,lm}$ with κ_4, which gives error terms of the order $O(n^{-1/2})$, $n\to\infty$. Hence,

$$\begin{aligned} T_3^{(n)} &= \frac{\kappa_4}{6n^2}\sum_{l,m=1}^{n}\mathbf{E}\{\xi_n^{A\circ}\cdot D_{lm}^3(U*A^{(n)}U)_{ml} + 3D_{lm}\xi_n^A\cdot D_{lm}^2(U*A^{(n)}U)_{ml} \\ &\quad + 3D_{lm}^2\xi_n^A\cdot D_{lm}(U*A^{(n)}U)_{ml} + (U*A^{(n)}U)_{ml}\cdot D_{lm}^3\xi_n^A\} \\ &=: \kappa_4[T_{31}^{(n)} + T_{32}^{(n)} + T_{33}^{(n)} + T_{34}^{(n)}] + O(n^{-1/2}), \quad n\to\infty. \end{aligned} \tag{4.47}$$

Treating $T_{31}^{(n)}$ similarly to $T_{21}^{(n)}$ of (4.30) (see (4.36)–(4.38)), one can get

$$T_{31}^{(n)} = O(n^{-1/2}), \quad n\to\infty. \tag{4.48}$$

Besides, it can be shown with the help of (2.26), (2.27) and (4.4) that all terms containing off-diagonal entries U_{lm} or $(UA^{(n)}U)_{lm}$ vanish in the limit $n\to\infty$. Thus,

$$T_{32}^{(n)} + T_{34} = O(n^{-1/2}),$$

and we are left with

$$\begin{aligned} T_3^{(n)} = &-\frac{i\kappa_4}{n^2}\sum_{l,m=1}^{n}\mathbf{E}\{\big(U_{ll}*(U*A^{(n)}U)_{mm}\big)(t_1) \\ &\times\big(U_{ll}*(U*C^{(n)}U)_{mm} + U_{mm}*(U*C^{(n)}U)_{ll}\big)(t_2)\big)\} + O(n^{-1/2}), \end{aligned}$$

as $n \to \infty$. Now it follows from (6.11) that

$$\lim_{n\to\infty} T_3^{(n)} = -2i\kappa_4 \big[K_A^{(3)}(v*v*v)(t_1)(v*v*v)(t_2) + 2T_A^2(v*tv)(t_1)(v*tv)(t_2) \big].$$

This and (4.44), (4.45) yield (4.28) after some calculations.

It remains to show (4.29). It is much simpler to do because in this case we have additional factors $n^{-1/2}$ (see (4.25)), so that treating T_j, $j = 4, 5, 6$ similarly to T_j, $j = 2, 3$, one can easily get (4.29). This completes the proof of the lemma. ■

5. Limiting Probability Law of Fluctuations of $\xi_n^A[\varphi]$

Theorem 5.1. *Consider the real symmetric Wigner random matrix of the form*

$$M^{(n)} = n^{-1/2} W^{(n)}, \quad W^{(n)} = \{W_{jk} \in \mathbb{R},\ W_{jk} = W_{kj} = (1+\delta_{jk})^{1/2} V_{jk}\}_{j,k=1}^n, \tag{5.1}$$

where $\{V_{jk}\}_{1\le j\le k<\infty}$ are i.i.d. random variables such that

$$\mathbf{E}\{V_{11}\} = 0, \quad \mathbf{E}\{V_{11}^2\} = w^2,$$

and the functions $\ln \mathbf{E}\{e^{itV_{11}}\}$ and $\mathbf{E}\{e^{it|V_{11}|}\}$ are entire.

Let $\{A^{(n)}\}_{n=1}^\infty$ satisfy (1.2), (1.3), *$C^{(n)} = A^{(n)} + A^{(n)T}$, and there exist*

$$A_p = \lim_{n\to\infty} n^{-p/2} \Big(\sum_{l,m=1}^n (C_{lm}^{(n)})^p + (2^{(2-p)/2} - 1) \sum_{m=1}^n (C_{mm}^{(n)})^p \Big)/2, \quad p \ge 3. \tag{5.2}$$

Then for any $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying (4.2), *the random variable $\xi_n^{A\circ}[\varphi]$ converges in distribution as $n \to \infty$ to the random variable $\xi^A[\varphi]$ such that*

$$\ln \mathbf{E}\{e^{ix\xi^A[\varphi]}\} = -x^2 V_W[\varphi]/2 + \sum_{p=3}^\infty \frac{\kappa_p A_p}{p!}(ix^*)^p, \tag{5.3}$$

$$x^* = \frac{x}{w^2} \int_{-2w}^{2w} \varphi(\mu)\mu\rho_{sc}(\mu)d\mu, \tag{5.4}$$

where ρ_{sc} is the density of the semicircle law (2.31), *and $V_W[\varphi]$ is given by* (4.11).

R e m a r k 5.2. It can be shown that in the case of the matrix $\widetilde{M}^{(n)} = n^{-1/2}\{V_{jk}\}_{j,k=1}^n$, Theorem 5.1 holds true with

$$\ln \mathbf{E}\{e^{ix\xi^A[\varphi]}\} = -V_{\widetilde{W}}^1[\varphi]x^2/2 + \sum_{p=3}^\infty \frac{\kappa_p \widetilde{A}_p}{p!}(ix^*)^p,$$

where $V^1_{\widetilde{W}}[\varphi]$ is given by (4.16) with $w_2 = 1$, and

$$\widetilde{A}_p = \lim_{n\to\infty} n^{-p/2}\Big(\sum_{l,m=1}^{n}(C_{lm}^{(n)})^p + (2^{(-2p+1)/2} - 1)\sum_{m=1}^{n}(C_{mm}^{(n)})^p\Big)/2.$$

R e m a r k 5.3. For matrix elements (1.5), $A_p = 2^{p/2}$, and we obtain Theorem 3.4 of [14]. In the case of bilinear forms (1.6), (1.7), we have for A_p of (5.2):

$$A_p = \lim_{n\to\infty}\Big(\big(\sum_{l=1}^{n}(\eta_l^{(n)})^p\big)^2 + (2^{(2-p)/2} - 1)\sum_{l=1}^{n}(\eta_l^{(n)})^{2p}\Big), \quad p \geq 3.$$

In particular, if for all $m = 1, \ldots, n$, $\eta_m^{(n)} = O(n^{-1/2})$, $n \to \infty$, then $A_p = 0$, $p \geq 3$, and the random variable $(\varphi(M^{(n)})^\circ\eta^{(n)}, \eta^{(n)})$ converges in distribution to the Gaussian random variable with zero mean and the variance $V_{GOE}^{(M\eta,\eta)}[\varphi]$ of (3.18).

R e m a r k 5.4. It follows from Theorem 5.1 that if φ is even, then the random variable $\xi_n^{A\circ}[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance $V_{GOE}[\varphi] + C_{\kappa 4}[\varphi, \varphi]$ (see (4.9)–(4.11)).

P r o o f of Theorem 5.1 We prove the theorem for a class of test functions satisfying the condition

$$\int |F[\varphi](t)||t|^l dt < C_\varphi l! \quad \forall l \in \mathbb{N}, \tag{5.5}$$

where $F[\varphi]$ is given by (3.5) and C_φ is an absolute constant. Then the theorem can be extended to the class of the functions satisfying (4.2) by using a standard approximation procedure (see, e.g., [17], Section 3.2). Consider the characteristic functions

$$Z_n^A(x) = \mathbf{E}\left\{e^{ix(\xi_n^A[\varphi])^\circ}\right\}, \quad \widehat{Z}_n^A(x) = \mathbf{E}\left\{e^{ix(\widehat{\xi}_n^A[\varphi])^\circ}\right\},$$

where $\widehat{\xi}_n^A[\varphi]$ corresponds to the GOE matrix $\widehat{M}^{(n)} = n^{-1/2}\widehat{W}^{(n)}$ (2.3). In view of Theorem 3.3, (4.11), and (5.3), it suffices to prove that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \ln Z_n^A(x)/\widehat{Z}_n^A(x) = -(C_{\kappa 3}[\varphi,\varphi] + C_{\kappa 4}[\varphi,\varphi])x^2/2 + \sum_{p=3}^{\infty}\frac{\kappa_p A_p}{p!}(ix^*)^p. \tag{5.6}$$

