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On the Boundary Value Problem with the Operator in Boundary Conditions for the Operator-Differential Equation of Second Order with Discontinuous Coefficients

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Sufficient conditions for regular solvability of the boundary value problem for an elliptic operator-differential equation of second order considered on the positive semi-axis are obtained. Note that the principal part of the equation contains a discontinuous coefficient, and the boundary condition involves a linear operator.

Key words: Hilbert space, operator-differential equation, discontinuous coefficient, regular solvability, Fourier transformation, intermediate derivatives.

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Dedicated to the 80th birthday of Professor F.S. Rofe-Beketov

1. Introduction

Let A be a self-adjoint positive definite operator in a separable Hilbert space H. It is known that the domain of the operator A^θ $(\theta \geq 0)$ becomes a Hilbert space H_θ with respect to the scalar product $(x,y)_\theta = (A^\theta x, A^\theta y)$, $x, y \in Dom(A^\theta)$. For $\theta = 0$, we consider that $H_0 = H$.

We denote by $L_2((a,b);H)$,$-\infty \leq a < b \leq +\infty$, the Hilbert space of all vector functions defined on (a,b), with the values in H, which have the finite norm

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f(t)\|_H^2 \, dt \right)^{\frac{1}{2}}$$

(see [1]). Further, we denote by $L(X,Y)$ a set of the linear bounded operators acting from the Hilbert space X to another Hilbert space Y. For $Y = X$, we consider $L(X,X) = L(X)$. We also denote the spectrum of the operator $(\cdot)$ by $\sigma(\cdot)$.

We introduce the linear space

$$W_2^2((a,b);H) = \left\{ u(t) : A^2 u(t) \in L_2((a,b);H),\, u''(t) \in L_2((a,b);H) \right\}$$

with the norm

$$\|u\|_{W_2^2((a,b);H)} = \left(\left\|A^2 u\right\|^2_{L_2((a,b);H)} + \left\|u''\right\|^2_{L_2((a,b);H)} \right)^{\frac{1}{2}}.$$

The lineal becomes a Hilbert space [2]. Here and further the derivatives are understood in the sense of distribution theory. For $a = -\infty$, $b = +\infty$, we will assume that

$$L_2((-\infty,+\infty);H) \equiv L_2(R;H),\; W_2^2((-\infty,+\infty);H) \equiv W_2^2(R;H),\; R = (-\infty,+\infty).$$

For $a = 0$, $b = +\infty$, we will suppose that

$$L_2((0,+\infty);H) \equiv L_2(R_+;H),\; W_2^2((0,+\infty);H) \equiv W_2^2(R_+;H),\; R_+ = (0,+\infty).$$

Besides the spaces introduced, we will use the following subspaces:

$$\overset{o}{W}{}_2^2(R_+;H) = \left\{ u(t) : u(t) \in W_2^2(R_+;H),\, u(0) = u'(0) = 0 \right\},$$

$$\overset{o}{W}{}_{2,T}^2(R_+;H) = \left\{ u(t) : u(t) \in W_2^2(R_+;H),\, u(0) = Tu'(0),\, T \in L(H_{\frac{1}{2}}, H_{\frac{3}{2}}) \right\}.$$

Now we pass to the statement of the boundary value problem studied. We consider the operator differential equation of the form

$$-u''(t) + \rho(t)A^2u(t) + A_1u'(t) = f(t), \ \ t \in R_+, \tag{1}$$

satisfying the boundary conditions at zero

$$u(0) = Tu'(0), \tag{2}$$

where $f(t)$, $u(t)$ are the functions with values in H, $T \in L(H_{\frac{1}{2}}, H_{\frac{3}{2}})$, A_1 is a linear unbounded operator, A is a self-adjoint positive definite operator in H, $\rho(t) = \alpha$ if $t \in (0,1)$, and $\rho(t) = \beta$ if $t \in (1, +\infty)$, and α, β are positive unequal numbers. For definiteness, we suppose that $\alpha \leq \beta$.

Definition. *Problem* (1), (2) *is called regularly solvable if for every function* $f(t) \in L_2(R_+; H)$ *there exists a function* $u(t) \in W_2^2(R_+; H)$ *satisfying equation* (1) *almost everywhere in* R_+, *boundary condition* (2) *holds in the sense of convergence of the space* $H_{\frac{3}{2}}$, *i.e.,*

$$\lim_{t \to 0} \left\| u(t) - Tu'(t) \right\|_{H_{\frac{3}{2}}} = 0,$$

and the estimate

$$\|u\|_{W_2^2(R_+;H)} \leq \text{const} \, \|f\|_{L_2(R_+;H)}$$

takes place.