Following the idea of the proof of Theorem 3.2, we introduce the "interpolating" random matrix $M^{(n)}(s)$ of (4.18), put

$$Z_n^A(x,s) = \mathbf{E}\left\{e_n(x,s)\right\}, \quad e_n(x,s) = \exp\{ix(\xi_n^{A,s}[\varphi])^\circ\}, \tag{5.7}$$

$$\xi_n^{A,s}[\varphi] = \operatorname{Tr}\varphi(M^{(n)}(s))A^{(n)}, \quad \xi_n^{A,s}(t) = \operatorname{Tr}U(t,s)A^{(n)}, \quad U(t,s) = e^{itM(s)}, \tag{5.8}$$

and write

$$\ln Z_n^A(x)/\widehat{Z}_n^A(x) = -\frac{x}{2}\int_0^1 \frac{ds}{Z_n^A(x,s)} \int \Big(\frac{1}{\sqrt{ns}} \sum_{l,m=1}^n \mathbf{E}\Big\{ W_{lm}^{(n)} \Psi_{lm} \Big\}$$

$$- \frac{1}{\sqrt{n(1-s)}} \sum_{l,m=1}^n \mathbf{E}\Big\{ \widehat{W}_{lm} \Psi_{lm} \Big\} \Big) F[\varphi](t) dt, \tag{5.9}$$

$$\Psi_{lm} = \Psi_{lm}(t,x,s) = (U * A^{(n)} U)_{ml}(t,s) e_n^{\circ}(x,s) \tag{5.10}$$

(cf (4.20)–(4.22)). Let us note that unlike the functions Φ_{lm} of (4.22), having all the derivatives $D_{lm}^p \Phi_{lm}$ of the order $O(n^{3/2})$ (see (4.23)), here we have $D_{lm}^p \Psi_{lm} = O(n^{(p+1)/2})$, and there is no such finite $p \in \mathbb{N}$ that ε_p of (2.8) vanishes as $n \to \infty$. Hence, instead of (2.8), used while treating (4.20), here for every term of the first sum of the r.h.s. of (5.9) we apply the infinite version of (2.8) given by (2.10). To do this, we check first that $\Psi_{lm}(x,t)$ satisfies condition (2.11). Using the Leibnitz rule, we obtain

$$D_{lm}^p \Psi_{lm}(x,t,s) = \sum_{q=0}^p \binom{p}{q} D_{lm}^{p-q} (U * A^{(n)} U)_{ml}(t,s) D_{lm}^q e_n^{\circ}(x,s), \tag{5.11}$$

where

$$D_{lm}^q e_n(x,s) = ix D_{lm}^{q-1} \big(e_n(x,s) D_{lm} \xi_n^{A,s}[\varphi] \big) \tag{5.12}$$

(see (5.7)), so that

$$D_{lm}^q e_n(x,s) = e_n(x,s) \sum_{r=1}^q (ix)^r \sum_{\substack{\overline{q} = (q_1,\dots,q_r): \\ q_1+\ldots+q_r = q}} C_{\overline{q},r} \prod_{t=1}^r D_{lm}^{q_t} \xi_n^{A,s}[\varphi], \ \sum_{\overline{q},r} C_{\overline{q},r} \le 2^q.$$

Hence,

$$|D_{lm}^q e_n(x,s)| \le \big(2(1+|x|)\big)^q \max_{1\le r\le q,\ \sum_{t=1}^r q_t = q} \prod_{t=1}^r |D_{lm}^{q_t} \xi_n^{A,s}[\varphi]|,$$

where

$$D_{lm}^q \xi_n^{A,s}[\varphi] = \int F[\varphi](\theta) D_{lm}^q \xi_n^{A,s}(\theta) d\theta \tag{5.13}$$

with $\xi_n^{A,s}$ of (5.8), and in view of (2.29) and (5.5),

$$|D_{lm}^q \xi_n^{A,s}[\varphi]| \le \int |F[\varphi](\theta)| |D_{lm}^q \xi_n^{A,s}(\theta)| d\theta \le C_A C_\varphi 2^{q+1}, \tag{5.14}$$

so that

$$|D^q_{lm} e_n(x,s)| \le \big(c\sqrt{n}(1+|x|)\big)^q. \tag{5.15}$$

Here and in what follows, c depends only on $A^{(n)}$ and φ. This, (2.26), and (5.11) yield

$$|D^p_{lm}\Psi_{lm}(x,t,s)| \le (c\sqrt{n}(1+|x|+t))^{p+1}, \quad x\in\mathbb{R},\ t>0. \tag{5.16}$$

Thus, Ψ_{lm} satisfies (2.11) for every $x\in\mathbb{R}$, $t>0$. Besides, since $\ln \mathbf{E}\{e^{itV_{11}}\}$ is entire, then we have

$$\sum_{p=1}^{\infty} \frac{x^p|\kappa_{p+1}|}{p!} < \infty, \quad \forall x>0, \tag{5.17}$$

where κ_p is the pth cumulant of V_{11}. This implies (2.12), $\forall x\in\mathbb{R}$, $t>0$. Applying differentiation formula (2.10) with $\zeta = W^{(n)}_{lm} = \beta^{-1/2}_{lm}V_{lm}$ and $\Phi=\Psi_{lm}$ to every term of the first sum and differentiation formula (2.4) to every term of the second sum in the r.h.s. of (5.9), we get

$$\ln Z^A_n(x)/\widehat{Z}^A_n(x) = -\frac{x}{2}\int_0^1 \frac{ds}{Z^A_n(x,s)}\int\sum_{p=2}^{\infty} s^{(p-1)/2}\frac{\kappa_{p+1}}{p!}S^{(n)}_p(x,t,s)F[\varphi](t)dt, \tag{5.18}$$

$$S^{(n)}_p(x,t,s) = \frac{1}{n^{(p+1)/2}}\sum_{l,m=1}^{n}\beta^{-(p+1)/2}_{lm}\mathbf{E}\{D^p_{lm}\Psi_{lm}(x,t,s)\}. \tag{5.19}$$

It was shown in [14] that in the case of matrix elements (1.5), the series in (5.18) converges uniformly in $n\in\mathbb{N}$, $(t,x)\in K$ for any compact set $K\subset\{(x,t)\in\mathbb{R}^2 : t>0\}$. For the general case, the proof is almost the same with the obvious modifications. It is based on (5.17), the estimate

$$A_p \le 2^{p/2}, \quad \forall p\in\mathbb{N}, \tag{5.20}$$

following from (5.2) and (1.2), and on the uniform bound

$$|S^{(n)}_p(x,t,s)| \le (C_K)^l, \quad \forall (t,x)\in K,\ n\in\mathbb{N}, \quad s\in[0,1], \tag{5.21}$$

which can be obtained with the help of (2.19)–(2.29). Here C_K is an absolute constant depending only on K. In view of the uniform convergence of the series, to perform the limiting transition as $n\to\infty$ in (5.18), it suffices to find the limits

$S_p = \lim_{n\to\infty} S_p^{(n)}$ for every fixed $p \in \mathbb{N}$. We have

$$S_p^{(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{-(p+1)/2} \mathbf{E}\Big\{ (U*AU)_{lm} D_{lm}^{p} e_n^{\circ} + pD_{lm}(U*AU)_{lm} D_{lm}^{p-1} e_n^{\circ}$$
$$+ D_{lm}^{2}(U*AU)_{lm} D_{lm}^{p-2} e_n^{\circ} \cdot p(p-1)/2 + D_{lm}^{3}(U*AU)_{lm} D_{lm}^{p-3} e_n^{\circ}$$
$$\times p(p-1)(p-2)/6 + \delta_{1,2,3} \sum_{q=0}^{p-4} \binom{p}{q} D_{lm}^{p-q}(U*A^{(n)}U)_{ml} D_{lm}^{q} e_n^{\circ} \Big\}$$
$$= S_{p1}^{(n)} + S_{p2}^{(n)} + S_{p3}^{(n)} \cdot p(p-1)/2 + S_{p4}^{(n)} \cdot p(p-1)(p-2)/6 + \delta_{1,2,3} \cdot S_{p5}^{(n)}, \tag{5.22}$$

where $\delta_{1,2,3} = 0$, if $p = 1,2,3$. It follows from (2.26)–(2.27) and (5.15) that $S_{p5}^{(n)} = O(n^{-1/2})$, $n \to \infty$. Since

$$D_{lm}^{q} e_n(x,s) = e_n(x,s)\big(ixD_{lm}\xi_n^{A,s}[\varphi]\big)^q + O(n^{(q-1)/2})$$
$$= e_n(x,s)\Big(-x\beta_{lm}\int \widehat{\varphi}(\theta)(U*C^{(n)}U)_{lm}(\theta)d\theta\Big)^q + O(n^{(q-1)/2}), \quad n\to\infty, \tag{5.23}$$

then

$$S_{p4}^{(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{-(p+1)/2} \mathbf{E}\big\{ D_{lm}^{3}(U*A^{(n)}U)_{lm}$$
$$\times \Big(-x\beta_{lm}\int \widehat{\varphi}(\theta)(U*C^{(n)}U)_{lm}(\theta)d\theta\Big)^{p-3} e_n\big\} + O(n^{-1/2}), \quad p>3,$$

and by (2.26), (2.27) $S_{p4}^{(n)} = O(n^{-1/2})$, $n\to\infty$, $p>3$. If $p=3$, then

$$S_{34}^{(n)} = \frac{1}{n^2} \sum_{l,m=1}^{n} \beta_{lm}^{-2} \mathbf{E}\big\{ D_{lm}^{3}(U*A^{(n)}U)_{lm} e_n^{\circ}(x,s)\big\}$$

(compare with T_{31}^{n} of (4.47)), and in addition to (2.26), (2.27), we use (6.10) to show that $\mathbf{Var}\big\{n^{-2}\sum_{l,m=1}^{n} D_{lm}^{3}(U*A^{(n)}U)_{lm} e\big\} = O(n^{-1})$, and so $S_{34}^{(n)} = O(n^{-1/2})$, $n\to\infty$. Thus,

$$S_{p4}^{(n)} = O(n^{-1/2}), \quad n\to\infty, \quad p\ge 3. \tag{5.24}$$

Consider now $S_{p3}^{(n)}$ of (5.22). Applying (5.23), we can write

$$S_{p3}^{(n)} = \frac{1}{n^{(p+1)/2}}$$
$$\times \sum_{l,m=1}^{n} \beta_{lm}^{(p-5)/2} \mathbf{E}\{D_{lm}^2 (U * A^{(n)} U)_{lm} \big(-x \int \widehat{\varphi}(\theta)(U * C^{(n)} U)_{lm}(\theta) d\theta \big)^{p-2} e_n\}$$
$$+O(n^{-1/2}), \quad n \to \infty, \quad p > 2.$$

There arise the sums of three types:

$$S_{p3}^{1(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{(p-1)/2} U_{ll} U_{mm} (UA^{(n)}U)_{lm} (UC^{(n)}U)_{lm}^{p-2},$$
$$S_{p3}^{2(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{(p-1)/2} U_{lm} U_{lm} (UA^{(n)}U)_{lm} (UC^{(n)}U)_{lm}^{p-2},$$
$$S_{p3}^{3(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{(p-1)/2} U_{ll} U_{lm} (UA^{(n)}U)_{mm} (UC^{(n)}U)_{lm}^{p-2},$$

where we omit the arguments of U and put $(UC^{(n)}U)_{lm}^q = \prod_{j=1}^q (U(t_{j_1}) C^{(n)} U(t_{j_2}))_{lm}$. If $p = 2$, then treating $S_{23}^{(n)}$ similarly to $T_{21}^{(n)}$ of (4.30) (see (4.30)–(4.38)), we get $S_{23}^{(n)} = O(n^{-1/2})$, $n \to \infty$. In the case $p > 2$, we use the asymptotic relations following from (2.25)–(2.27):

$$\sum_{l,m=1}^{n} |U_{lm}||(UA^{(n)}U)_{lm}| = O(n), \tag{5.25}$$
$$\sum_{l,m=1}^{n} |(UA^{(n)}U)_{lm}||(UA^{(n)}U)_{lm}| = O(n), \tag{5.26}$$

$$\sum_{l,m=1}^{n} |(UA^{(n)}U)_{mm}||(UA^{(n)}U)_{lm}| = O(n\sqrt{n}), \tag{5.27}$$
$$\sum_{m=1}^{n} |(UA^{(n)}U)_{mm}||(UA^{(n)}U)_{mm}| = O(n) \tag{5.28}$$

as $n \to \infty$. They, together with (2.26), allow to show that $S_{p3}^{j(n)}$, $j = 1, 2, 3$ are of the order $O(n^{-1/2})$, $n \to \infty$, so that

$$S_{p3}^{(n)} = O(n^{-1/2}), \quad n \to \infty. \tag{5.29}$$

Consider $S_{p2}^{(n)}$ of (5.22). Applying (2.23) and (5.12), we can write

$$S_{p2}^{(n)} = -\frac{2px}{n^{(p+1)/2}}$$
$$\times \sum_{l,m=1}^{n} \beta_{lm}^{-(p-1)/2} \mathbf{E}\{\big(U_{ll} * (U * A^{(n)}U)_{mm} + U_{lm} * (U * A^{(n)}U)_{lm}\big)(t,s)$$
$$\times D_{lm}^{p-2}\big(e_n(x,s) D_{lm}\xi_n^{A,s}[\varphi]\big)\} = S_{p2}^{1(n)} + S_{p2}^{2(n)}. \tag{5.30}$$

Since

$$D_{lm}^{q}\big(e_n(x,s) D_{lm}\xi_n^{A,s}[\varphi]\big) = D_{lm}^{q} e_n(x,s) \cdot D_{lm}\xi_n^{A,s}[\varphi]$$
$$+ q D_{lm}^{q-1} e_n(x,s) \cdot D_{lm}^{2}\xi_n^{A,s}[\varphi] + O(n^{(q-1)/2}), \quad n \to \infty, \tag{5.31}$$

where

$$D_{lm}\xi_n^{A,s}[\varphi] = i\beta_{lm} \int (U * C^{(n)}U)_{lm}(\theta,s) F[\varphi](\theta) d\theta, \tag{5.32}$$
$$D_{lm}^{2}\xi_n^{A,s}[\varphi] = -\beta_{lm}^{2} \int \big(U_{ll} * (U * C^{(n)}U)_{mm} + U_{mm} * (U * C^{(n)}U)_{ll}$$
$$+ 2U_{lm} * (U * C^{(n)}U)_{lm}\big)(\theta,s) F[\varphi](\theta) d\theta,$$

then putting (5.31) with $q = p-2$ in $S_{p2}^{2(n)}$ of (5.30) and applying (5.15), (5.26), and (5.28), we get as $n \to \infty$

$$S_{p2}^{2(n)} = O(n^{-1/2}), \tag{5.33}$$

$$S_{p2}^{1(n)} = -2px \int F[\varphi](\theta) d\theta \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{-(p+1)/2} \mathbf{E}\{\big(U_{ll} * (U * A^{(n)}U)_{mm}\big)(t,s)$$
$$\times \big[D_{lm}^{p-2} e_n(x,s) \cdot i(U * C^{(n)}U)_{lm}(\theta,s)$$
$$- (p-2) D_{lm}^{p-3} e_n(x,s) \cdot \big(U_{ll} * (U * C^{(n)}U)_{mm}$$
$$+ U_{mm} * (U * C^{(n)}U)_{ll}\big)(\theta,s)\big]\} + O(n^{-1/2}).$$