A review article of A.A. Shkalikov [3] contains a detailed analysis of the results of both the author himself and other authors obtained on the problems of solvability and spectral problems, mainly the boundary value problems for operator differential equations when the coefficients in the boundary conditions are only complex numbers. Among the results, we especially mark out the papers of M.G. Gasymov [4–6] and his followers. Note that these boundary value problems do not lose their actuality (see, for example, the recent papers of S.S. Mirzoev and M.Yu. Salimov [7], A.R. Aliev and S.S. Mirzoev [8], A.R. Aliev [9]). Nevertheless, there are rather few works on solvability and the studying of the spectrum, the completeness of root vectors and elementary solutions of boundary value problems for operator-differential equations when the coefficients of the boundary conditions are operators. The first works on this subject were the papers of F.S. Rofe-Beketov [10], V.A. Ilyin and A.F. Filippov [11], M.L.Gorbachuk [12], S.Y. Yakubov and B.A. Aliev [13]. Later in this direction there appeared an interesting paper of M.G. Gasymov and S.S. Mirzoev [14], in which both the problems of solvability and some spectral aspects of the boundary value problems for elliptic type operator differential equations of the second order considered on the semiaxis were studied by using original methods. This work found its proper development in the papers of S.S. Mirzoev and his followers (see [15–19]).

The present paper aims to obtain appropriate solvability results of the paper by M.G. Gasymov and S.S. Mirzoev [14] for the case when the principal part of the equation contains discontinuous (piecewise constant) coefficient. Such problems are of interest not only because they contain appropriate boundary value problems, in the boundary conditions in which the coefficients are complex numbers, but also because they can be applied to a wider range of the problems for partial differential equations and a number of problems in mechanics, in particular, non-standard problems in the theory of elasticity of multilayered bodies. For simplicity, a point of discontinuity is taken. Here we note that a regular solvability of the boundary value problems for operator differential equations of the second order with discontinuous coefficients, when the coefficients in the boundary condition are only complex numbers, is studied in paper [20] and developed in [21].

2. Main Results

We begin with considering the problem

$$-u''(t) + \rho(t)A^2u(t) = f(t), t \in R_+, \tag{3}$$

$$u(0) = Tu'(0), \tag{4}$$

where $f(t) \in L_2(R_+; H)$, $u(t) \in W_2^2(R_+; H)$.

As can be seen, equation (3) is obtained from (1) at $A_1 = 0$.

The following theorem is true.

Theorem 1. *Let the operator*

$$T_{\alpha,\beta} = E + \sqrt{\alpha}TA + \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\beta}}\left(\sqrt{\alpha}TA - E\right)e^{-2\sqrt{\alpha}A}$$

have a bounded inverse operator in $H_{\frac{3}{2}}$, *where* E *is an identity operator in* H *and* e^{-tA} *is a semigroup of bounded linear operators generated by the operator* $-A$. *Then the operator* P_0, *acting from the space* $\overset{o}{W}{}^2_{2,T}(R_+; H)$ *to the space* $L_2(R_+; H)$,

$$P_0u(t) \equiv -u''(t) + \rho(t)A^2u(t), \;\; u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H),$$

induces an isomorphism between these spaces.

P r o o f. We will show that the equation $P_0u(t) = 0$ has only the trivial solution in the space $\overset{o}{W}{}^2_{2,T}(R_+; H)$. Indeed, the general solution of the equation

$P_0u(t) = 0$ from the space $W_2^2(R_+; H)$ has the form

$$u_0(t) = \begin{cases} v_1(t) = e^{-\sqrt{\alpha}tA}\varphi_1 + e^{-\sqrt{\alpha}(1-t)A}\varphi_2 \,, & t \in (0,1), \\ v_2(t) = e^{-\sqrt{\beta}(t-1)A}\varphi_3 \,, & t \in (1, +\infty), \end{cases}$$

where the vectors φ_1, φ_2, $\varphi_3 \in H_{\frac{3}{2}}$. From the condition $u_0(t) \in \overset{o}{W}{}_{2,T}^2(R_+; H)$ we have