It follows from (5.23) and (5.25)–(5.28) that $S_{p2}^{1(n)}$ does not vanish only if $p = 2$ or $p = 3$. Hence, putting $e_n(x,s) = Z_n^{A}(x,s) + e_n^{\circ}(x,s)$ and using (6.10), (6.12), and (5.33), we get

$$S_{p2}^{(n)} = xZ_n^A(x,s)$$
$$\times \int \Big[-\frac{4i\delta_{p2}}{n^{3/2}} \sum_{l,m=1}^{n} \mathbf{E}\{\big(U_{ll} * (U * A^{(n)}U)_{mm}\big)(t,s)(U * C^{(n)}U)_{lm}(\theta,s)\}$$
$$+\frac{6\delta_{p3}}{n^2} \sum_{l,m=1}^{n} \mathbf{E}\{\big(U_{ll} * (U * A^{(n)}U)_{mm}\big)(t,s)\Big(U_{ll} * (U * C^{(n)}U)_{mm}$$
$$+ U_{mm} * (U * C^{(n)}U)_{ll}\Big)(\theta,s)\}\Big] F[\varphi](\theta)d\theta + O(n^{-1/2}) \tag{5.34}$$

as $n \to \infty$. Such expressions were considered in the proof of Theorem 4.2 (see Lemma 4.6). Treating $S_{p2}^{(n)}$, $p = 2, 3$ in the same way and using (6.11), (6.13) and (4.43)–(4.45), we get

$$\lim_{n\to\infty} -\frac{x}{2} \int_0^1 \frac{ds}{Z_n^A(x,s)} \int \Big[\frac{\kappa_3\sqrt{s}}{2} S_{22}^{(n)}(x,t,s) + \frac{\kappa_4 s}{6} S_{32}^{(n)}(x,t,s)\Big] F[\varphi](t)dt$$
$$= -\Big(\frac{2}{3} C_{\kappa_3}[\varphi,\varphi] + C_{\kappa_4}[\varphi,\varphi]\Big)x^2/2 \tag{5.35}$$

with $C_{\kappa_3}[\varphi,\varphi]$, $C_{\kappa_4}[\varphi,\varphi]$ of (4.9), (4.10) (see also (5.6), (5.18)).

At last consider $S_{p1}^{(n)}$ of (5.22):

$$S_{p1}^{(n)} = \frac{ix}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t,s) D_{lm}^{p-1}\big(e_n(x,s) D_{lm}\xi_n^{A,s}[\varphi]\big)\}$$
$$= \frac{ix}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t,s) \big[D_{lm}^{p-1} e_n(x,s) \cdot D_{lm}\xi_n^{A,s}[\varphi]$$
$$+ (p-1) D_{lm}^{p-2} e_n(x,s) \cdot D_{lm}^2 \xi_n^{A,s}[\varphi]\big]\} + O(n^{-1/2}) = S_{p1}^{1(n)} + S_{p1}^{2(n)} + O(n^{-1/2}), \tag{5.36}$$

$n \to \infty$. Here we used (5.12), (5.31), and then (2.29), (5.15), and (5.25)–(5.28) to estimate the vanishing term. It follows from (5.23) and (5.26)–(5.28) that if $p > 2$, then

$$S_{p1}^{2(n)} = O(n^{-1/2}), \quad n \to \infty, \quad p > 2. \tag{5.37}$$

If $p = 2$, we have

$$S_{21}^{2(n)} = -ix \int \frac{1}{n^{3/2}} \sum_{l,m=1}^{n} \mathbf{E}\big\{e_n(x,s)(U * A^{(n)}U)_{lm}(t,s)\big(U_{ll} * (U * C^{(n)}U)_{mm}$$
$$+ U_{mm} * (U * C^{(n)}U)_{ll}\big)(\theta,s)\big\} F[\varphi](\theta)d\theta + O(n^{-1/2})$$

and, similarly to (5.35),

$$\lim_{n\to\infty} -\frac{x}{2}\int\limits_0^1 \frac{ds}{Z_n^A(x,s)}\int \frac{\kappa_3\sqrt{s}}{2} S_{21}^{2(n)}(x,t,s)F[\varphi](t)dt = -\Big(\frac{1}{3}C_{\kappa_3}[\varphi,\varphi]\Big)x^2/2. \tag{5.38}$$

Using (5.23) with $q = p-1$ and (5.32), for $S_{p1}^{1(n)}$ of (5.36) we get

$$\begin{aligned} S_{p1}^{1(n)} =& \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t,s)e_n(x,s) \\ &\times \big(-x\int \widehat{\varphi}(\theta)(U*C^{(n)}U)_{mm}(\theta)d\theta\big)^p\} + O(n^{-1/2}), \quad n\to\infty, \end{aligned} \tag{5.39}$$

where we estimate the vanishing term with the help of (5.26) and (5.28). Putting here

$$\beta_{lm}^{(p-1)/2} = 1 + \delta_{lm}(2^{(1-p)/2}-1)$$

and $e_n(x,s) = Z_n^A(x,s) + e_n^\circ(x,s)$, and then applying first parts of (6.14), (6.15), we get

$$\begin{aligned} S_{p1}^{1(n)} =& \frac{Z_n^A(x,s)}{n^{(p+1)/2}} \sum_{l,m=1}^n \mathbf{E}\{(U*A^{(n)}U)_{lm}(t,s)\big(-x\int \widehat{\varphi}(\theta)(U*C^{(n)}U)_{lm}(\theta)d\theta\big)^p\} \\ &+ 2(2^{(1-p)/2}-1)\frac{Z_n^A(x,s)}{n^{(p+1)/2}} \sum_{m=1}^n \mathbf{E}\{(U*A^{(n)}U)_{mm}(t,s) \\ &\times \big(-x\int \widehat{\varphi}(\theta)(U*C^{(n)}U)_{mm}(\theta)d\theta\big)^p\} + O(n^{-1/2}), \quad n\to\infty. \end{aligned}$$

This and second parts of (6.14), (6.15) yield for $p \geq 2$

$$\lim_{n\to\infty} -\frac{x}{2}\frac{\kappa_{p+1}}{p!}\int\limits_0^1 \frac{ds}{Z_n^A(x,s)}\int s^{(p-1)/2}S_{p1}^{1(n)}(x,t,s)F[\varphi](t)dt = \frac{\kappa_{p+1}A_{p+1}}{(p+1)!}(ix^*)^{p+1} \tag{5.40}$$

with A_p and x^* defined in (5.2) and (5.4). Now putting (5.35), (5.38), and (5.40) in (5.9), we get (5.6) and finish the proof of the theorem. ■

6. Auxiliary Results

Lemma 6.1. *Consider a matrix* $A^{(n)}$ *satisfying* (1.2), (1.3), $C^{(n)} = A^{(n)} + A^{(n)T}$, *and a unitary matrix* $U(t) = U^{(n)}(t) = e^{itM^{(n)}}$ *with the Wigner matrix*

$M^{(n)}$ of (2.1), (2.2). Denote $U^j = U(t_j)$, $\overline{t^{(p)}} = (t_1, \ldots, t_p)$, $\overline{f} = \mathbf{E}\{f\}$, and define

$$\eta_n^A(t_1, t_2) = n^{-3/2} \sum_{l,m=1}^{n} (U^1 A^{(n)} U^2)_{lm}, \tag{6.1}$$

$$v_n^I(t_1, t_2, t_3) = n^{-1} \sum_{m=1}^{n} U^1_{mm} (U^2 A^{(n)} U^3)_{mm}, \tag{6.2}$$

$$v_n^C(\overline{t^{(4)}}) = n^{-1} \sum_{m=1}^{n} (U^1 C^{(n)} U^2)_{mm} (U^3 A^{(n)} U^4)_{mm}, \tag{6.3}$$

$$\omega_n(\overline{t^{(5)}}) = n^{-3/2} \sum_{l,m=1}^{n} U^1_{ll} (U^2 A^{(n)} U^3)_{mm} (U^4 C^{(n)} U^5)_{lm}, \tag{6.4}$$

$$\gamma_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{l,m=1} (U^1 A^{(n)} U^2)_{lm} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm}, \tag{6.5}$$

$$\gamma_n^{(2)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{m=1} (U^1 A^{(n)} U^2)_{mm} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{mm}, \tag{6.6}$$