$$v_1(0) = Tv_1'(0), \;\; v_1(1) = v_2(1), \;\; v_1'(1) = v_2'(1).$$

Thus for the vectors φ_1, φ_2, φ_3 we get the system of equations

$$\begin{cases} \varphi_1 + e^{-\sqrt{\alpha}A}\varphi_2 = T\left(-\sqrt{\alpha}A\varphi_1 + \sqrt{\alpha}Ae^{-\sqrt{\alpha}A}\varphi_2\right) \,, \\ e^{-\sqrt{\alpha}A}\varphi_1 + \varphi_2 = \varphi_3, \\ -\sqrt{\alpha}Ae^{-\sqrt{\alpha}A}\varphi_1 + \sqrt{\alpha}A\varphi_2 = -\sqrt{\beta}A\varphi_3. \end{cases}$$

From this system, in turn, it implies that

$$\varphi_3 = \sqrt{\frac{\alpha}{\beta}}e^{-\sqrt{\alpha}A}\varphi_1 - \sqrt{\frac{\alpha}{\beta}}\varphi_2, \;\; \varphi_2 = \frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}e^{-\sqrt{\alpha}A}\varphi_1,$$

$$\varphi_1 + \frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}e^{-2\sqrt{\alpha}A}\varphi_1 + \sqrt{\alpha}TA\varphi_1 - \frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\sqrt{\alpha}TAe^{-2\sqrt{\alpha}A}\varphi_1 = 0.$$

Consequently,

$$T_{\alpha,\beta}\varphi_1 \equiv \left(E + \sqrt{\alpha}TA + \frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\left(\sqrt{\alpha}TA - E\right)e^{-2\sqrt{\alpha}A}\right)\varphi_1 = 0.$$

Since by the condition of the theorem the operator $T_{\alpha,\beta}$ has a bounded inverse operator in $H_{\frac{3}{2}}$, then from the last equation it follows that $\varphi_1 = 0$. Consequently, $\varphi_2 = \varphi_3 = 0$, i.e., $u_0(t) = 0$. From the condition of the theorem it is clear that at any $f(t) \in L_2(R_+; H)$ the equation $P_0u(t) = f(t)$ has a solution from the space $\overset{o}{W}{}_{2,T}^2(R_+; H)$ and this solution has the representation

$$u(t) = \begin{cases} u_1(t)\,, & t \in (0,1), \\ u_2(t)\,, & t \in (1, +\infty), \end{cases}$$

where

$$u_1(t) = \frac{1}{2\sqrt{\alpha}}\int_0^1 e^{-\sqrt{\alpha}|t-s|A}A^{-1}f(s)ds + T_{\alpha,\beta}^{-1}(\sqrt{\alpha}TA - E)e^{-\sqrt{\alpha}(t+1)A}$$

$$\times\left[\frac{\sqrt{\alpha}-\sqrt{\beta}}{2\sqrt{\alpha}\left(\sqrt{\alpha}+\sqrt{\beta}\right)}\int_0^1 e^{-\sqrt{\alpha}(1-s)A}A^{-1}f(s)ds+\frac{1}{2\sqrt{\alpha}}\int_0^1 e^{-\sqrt{\alpha}(s-1)A}A^{-1}f(s)ds\right.$$

$$\left.+\frac{1}{\sqrt{\alpha}+\sqrt{\beta}}\int_1^{+\infty} e^{-\sqrt{\beta}(s-1)A}A^{-1}f(s)ds\right]+T_{\alpha,\beta}^{-1}(E+\sqrt{\alpha}TA)e^{-\sqrt{\alpha}(1-t)A}$$

$$\times\left[\frac{\sqrt{\alpha}-\sqrt{\beta}}{2\sqrt{\alpha}\left(\sqrt{\alpha}+\sqrt{\beta}\right)}\int_0^1 e^{-\sqrt{\alpha}(1-s)A}A^{-1}f(s)ds+\frac{1}{\sqrt{\alpha}+\sqrt{\beta}}\int_1^{+\infty} e^{-\sqrt{\beta}(s-1)A}A^{-1}f(s)ds\right]$$

$$+T_{\alpha,\beta}^{-1}(\sqrt{\alpha}TA-E)e^{-\sqrt{\alpha}(2-t)A}\frac{\sqrt{\alpha}-\sqrt{\beta}}{2\sqrt{\alpha}\left(\sqrt{\alpha}+\sqrt{\beta}\right)}\int_0^1 e^{-\sqrt{\alpha}sA}A^{-1}f(s)ds,$$