$p \geq 2$. Then under the conditions of Theorem 4.2, we have

$$(i)\ \mathbf{Var}\{\xi_n^A(t)\} \leq c(1 + |t|)^8, \qquad \lim_{n\to\infty} \overline{\xi}_n^A(t) = T_A \cdot v(t), \tag{6.7}$$

$$(ii)\ \mathbf{Var}\{\eta_n^A(t_1, t_2)\} = O(n^{-1}) \qquad \lim_{n\to\infty} \overline{\eta}_n^A(t_1, t_2) = K_A^{'(2)} \cdot v(t_1) v(t_2), \tag{6.8}$$

$$(iii)\ \mathbf{Var}\{v_n^I(t_1, t_2, t_3)\} = O(n^{-1}), \qquad \lim_{n\to\infty} \overline{v}_n^I(t_1, t_2, t_3) = T_A \cdot v(t_1) v(t_2 + t_3), \tag{6.9}$$

$$(iv)\ \mathbf{Var}\{v_n^C(\overline{t^{(4)}})\} = O(n^{-1}), \tag{6.10}$$

$$\lim_{n\to\infty} \overline{v}_n^C(\overline{t^{(4)}}) = 2K_A^{(3)} \prod_{j=1}^{4} v(t_j) + 2T_A^2 \cdot v(t_1 + t_2) v(t_3 + t_4), \tag{6.11}$$

$$(v)\ \mathbf{Var}\{\omega_n(\overline{t^{(5)}})\} = O(n^{-1/2}), \tag{6.12}$$

$$\lim_{n\to\infty} \overline{\omega}_n(\overline{t^{(5)}}) = (K_A^{(1)} - K_A^{(2)}) \prod_{j=1}^{5} v(t_j) + K_A^{(2)} v(t_1) v(t_4) v(t_5) v(t_2 + t_3), \tag{6.13}$$

$$(vi)\ \mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\} = O(n^{-1/2}), \qquad \lim_{n\to\infty} \overline{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = K_A^{(4)} \prod_{j=1}^{p+1} v(t_j), \tag{6.14}$$

$$(vii)\ \mathbf{Var}\{\gamma_n^{(2)}(\overline{t^{(2p+2)}})\} = O(n^{-1/2}),\ \lim_{n\to\infty} \overline{\gamma}_n^{(2)}(\overline{t^{(2p+2)}}) = K_A^{(5)} \prod_{j=1}^{p+1} v(t_j), \quad (6.15)$$

where $O(n^\alpha)$, $n \to \infty$, *can depend on* $\overline{t^{(p)}}$, v *and* $K_A^{(1,2,3)}$ *are defined in* (2.30), (4.5)–(4.7), *and*

$$K_A^{\prime(2)} = \lim_{n\to\infty} n^{-3/2} \sum_{l,m=1}^{n} A_{lm}^{(n)}, \quad (6.16)$$

$$K_A^{(4)} = \lim_{n\to\infty} n^{-(p+1)/2} \sum_{l,m=1}^{n} A_{lm}^{(n)} (C_{lm}^{(n)})^p, \quad K_A^{(5)} = \lim_{n\to\infty} n^{-(p+1)/2} \sum_{m=1}^{n} A_{mm}^{(n)} (C_{mm}^{(n)})^p. \quad (6.17)$$

R e m a r k 6.2. All the statements of the lemma remain valid under the conditions of Theorem 5.1.

P r o o f. 1. Firstly we prove the lemma supposing that the matrix $M^{(n)}$ belongs to the GOE.

(i) Statement (i) in the GOE case was proved in Lemma 2.3.

(ii) By Poincaré inequality (2.5), we have

$$\mathbf{Var}\{\eta_n^A(t_1,t_2)\} \le \frac{w^2}{n^4} \sum_{1\le j\le k\le n} \beta_{jk}^{-1} \mathbf{E}\{\big|D_{jk} \sum_{l,m=1}^{n} (U^1 A^{(n)} U^2)_{lm}\big|^2\}.$$

This and (2.22) show that it suffices to estimate

$$T_n = \frac{1}{n^4} \sum_{j,k=1}^{n} \big| \sum_{l,m=1}^{n} U_{lj}^1 (U^2 A^{(n)} U^3)_{km}\big|^2.$$

We have

$$\begin{aligned} T_n &= \frac{1}{n^4} \sum_{j,k=1}^{n} \sum_{l,l',m,m'=1}^{n} U_{lj}^1 \overline{U}_{jl'}^1 (\overline{U}^3 A^{(n)T} \overline{U^2})_{mk} (U^2 A^{(n)} U^3)_{km} \\ &= \frac{1}{n^3} \sum_{m,m'=1}^{n} (\overline{U}^3 A^{(n)T} A^{(n)} U^3)_{mm'} = \frac{1}{n^3} \Big| \sum_{m,p=1}^{n} (A^{(n)} U^3)_{pm} \Big|^2 \\ &\le \frac{1}{n^2} \mathrm{Tr}\, A A^{(n)T} = O(n^{-1}), \end{aligned}$$

and hence $\mathbf{Var}\{\eta_n^A(t_1,t_2)\} = O(n^{-1})$, $n \to \infty$. Now applying (2.13), (2.4), and (2.23), and then estimating the error terms with the help of (3.8), one can get as $n \to \infty$

$$\overline{\eta}_n^A(t_1,t_2) = n^{-3/2}\sum_{l,m=1}^{n}\mathbf{E}\{(A^{(n)}U(t_2))_{lm}\}$$
$$-w^2\int_0^{t_1}dt_3\int_0^{t_3}\overline{v}_n(t_3-t_4)\overline{\eta}_n^A(t_3,t_2)dt_4+o(1),$$

where by (3.1) and (2.30),

$$\lim_{n\to\infty} n^{-3/2}\sum_{l,m=1}^{n}\mathbf{E}\{(A^{(n)}U(t_2))_{lm}\} = \lim_{n\to\infty}\overline{v}_n(t_2)n^{-3/2}\sum_{l,m=1}^{n}A_{lm}^{(n)} = K_A^{'(2)}v(t_2)$$

with $K_A^{'(2)}$ of (6.16). Thus, for $\eta^A = \lim_{n\to\infty}\overline{\eta}_n^A$, we have

$$\eta^A(t_1,t_2)+w^2\int_0^{t_1}dt_3\int_0^{t_3}v(t_3-t_4)\eta^A(t_4,t_2)dt_4 = K_A^{'(2)}v(t_1),$$

and by (2.36),

$$\eta^A(t_1,t_2) = K_A^{'(2)}v(t_1)v(t_2). \tag{6.18}$$

(iii) Putting $U_{mm} = U_{mm}^{\circ}+\overline{U}_{mm}$ and using (3.1), we get

$$\overline{v}_n^I(t_1,t_2,t_3) = \overline{v}_n(t_1)\overline{\xi}_n^A(t_2+t_3)+\overline{r}_n, \quad r_n = n^{-1}\sum_{m=1}^{n}(U_{mm}^1)^{\circ}(U^2A^{(n)}U^3)_{mm}. \tag{6.19}$$

By the Schwarz inequality and (2.27), we have

$$\begin{aligned}|r_n| &\le \Big(\sum_{m=1}^{n}|(U^2A^{(n)}U^3)_{mm}|^2\Big)^{1/2}\Big(\sum_{m=1}^{n}|(U_{mm}^1)^{\circ}|^2\Big)^{1/2}\\ &= O(n^{-1/2})\Big(\sum_{m=1}^{n}|(U_{mm}^1)^{\circ}|^2\Big)^{1/2},\end{aligned} \tag{6.20}$$

where, as it follows from (3.8),

$$\mathbf{E}\Big\{\sum_{m=1}^{n}|(U_{mm}^1)^{\circ}|^2\Big\} = O(1), \quad n\to\infty. \tag{6.21}$$

This and the Schwarz inequality for the expectations yield

$$|\overline{r}_n| = O(n^{-1/2}), \quad n \to \infty. \tag{6.22}$$

Now (6.7), (6.19), and (6.22) give $\lim_{n\to\infty} \overline{v}_n^I(t_1, t_2, t_3) = T_A \cdot v(t_1)v(t_2 + t_3)$. We also have

$$\begin{aligned}\mathbf{Var}\{v_n^I\} &= \mathbf{E}\{n^{-1}\sum_{m=1}^n U^1_{mm}(U^2A^{(n)}U^3)_{mm} \cdot v_n^{I\circ}\} \\ &= \overline{v}_n(t_1)\mathbf{E}\{n^{-1}\xi_n^A(t_2+t_3)v_n^{I\circ}\} + \mathbf{E}\{r_n v_n^{I\circ}\}\end{aligned}$$

with r_n of (6.19). It follows from the Schwarz inequality, (3.8) and (6.20)–(6.22) that $\mathbf{Var}\{v_n^I\} \le O(n^{-1/2})\mathbf{Var}\{v_n^I\}^{1/2}$, $n \to \infty$, which finishes the proof of (6.9)