$$u_2(t)=\frac{1}{2\sqrt{\beta}}\int_1^{+\infty} e^{-\sqrt{\beta}|t-s|A}A^{-1}f(s)ds+T_{\alpha,\beta}^{-1}(E+\sqrt{\alpha}TA)\frac{e^{-\sqrt{\beta}(t-1)A}}{\sqrt{\alpha}+\sqrt{\beta}}$$

$$\times\left[\int_0^1 e^{-\sqrt{\alpha}(1-s)A}A^{-1}f(s)ds+\frac{\sqrt{\beta}-\sqrt{\alpha}}{2\sqrt{\beta}}\int_1^{+\infty} e^{-\sqrt{\beta}(s-1)A}A^{-1}f(s)ds\right]$$

$$+T_{\alpha,\beta}^{-1}(\sqrt{\alpha}TA-E)e^{-\sqrt{\beta}(t-1)A}e^{-2\sqrt{\alpha}A}\left[\frac{1}{\sqrt{\alpha}+\sqrt{\beta}}\int_0^1 e^{-\sqrt{\alpha}(s-1)A}A^{-1}f(s)ds\right.$$

$$\left.+\frac{1}{2\sqrt{\beta}}\int_1^{+\infty} e^{-\sqrt{\beta}(s-1)A}A^{-1}f(s)ds\right].$$

We note that the fulfillment of boundary condition (4) is verified directly. In addition, for any $u(t)\in \overset{o}{W}{}_{2,T}^{2}(R_+;H)$ we have

$$\begin{aligned}\|P_0u\|_{L_2(R_+;H)}^2&=\left\|u''+\rho A^2u\right\|_{L_2(R_+;H)}^2\\&\le 2\left(\left\|u''\right\|_{L_2(R_+;H)}^2+\max(\alpha^2;\beta^2)\left\|A^2u\right\|_{L_2(R_+;H)}^2\right)\\&\le 2\max(1;\alpha^2;\beta^2)\,\|u\|_{W_2^2(R_+;H)}^2,\end{aligned}$$

i.e., the operator $P_0:\overset{o}{W}{}_{2,T}^{2}(R_+;H)\to L_2(R_+;H)$ is bounded. Then the assertion of the theorem follows from the Banach theorem on the inverse operator. The theorem is proved.

From Theorem 1 it implies that the norms $\|P_0u\|_{L_2(R_+;H)}$ and $\|u\|_{W_2^2(R_+;H)}$ are equivalent in the space $\overset{o}{W}{}^2_{2,T}(R_+;H)$.

The following theorem is true.

Theorem 2. *Suppose that the operator $T_{\alpha,\beta}$ has a bounded inverse operator in $H_{\frac{3}{2}}$, and the operator $B = A_1A^{-1}$ is bounded in H, moreover, $\|B\| < N_T^{-1}$, where*

$$N_T = \sup_{0\neq u(t)\in \overset{o}{W}{}^2_{2,T}(R_+;H)} \left\|Au'\right\|_{L_2(R_+;H)} \|P_0u\|^{-1}_{L_2(R_+;H)} .$$

Then problem (1), (2) *is regularly solvable.*

P r o o f. Denoting by P_1 the operator acting from the space $\overset{o}{W}{}^2_{2,T}(R_+;H)$ to the space $L_2(R_+;H)$ in the following way:

$$P_1u(t) \equiv A_1u'(t), \ \ u(t) \in \overset{o}{W}{}^2_{2,T}(R_+;H),$$

we can rewrite problem (1), (2) in the form of the operator equation

$$P_0u(t) + P_1u(t) = f(t),$$

where $f(t) \in L_2(R_+;H)$, $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+;H)$. Since by Theorem 1 the operator P_0 has a bounded inverse P_0^{-1} acting from the space $L_2(R_+;H)$ to the space $\overset{o}{W}{}^2_{2,T}(R_+;H)$, then after substitution $u(t) = P_0^{-1}v(t)$ we obtain the equation

$$(E + P_1P_0^{-1})v(t) = f(t)$$

in the space $L_2(R_+;H)$. And due to the fact that for any $v(t) \in L_2(R_+;H)$