(iv) The proof of (6.10) repeats with the obvious modifications that one of (6.8). Let us prove (6.11). Applying Duhamel formula (2.13), differentiation formulas (2.4), (2.19)–(2.23), and then estimating the error terms with the help of (3.8), one can get as $n \to \infty$:

$$\begin{aligned}&n^{-1}\sum_{m=1}^n \mathbf{E}\{(U^1A^{(n)}U^2)_{mm}(U^3C^{(n)}U^4)_{mm}\} \\ &= n^{-1}\sum_{m=1}^n \mathbf{E}\{(A^{(n)}U^2)_{mm}(U^3C^{(n)}U^4)_{mm}\} \\ &- w^2\int_0^{t_1} dt_5 \int_0^{t_5} \mathbf{E}\Big\{v_n(t_5-t_6)n^{-1}\sum_{m=1}^n (U^6A^{(n)}U^2)_{mm}(U^3C^{(n)}U^4)_{mm}\Big\}dt_6 \\ &- w^2\int_0^{t_1} dt_5 \int_0^{t_5} \mathbf{E}\Big\{n^{-1}\xi_n^A(t_5+t_6)n^{-1}\sum_{m=1}^n U_{mm}(t_2-t_6)(U^3C^{(n)}U^4)_{mm}\Big\}dt_6 + o(1),\end{aligned} \tag{6.23}$$

$$\begin{aligned}n^{-1}\sum_{m=1}^n \mathbf{E}\{(A^{(n)}U^2)_{mm}(U^3C^{(n)}U^4)_{mm}\} &= n^{-1}\sum_{m=1}^n A^{(n)}_{mm}\mathbf{E}\{(U^3C^{(n)}U^4)_{mm}\} \\ - w^2\int_0^{t_2} dt_5 \int_0^{t_5} \mathbf{E}\Big\{v_n(t_5-t_6)n^{-1}&\sum_{m=1}^n (A^{(n)}U^6)_{mm}(U^3C^{(n)}U^4)_{mm}\Big\}dt_6 + o(1),\end{aligned} \tag{6.24}$$

$$n^{-1}\sum_{m=1}^n A^{(n)}_{mm}\mathbf{E}\{(U^3C^{(n)}U^4)_{mm}\} = n^{-1}\sum_{m=1}^n A^{(n)}_{mm}\mathbf{E}\{(C^{(n)}U^4)_{mm}\}$$

$$- w^2 \int_0^{t_3} dt_5 \int_0^{t_5} \mathbf{E}\Big\{ v_n(t_5 - t_6) n^{-1} \sum_{m=1}^{n} A_{mm}^{(n)} (U^6 C^{(n)} U^4)_{mm} \Big\} dt_6$$

$$- w^2 \int_0^{t_3} dt_5 \int_0^{t_4} \mathbf{E}\Big\{ n^{-1} \xi_n^C (t_5 + t_6) n^{-1} \sum_{m=1}^{n} A_{mm}^{(n)} U_{mm}(t_4 - t_6) \Big\} dt_6 + o(1), \quad (6.25)$$

where by (3.1),

$$\lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} A_{mm}^{(n)} \mathbf{E}\{(C^{(n)}U^4)_{mm}\} = 2D_A \cdot v(t_4), \quad D_A = \lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} (A_{mm}^{(n)})^2. \quad (6.26)$$

Denote

$$v^C(\overline{t^{(4)}}) = \lim_{n\to\infty} \overline{v}_n^C(\overline{t^{(4)}}), \quad H(t_3, t_4) = \lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} A_{mm}^{(n)} \mathbf{E}\{(U^3 C^{(n)} U^4)_{mm}\},$$

$$G(t_2, t_3, t_4) = \lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} \mathbf{E}\{(A^{(n)}U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\}.$$

It follows from (6.23)–(6.26), (6.7) and (6.9) that v^C, G, and H satisfy the integral equations:

$$v^C(\overline{t^{(4)}}) + w^2 \int_0^{t_1} dt_5 \int_0^{t_5} v(t_5 - t_6) v^C(t_6, t_2, t_3, t_4) dt_6$$

$$= G(t_2, t_3, t_4) - 2w^2 T_A^2 v(t_3 + t_4) \int_0^{t_1} dt_5 \int_0^{t_5} v(t_5 + t_6) v(t_2 - t_6) dt_6,$$

$$G(t_2, t_3, t_4) + w^2 \int_0^{t_2} dt_5 \int_0^{t_5} v(t_5 - t_6) G(t_6, t_3, t_4) dt_6 = H(t_3, t_4),$$

$$H(t_3, t_4) + w^2 \int_0^{t_3} dt_5 \int_0^{t_5} v(t_5 - t_6) H(t_6, t_4) dt_6$$

$$= 2D_A \cdot v(t_4) - 2w^2 T_A^2 \int_0^{t_3} dt_5 \int_0^{t_4} v(t_5 + t_6) v(t_4 - t_6) dt_6.$$

Solving the equations with the help of (2.36), we get

$$H(t_3, t_4) = 2K_A^{(3)} v(t_3) v(t_4) + 2T_A^2 v(t_3 + t_4), \quad G(t_2, t_3, t_4) = v(t_2) H(t_3, t_4),$$

$$v^C(\overline{t^{(4)}}) = v(t_1) G(t_2, t_3, t_4) + 2T_A^2 (v(t_1 + t_2) - v(t_1) v(t_2)) v(t_3 + t_4).$$

This leads to (6.11) and finishes the proof of (iv).

(v) Similarly to (6.19)–(6.22), we have

$$\overline{\omega}_n(\overline{t^{(5)}}) = \overline{v}_n(t_1)\overline{\Gamma}_n(t_2,t_3,t_4,t_5) + \overline{r}_n, \tag{6.27}$$

$$\Gamma_n(t_2,t_3,t_4,t_5) = n^{-3/2}\sum_{l,m=1}^{n}(U^2A^{(n)}U^3)_{mm}(U^4C^{(n)}U^5)_{lm}, \tag{6.28}$$

$$r_n(\overline{t^{(5)}}) = n^{-3/2}\sum_{l,m=1}^{n}(U^1_{ll})^\circ(U^2A^{(n)}U^3)_{mm}(U^4C^{(n)}U^5)_{lm},$$

where by (2.15), the Schwarz inequality, (2.27), and (6.21),

$$\begin{aligned}|\overline{r}_n| &\le n^{-3/2}||C^{(n)}|| \\ &\times\Big(\mathbf{E}\Big\{\sum_{l=1}^{n}|(U^1_{ll})^\circ|^2\Big\}\Big)^{1/2}\Big(\mathbf{E}\Big\{\sum_{m=1}^{n}|(U^2A^{(n)}U^3)_{mm}|^2\Big\}\Big)^{1/2} = O(n^{-1/2}),\end{aligned} \tag{6.29}$$

$n\to\infty$. Applying (2.13), (2.4), (2.19)–(2.23) and then estimating the error terms with the help of (3.8), one can get for Γ_n as $n\to\infty$ (cf (6.23)–(6.25)):

$$\begin{aligned}\overline{\Gamma}_n(t_2,t_3,t_4,t_5) = &\,\overline{B}_n(t_3,t_4,t_5) \\ &- w^2\int_0^{t_2}dt_6\int_0^{t_6}\mathbf{E}\big\{v_n(t_6-t_7)\Gamma_n(t_7,t_3,t_4,t_5)\big\}dt_7 \\ &- w^2\int_0^{t_2}dt_6\int_0^{t_3}\mathbf{E}\big\{n^{-1}\xi_n^A(t_6+t_7)D_n(t_3-t_7,t_4,t_5)\big\}dt_7 + o(1),\end{aligned}$$

$$B_n(t_3,t_4,t_5) = n^{-3/2}\sum_{l,m=1}^{n}(A^{(n)}U^3)_{mm}(U^2C^{(n)}U^3)_{lm},$$

$$D_n(\tau,t_4,t_5) = n^{-3/2}\sum_{l,m=1}^{n}U_{mm}(\tau)(U^2C^{(n)}U^3)_{lm}.$$