$$\left\|P_1P_0^{-1}v\right\|_{L_2(R_+;H)} = \|P_1u\|_{L_2(R_+;H)} \leq \|B\| \left\|Au'\right\|_{L_2(R_+;H)}$$

$$\leq N_T \|B\| \|P_0u\|_{L_2(R_+;H)} = N_T \|B\| \|v\|_{L_2(R_+;H)} ,$$

then for $N_T \|B\| < 1$ the operator $E+P_1P_0^{-1}$ is invertible in the space $L_2(R_+;H)$ and

$$u(t) = P_0^{-1}(E + P_1P_0^{-1})^{-1}f(t),$$

thus

$$\|u\|_{W_2^2(R_+;H)} \leq const \|f\|_{L_2(R_+;H)} .$$

The theorem is proved.

We note that the problem of estimating the number N_T arises here. For this purpose we will use the idea of the procedure offered in [22].

First we prove the following statement.

Lemma 1. *For any* $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H)$ *there takes place the inequality*

$$\|P_0 u\|^2_{L_2(R_+;H)} \geq \frac{\alpha}{\beta} \left(\|u''\|^2_{L_2(R_+;H)} + \alpha\beta \left\|A^2 u\right\|^2_{L_2(R_+;H)} \right.$$

$$\left. + 2\beta \left\|Au'\right\|^2_{L_2(R_+;H)} + 2\beta Re(ATx, x)_{\frac{1}{2}} \right),$$

where $x = u'(0) \in H_{\frac{1}{2}}$.

P r o o f. Multiplying both sides of equation (3) by $\rho^{-\frac{1}{2}}(t)$, we get

$$\left\|\rho^{-\frac{1}{2}} f\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-\frac{1}{2}} P_0 u\right\|^2_{L_2(R_+;H)} = \left\|-\rho^{-\frac{1}{2}} u'' + \rho^{\frac{1}{2}} A^2 u\right\|^2_{L_2(R_+;H)}$$

$$= \left\|\rho^{-\frac{1}{2}} u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{\frac{1}{2}} A^2 u\right\|^2_{L_2(R_+;H)} - 2Re\left(u'', A^2 u\right)_{L_2(R_+;H)}.$$

Now, integrating by parts, we have

$$-2Re\left(u'', A^2 u\right)_{L_2(R_+;H)} = 2Re\left(A^{\frac{1}{2}} u'(0), A^{\frac{3}{2}} u(0)\right) + 2\left\|Au'\right\|^2_{L_2(R_+;H)}$$

$$= 2Re\left(ATx, x\right)_{\frac{1}{2}} + 2\left\|Au'\right\|^2_{L_2(R_+;H)}.$$

Then

$$\left\|\rho^{-\frac{1}{2}} P_0 u\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-\frac{1}{2}} u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{\frac{1}{2}} A^2 u\right\|^2_{L_2(R_+;H)}$$

$$+ 2\left\|Au'\right\|^2_{L_2(R_+;H)} + 2Re\left(ATx, x\right)_{\frac{1}{2}}. \tag{5}$$

On the other hand, for $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H)$ we have

$$\|P_0 u\|^2_{L_2(R_+;H)} \geq \alpha \left\|\rho^{-\frac{1}{2}} P_0 u\right\|^2_{L_2(R_+;H)}. \tag{6}$$

Taking into account equality (5) in (6), we obtain

$$\|P_0 u\|^2_{L_2(R_+;H)} \geq \alpha \left(\left\|\rho^{-\frac{1}{2}} u''\right\|^2_{L_2(R_+;H)} + \left\|\rho^{\frac{1}{2}} A^2 u\right\|^2_{L_2(R_+;H)} \right.$$

$$\left. + 2\left\|Au'\right\|^2_{L_2(R_+;H)} + 2Re(ATx, x)_{\frac{1}{2}} \right)$$

$$\geq \alpha \left(\frac{1}{\beta} \left\|u''\right\|^2_{L_2(R_+;H)} + \alpha \left\|A^2 u\right\|^2_{L_2(R_+;H)} + 2\left\|Au'\right\|^2_{L_2(R_+;H)} \right.$$

$$\left. + 2Re(ATx, x)_{\frac{1}{2}} \right) = \frac{\alpha}{\beta} S(u),$$

where

$$S(u) = \left\| u'' \right\|^2_{L_2(R_+;H)} + \alpha\beta \left\| A^2 u \right\|^2_{L_2(R_+;H)}$$
$$+ 2\beta \left\| Au' \right\|^2_{L_2(R_+;H)} + 2\beta Re(ATx, x)_{\frac{1}{2}}. \tag{7}$$

Lemma 1 is proved.