Similarly to (6.27)–(6.29), it can be shown that $\overline{D}_n(\tau,t_4,t_5) = \overline{v}_n(\tau)\overline{\eta}_n^C(t_4,t_5) + O(n^{-1/2})$, $n\to\infty$, where η_n^C is defined in (6.1), so that by (6.8),

$$\lim_{n\to\infty}\overline{D}_n(\tau,t_4,t_5) = 2K_A^{'(2)}v(\tau)v(t_4)v(t_5). \tag{6.30}$$

For B_n, we also have

$$\overline{B}_n(t_3, t_4, t_5) = \overline{v}_n(t_3) n^{-3/2} \sum_{l,m=1}^{n} \mathbf{E}\Big\{ A^{(n)}_{mm} (U^4 C^{(n)} U^5)_{lm} \Big\} + R_n, \tag{6.31}$$

$$|R_n| \le n^{-1} ||A^{(n)}|| \Big(\mathbf{E}\Big\{ \sum_{m=1}^{n} |(AU^3)^{\circ}_{mm}|^2 \Big\} \Big)^{1/2}.$$

By the standard argument based on Poincaré inequality (2.5), one can easily get

$$\mathbf{Var}\{(AU)_{mm}(t)\} \le C|t|^2 n^{-1} (AA^{(n)T})_{mm},$$

and hence

$$R_n = O(n^{-1/2}), \quad n \to \infty. \tag{6.32}$$

Besides, repeating with the obvious modifications the steps leading to (6.18), we get

$$\lim_{n\to\infty} n^{-3/2} \sum_{l,m=1} \mathbf{E}\Big\{ A^{(n)}_{mm} (U^4 C^{(n)} U^5)_{lm} \Big\} = K^{(1)}_A v(t_4) v(t_5).$$

This and (6.31), (6.32) yield

$$B(t_3, t_4, t_5) := \lim_{n\to\infty} \overline{B}_n(t_3, t_4, t_5) = K^{(1)}_A v(t_3) v(t_4) v(t_5). \tag{6.33}$$

Plugging (6.30), (6.33) in (6.28), we get the equation with respect to $\Gamma = \lim_{n\to\infty} \overline{\Gamma}_n$,

$$\Gamma(t_2, t_3, t_4, t_5) + w^2 \int_0^{t_2} dt_6 \int_0^{t_6} v(t_6 - t_7) \Gamma(t_7, t_3, t_4, t_5) dt_7$$

$$= B(t_3, t_4, t_5) - w^2 K^{(2)}_A v(t_4) v(t_5) \int_0^{t_2} dt_6 \int_0^{t_3} v(t_6 + t_7) v(t_3 - t_7) dt_7,$$

where we put $K^{(2)}_A = 2T_A K^{'(2)}_A$ (see (4.6), (6.16)). Hence, by (2.36),

$$\Gamma(t_2, t_3, t_4, t_5) = v(t_2) B(t_3, t_4, t_5) + K^{(2)}_A (v(t_2 + t_3) - v(t_2) v(t_3)) v(t_4) v(t_5)$$

$$= (K^{(1)}_A - K^{(2)}_A) \prod_{j=1}^{5} v(t_j) + K^{(2)}_A v(t_2 + t_3) v(t_4) v(t_5).$$

This and (6.27)–(6.29) lead to (6.13).

(vi) It follows from Poincaré inequality (2.5) that

$$\mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\}$$
$$\leq \frac{w^2}{n^{(p+2)}}\sum_{1\leq j\leq k\leq n}\beta_{jk}^{-1}\mathbf{E}\Big\{\Big|D_{jk}\sum_{l,m=1}^{n}(U^1A^{(n)}U^2)_{lm}\prod_{j=2}^{p+1}(U^{2j-1}C^{(n)}U^{2j})_{lm}\Big|^2\Big\}.$$

Taking into account (2.22), it is seen that to get the first part of (6.14), it suffices to show that

$$R_n := \frac{1}{n^{(p+2)}}\sum_{j,k=1}^{n}\Big|\sum_{l,m=1}^{n}U_{jl}^0(U^1A^{(n)}U^2)_{km}\prod_{j=2}^{p+1}(U^{2j-1}C^{(n)}U^{2j})_{lm}\Big|^2 = O(n^{-1})$$

as $n\to\infty$ (here $U^0 = U(t_0)$). Since by (2.15),

$$\sum_{j=1}^{n}U_{jl}\overline{U_{jl'}} = \delta_{ll'},$$
$$\sum_{k=1}^{n}(U^2A^{(n)T}U^1)_{mk}\overline{(U^1A^{(n)}U^2)_{km'}} = \sum_{k=1}^{n}(U^2A^{(n)T})_{mk}\overline{(A^{(n)}U^2)_{km'}},$$

then

$$R_n = \frac{1}{n^{(p+2)}}\sum_{k,l=1}^{n}\Big|\sum_{m=1}^{n}(A^{(n)}U^2)_{km}\prod_{j=2}^{p+1}(U^{2j-1}C^{(n)}U^{2j})_{lm}\Big|^2$$
$$\leq \frac{1}{n^{(p+2)}}\sum_{k,l=1}^{n}\sum_{m=1}^{n}|(A^{(n)}U^2)_{km}|^2\sum_{m'=1}^{n}\Big|\prod_{j=2}^{p+1}(U^{2j-1}C^{(n)}U^{2j})_{lm'}\Big|^2.$$

By (2.26), (2.27), we have

$$\sum_{l,m'=1}^{n}\Big|\prod_{j=2}^{p+1}(U^{2j-1}C^{(n)}U^{2j})_{lm'}\Big|^2$$
$$= O(n^{p-1})\sum_{l,m'=1}^{n}\Big|(U^3C^{(n)}U^4)_{lm'}\Big|^2 = O(n^p),\quad n\to\infty.$$

This and (2.27) yield $R_n = O(n^{-1})$, $n\to\infty$. Therefore,

$$\mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\} = O(n^{-1}),\quad n\to\infty. \tag{6.34}$$

To prove (6.14), we must show that every $U(t)$ in

$$\overline{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2}\sum_{l,m=1}^{n}\mathbf{E}\Big\{(U(t_1)A^{(n)}U(t_2))_{lm}\prod_{j=2}^{p+1}(U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm}\Big\} \tag{6.35}$$

can be replaced with $\overline{v}_n$ with the error term that vanishes as $n \to \infty$. For this purpose it suffices to show that

$$\overline{\gamma}_n^{(1)}(\overline{t^{(p)}}) = \overline{v}_n(t_1)\overline{\delta}_n(t_2,\dots,t_{2p+2}) + o(1), \quad n \to \infty, \tag{6.36}$$

$$\delta_n(t_2,\dots,t_{2p+2}) = n^{-(p+1)/2}\sum_{l,m=1}^{n}(A^{(n)}U(t_2))_{lm}\prod_{j=2}^{p+1}(U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm}. \tag{6.37}$$