For further operations we factorize the considered in the space H_4 polynomial operator pencil of the form

$$P(\lambda;\gamma;A) = \lambda^4 E + \alpha\beta A^4 - 2\beta\lambda^2 A^2 + \gamma\lambda^2 A^2.$$

Lemma 2. *For $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ the operator pencil $P(\lambda;\gamma;A)$ is invertible on an imaginary axis and is represented in the form*

$$P(\lambda;\gamma;A) = F(\lambda;\gamma;A)F(-\lambda;\gamma;A),$$

where

$$F(\lambda;\gamma;A) = (\lambda E - \omega_1(\gamma)A)(\lambda E - \omega_2(\gamma)A)$$
$$\equiv \lambda^2 E + a_1(\gamma)\lambda A + a_2(\gamma)A^2, \tag{8}$$

and $Re\omega_k(\gamma) < 0$, $k = 1, 2$, $a_1(\gamma) = \sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) - \gamma}$, $a_2(\gamma) = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}$.

P r o o f. Let $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$. Then for $\sigma \in \sigma(A)$ and $\lambda = i\xi$, $\xi \in R$, we have

$$P(i\xi;\gamma;\sigma) = \xi^4 + \alpha\beta\sigma^4 + 2\beta\xi^2\sigma^2 - \gamma\xi^2\sigma^2$$
$$= \sigma^4\left(\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2} - \gamma\frac{\xi^2}{\sigma^2}\right) = \sigma^4\left(\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2}\right)$$
$$\times\left(1 - \gamma\frac{\frac{\xi^2}{\sigma^2}}{\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2}}\right) \geq \sigma^4\left(\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2}\right)$$
$$\times\left(1 - \gamma\sup_{\frac{\xi^2}{\sigma^2}\geq 0}\frac{\frac{\xi^2}{\sigma^2}}{\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2}}\right).$$

Since

$$\sup_{\frac{\xi^2}{\sigma^2}\geq 0}\frac{\frac{\xi^2}{\sigma^2}}{\frac{\xi^4}{\sigma^4} + \alpha\beta + 2\beta\frac{\xi^2}{\sigma^2}} = \frac{1}{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})},$$

it follows that $P(i\xi;\gamma;\sigma) > 0$ for $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$. Consequently, the polynomial $P(i\xi;\gamma;\sigma)$ has no roots on the imaginary axis for $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$. Therefore, this polynomial has two roots $\omega_1(\gamma)\sigma$ and $\omega_2(\gamma)\sigma$ from the left half-plane and two roots $-\omega_1(\gamma)\sigma$ and $-\omega_2(\gamma)\sigma$ from the right half-plane, i.e., $Re\omega_k(\gamma) < 0$, $k = 1, 2$, and $\omega_1(\gamma) = \overline{\omega_2(\gamma)}$. Thus,

$$P(\lambda; \gamma; \sigma) = F(\lambda; \gamma; \sigma)F(-\lambda; \gamma; \sigma), \tag{9}$$

where

$$F(\lambda; \gamma; \sigma) = (\lambda - \omega_1(\gamma)\sigma)(\lambda - \omega_2(\gamma)\sigma) \\ = \lambda^2 + a_1(\gamma)\lambda\sigma + a_2(\gamma)\sigma^2.$$

Since $Re\omega_k(\gamma) < 0$, $k = 1, 2$, then

$$a_1(\gamma) = -(\omega_1(\gamma) + \omega_2(\gamma)) = -(\omega_1(\gamma) + \overline{\omega_1(\gamma)}) = -2Re\omega_1(\gamma) > 0.$$

And as $a_2^2(\gamma) = \alpha\beta$ and $a_2(\gamma) = \omega_1(\gamma)\omega_2(\gamma) = \omega_1(\gamma)\overline{\omega_1(\gamma)} = |\omega_1(\gamma)|^2 > 0$, we obtain that $a_2(\gamma) = \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}$. Further, from equality (9) it follows that $a_1^2(\gamma) - 2a_2(\gamma) = -\gamma + 2\beta$, i.e., $a_1^2(\gamma) = 2a_2(\gamma) + 2\beta - \gamma = 2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + 2\beta - \gamma > 0$. Consequently, $a_1(\gamma) = \sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) - \gamma}$. Now, using the spectral decomposition of the operator A, from equality (9) we obtain the assertion of the lemma. Lemma 2 is proved.

Now we prove the lemma which plays an important role in our investigation.

Lemma 3. *For any* $u(t) \in \overset{o}{W}{}_{2,T}^2(R_+; H)$ *and* $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ *the following identity holds:*

$$S(u) - \gamma \left\|Au'\right\|^2_{L_2(R_+;H)} = \langle R(\gamma)x, x\rangle + \left\|F\left(\frac{d}{dt}; \gamma; A\right)u\right\|^2_{L_2(R_+;H)}, \tag{10}$$

where $F(\lambda; \gamma; A)$ *is defined from equality (8), and*

$$\langle R(\gamma)x, x\rangle = 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})Re(ATx, x)_{\frac{1}{2}} + \sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) - \gamma} \\ \times \left(\|x\|^2_{H_{\frac{1}{2}}} + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\|ATx\|^2_{H_{\frac{1}{2}}}\right). \tag{11}$$

P r o o f. First we denote by $D_T(R_+; H_2)$ the lineal of all infinitely differentiable functions with values in H_2, having compact supports on R_+ and satisfying boundary condition (2) at zero. By density and trace theorems [2], this lineal is dense everywhere in $\overset{o}{W}{}_{2,T}^2(R_+; H)$. Consequently, it is enough to prove (10) for the functions from $D_T(R_+; H_2)$. It is obvious that for $u(t) \in D_T(R_+; H_2)$ and $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ the following equality holds:

$$\left\|F\left(\frac{d}{dt}; \gamma; A\right)u\right\|^2_{L_2(R_+;H)} = \left\|u'' + a_1(\gamma)Au' + a_2(\gamma)A^2u\right\|^2_{L_2(R_+;H)} = \left\|u''\right\|^2_{L_2(R_+;H)}$$

$$+a_1^2(\gamma)\left\|Au'\right\|^2_{L_2(R_+;H)}+a_2^2(\gamma)\left\|A^2u\right\|^2_{L_2(R_+;H)}+2a_1(\gamma)Re\left(u'',Au'\right)_{L_2(R_+;H)}$$

$$+2a_2(\gamma)Re\left(u'',A^2u\right)_{L_2(R_+;H)}+2a_1(\gamma)a_2(\gamma)Re\left(Au',A^2u\right)_{L_2(R_+;H)}. \qquad (12)$$

On the other hand, applying integration by parts, we obtain the equalities

$$Re(u'',A^2u)_{L_2(R_+;H)}=-\left\|Au'\right\|^2_{L_2(R_+;H)}-Re(ATx,x)_{\frac{1}{2}},$$

$$2Re(u'',Au')_{L_2(R_+;H)}=-\left\|x\right\|^2_{H_{\frac{1}{2}}},\ \ 2Re(Au',A^2u)_{L_2(R_+;H)}=-\left\|ATx\right\|^2_{H_{\frac{1}{2}}}.$$

Taking into account equalities from (12) and the values $a_1(\gamma)$ and $a_2(\gamma)$ from Lemma 2, we have

$$\left\|F\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}=\left\|u''\right\|^2_{L_2(R_+;H)}+\alpha\beta\left\|A^2u\right\|^2_{L_2(R_+;H)}$$

$$+(2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})-\gamma)\left\|Au'\right\|^2_{L_2(R_+;H)}-2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\left\|Au'\right\|^2_{L_2(R_+;H)}-2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}Re(ATx,x)_{\frac{1}{2}}$$

$$-\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})-\gamma}\left\|x\right\|^2_{H_{\frac{1}{2}}}-\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})-\gamma}\left\|ATx\right\|^2_{H_{\frac{1}{2}}}$$

$$=S(u)-\gamma\left\|Au'\right\|^2_{L_2(R_+;H)}-\langle R(\gamma)x,x\rangle, \qquad (13)$$

where $S(u)$ and $\langle R(\gamma)x,x\rangle$ are defined from (7) and (11), respectively. The validity of (10) follows from (13). Lemma 3 is proved.

Obviously that $S(u)$ is the norm in the space $\overset{o}{W}{}_2^2(R_+;H)$ which is equivalent to the initial norm $\|u\|_{W_2^2(R_+;H)}$.