Applying (2.13) and then (2.4), (2.19)–(2.23), we get

$$\overline{\gamma}_n^{(1)}(\overline{t^{(p)}}) = \overline{\delta}_n(t_2,\dots,t_{2p+2})$$

$$-w^2\int_0^{t_1}d\tau_1\int_0^{\tau_1}v(\tau_1-\tau_2)\overline{\gamma}_n^{(1)}(\tau_2,t_2,\dots,t_{2p+2})d\tau_2 - w^2\int_0^{t_1}R_n(\tau_1,t_2,\dots,t_{2p+2})d\tau_1,$$

$$R_n(\tau_1,t_2,\dots,t_{2p+2}) = \int_0^{\tau_1}\mathbf{E}\big\{v_n^\circ(\tau_2)\gamma_n^{(1)}(\tau_1-\tau_2,t_2,\dots,t_{2p+2})\big\}d\tau_2$$

$$+\int_0^{\tau_1}(\overline{v}_n(\tau_2)-v(\tau_2))\overline{\gamma}_n^{(1)}(\tau_1-\tau_2,t_2,\dots,t_{2p+2})d\tau_2$$

$$+n^{-1}\tau_1\overline{\gamma}_n^{(1)}(\tau_1,t_2,\dots,t_{2p+2}) + n^{-1}\int_0^{t_2}\overline{\gamma}_n^{(1)}(\tau_1+\tau_2,t_2-\tau_2,\dots,t_{2p+2})d\tau_2$$

$$+\int_0^{t_2}\mathbf{E}\Big\{n^{-1}\xi_n^A(\tau_1+\tau_2)\frac{1}{n^{(p+1)/2}}\sum_{l,m=1}^{n}U_{lm}(t_2-\tau_2)\prod_{j=2}^{p+1}(U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm}\Big\}d\tau_2$$

$$+\frac{1}{n}\cdot\frac{1}{n^{(p+1)/2}}\sum_{l,m,k=1}^{n}\beta_{lk}^{-1}\mathbf{E}\Big\{(U(\tau_1)A^{(n)}U(t_2))_{km}D_{lk}\prod_{j=2}^{p+1}(U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm}\Big\}. \tag{6.38}$$

It follows from (2.32), (2.33), (2.36) that

$$\overline{\gamma}_n^{(1)}(\overline{t^{(p)}}) = v(t_1)\overline{\delta}_n(t_2,\dots,t_{2p+2}) - w^2\int_0^{t_1}v(t_1-\tau_1)R_n(\tau_1,t_2,\dots,t_{2p+2})d\tau_1.$$

In fact, to get (6.36), it suffices to show that $R_n = o(1)$, $n \to \infty$. Indeed, the first four terms of the r.h.s. of (6.38) vanish because of (6.7), the fifth term is of the order $O(n^{-1/2})$, $n \to \infty$, because of (2.26), (2.27) and the boundedness of $n^{-1}\xi_n^A(\tau_1 + \tau_2)$. Besides, the last term after differentiation gives the terms of the form $n^{-1}\overline{\gamma}_n^{(1)}$ or

$$\frac{1}{n} \cdot \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^{n} \mathbf{E}\Big\{(UA^{(n)}UC^{(n)}U)_{lm} \prod_{j=2}^{p+1} (UC^{(n)}U)_{lm}\Big\},$$

which are of the order $O(n^{-1/2})$, $n \to \infty$ (see (2.26), (2.27)). Hence, $R_n = o(1)$, $n \to \infty$, and thus (6.36) is proved. It remains to note that (6.36) holds true for

$$\overline{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{l,m=1}^{n} \mathbf{E}\Big\{(V(t_1)A^{(n)}V(t_2))_{lm} \prod_{j=2}^{p+1} (V(t_{2j-1})C^{(n)}V(t_{2j}))_{lm}\Big\}$$

(cf (6.35)), where V is equal to U or identity matrix I_n. In the limit $n \to \infty$, we can replace all U of (6.35) with v to get (6.14). The proof of (vii) repeats essentially that one of (vi). This finishes the proof of the lemma for the case when the matrix $M^{(n)}$ belongs to the GOE.

2. Consider now the general case of the Wigner matrix $M^{(n)}$ satisfying the conditions of the lemma. For the general case, the proofs of all statements (i)–(vii) follow the same scheme based on the known facts for the GOE matrices and interpolation procedure proposed in the proofs of Theorems 4.2 and 5.1. We demonstrate this scheme by proving (i).

Consider $V_n(t) := \mathbf{Var}\{\xi_n^A(t)\}$ and note that

$$V_n(t) = \mathbf{Var}\{\widehat{\xi}_n^A(t)\} + C_n^{\Delta}(t, -t), \tag{6.39}$$

where $\widehat{\xi}_n^A$ and C_n^{Δ} are defined in (4.17) and (4.20), respectively. By (4.4), we have

$$\mathbf{Var}\{\widehat{\xi}_n^A(t)\} \le ct^2. \tag{6.40}$$

Repeating the steps leading from (4.20) to (4.24)–(4.26), but using here (2.8) with $p = 5$ instead of $p = 6$ in (4.24), we get

$$c_n^{\Delta}(t, -t) = \frac{i}{2} \int_0^1 \Big[\sum_{j=2}^{5} s^{(j-1)/2} T_j^{(n)} + \varepsilon_5\Big] ds \tag{6.41}$$

with $T_j^{(n)}$ of (4.25), and

$$|\varepsilon_5| \le \frac{C_5 w_8^{7/8}}{n^{7/2}} \sum_{l,m=1}^{n} \sup_{M \in \mathcal{S}_n} |D_{lm}^6 \Phi_{lm}| \le c(1 + |t|)^7. \tag{6.42}$$

Consider $T_1^{(n)}$. It is given by (4.30) with $t_1 = t$, $t_2 = -t$. Since $T_{21}^{j(n)}$, $j = 1, 2, 3$ of (4.31)–(4.36) are bounded uniformly in $n \in \mathbb{N}$, and every derivative D_{lm} of $U(t) = e^{itM^{(n)}}$ gives the factor t, then

$$\Big| n^{-3/2} \sum_{l,m=1}^{n} D_{lm}^2 (U * A^{(n)} U)_{lm} \Big| \le c(1 + |t|)^3.$$

By the Schwarz inequality, for $T_{21}^{(n)}$ of (4.30), we have

$$|T_{21}^{(n)}| = \Big| \mathbf{E}\Big\{ n^{-3/2} \sum_{l,m=1}^{n} D_{lm}^2 (U * A^{(n)} U)_{lm} \cdot \xi_n^{A\circ}(t) \Big\} \Big| \le c(1 + |t|)^3 V_n^{1/2}.$$

We also have for $T_{22}^{(n)}$ and $T_{23}^{(n)}$of (4.30) (see (4.40) and (2.27)):

$$|T_{22}^{(n)} + T_{23}^{(n)}| \le c(1 + |t|)^3.$$

Hence,

$$|T_2^{(n)}| \le c(1 + |t|)^3 (V_n^{1/2} + 1). \tag{6.43}$$

Treating $T_3^{(n)}$of (4.47) and $T_j^{(n)}$, $j = 4, 5$ of (4.25) in the similar way, one can get

$$|T_3^{(n)}| \le c(1 + |t|)^4 (V_n^{1/2} + 1), \tag{6.44}$$

$$|T_j^{(n)}| \le c(1 + |t|)^{j+1}, \quad j = 4, 5. \tag{6.45}$$

Putting (6.42)–(6.45) in (6.41), and then together with (6.40) in (6.39), we get the quadratic inequality with respect to $V_n^{1/2}$

$$V_n - c(1 + |t|)^4 V_n^{1/2} - c(1 + |t|)^7 \le 0,$$

solving which we get $V_n \le c(1 + |t|)^8$. To finish the proof of (i), it remains to show that

$$\lim_{n\to\infty} n^{-1} \mathbf{E}\{\xi_n^A(t)\} = T_A v(t). \tag{6.46}$$

Similarly to (6.41), (6.42), we have

$$n^{-1}\mathbf{E}\{\xi_n^A(t)\} - n^{-1}\mathbf{E}\{\widehat{\xi}_n^A(t)\} = \frac{i}{2} \int_0^1 \Big[s^{1/2} T_2'^{(n)} + \varepsilon_2 \Big] ds,$$

$$T_2'^{(n)} = \frac{\kappa_3}{j! n^{5/2}} \sum_{l,m=1}^{n} \mathbf{E}\big\{ D_{lm}^2 (U * A^{(n)} U)_{lm}(t, s) \big\} = O(n^{-1}), \quad n \to \infty,$$

$$|\varepsilon_3| \le \frac{C_3\sqrt{w_8}}{n^6}\sum_{l,m=1}^{n}\sup_{M\in\mathcal{S}_n}|D^4_{lm}(U*A^{(n)}U)_{lm}(t,s)| = O(n^{-1/2}), \quad n\to\infty.$$

Hence, $n^{-1}\mathbf{E}\{\xi^A_n(t)\} - n^{-1}\mathbf{E}\{\widehat{\xi}^A_n(t)\} = O(n^{-1/2})$, $n\to\infty$. This and (3.4) yield (6.46). ■

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