The studies above allow us to assert that the following theorem is true.

Theorem 3. *The number S_0, defined as*

$$S_0=\sup_{0\neq u\in \overset{o}{W}{}_2^2(R_+;H)}\left\|Au'\right\|_{L_2(R_+;H)}S^{-\frac{1}{2}}(u),$$

is finite and $S_0=\frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}}$.

P r o o f. Clearly, for $u(t)\in \overset{o}{W}{}_2^2(R_+;H)$, for any $\gamma\in[0,2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}))$ from equality (10) we obtain

$$S(u)-\gamma\left\|Au'\right\|^2_{L_2(R_+;H)}=\left\|F\left(\frac{d}{dt};\gamma;A\right)u\right\|^2_{L_2(R_+;H)}. \qquad (14)$$

Passing in (14) to the limit as $\gamma \to 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})$, we have

$$S(u) \geq 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}) \left\|Au'\right\|^2_{L_2(R_+;H)},$$

i.e.,

$$\left\|Au'\right\|_{L_2(R_+;H)} \leq \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}} S^{\frac{1}{2}}(u).$$

Consequently, the number $S_0 \leq \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}}$. We show that here the equality holds, i.e., $S_0 = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}}$. For this purpose, for any $\varepsilon > 0$ it is sufficient to construct a vector function $u_\varepsilon(t) \in \overset{o}{W}{}^2_2(R_+;H)$ such that

$$\mathrm{E}(u_\varepsilon) \equiv S(u_\varepsilon) - (2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})+\varepsilon) \left\|Au'\right\|^2_{L_2(R_+;H)} < 0.$$

Let $\psi \in H_4$, $\|\psi\| = 1$, and $g(t)$ be a scalar function having a twice continuous derivative in R, and $g(t)$, $g''(t) \in L_2(R)$. Then, using Plancherel's theorem on the Fourier transformation, we obtain

$$\mathrm{E}(g(t)\psi) \equiv S(g(t)\psi) - (2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})+\varepsilon) \left\|Ag'(t)\psi\right\|^2_{L_2(R;H)}$$

$$= \int_{-\infty}^{+\infty} ([\xi^4 E + \alpha\beta A^4 + 2\beta\xi^2 A^2 - (2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})+\varepsilon)\xi^2 A^2]\psi,\psi) \left|\hat{g}(\xi)\right|^2 d\xi$$

$$= \int_{-\infty}^{+\infty} q_\varepsilon(\xi,\psi) \left|\hat{g}(\xi)\right|^2 d\xi \quad (\|\psi\| = 1), \tag{15}$$

where

$$q_\varepsilon(\xi,\psi) = \xi^4 + \alpha\beta \left\|A^2\psi\right\|^2 - (2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\varepsilon)\xi^2 \left\|A\psi\right\|^2.$$

It is obvious that for fixed ψ, at the points $\xi = \pm\left(\frac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\varepsilon}{2}\right)^{\frac{1}{2}} \|A\psi\|$, the function $q_\varepsilon(\xi,\psi)$ takes its minimum value which is equal to

$$h_\varepsilon(\psi) = \alpha\beta \left\|A^2\psi\right\|^2 - \left(\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + \frac{\varepsilon}{2}\right)^2 \left\|A\psi\right\|^4.$$

Since $\psi \in H_4$ is an arbitrary vector ($\|\psi\| = 1$), it can be chosen. Namely, if the operator A has an eigenvector, then we may chose this vector as ψ. Indeed, in this case $A\psi = \mu\psi$, and $h_\varepsilon(\psi) = \alpha\beta\mu^4 - \left(\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + \frac{\varepsilon}{2}\right)^2 \mu^4 < 0$. If μ is a continuous spectrum, then it is possible to find a vector $\psi \in H_4$ such that $A^k\psi = \mu^k\psi + o(\delta)$

at $\delta \to 0$, $k = 1, 2, 3, 4$. Then $h_\varepsilon(\psi) = \alpha\beta\mu^4 - \left(\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + \frac{\varepsilon}{2}\right)^2 \mu^4 + o(\delta) < 0$ for sufficiently small $\delta > 0$. Thus, it is always possible to find a vector $\psi \in H_4$ ($\|\psi\| = 1$) such that $\min_\xi q_\varepsilon(\xi, \psi) < 0$. After choosing the vector ψ, by using the continuity of the function $q_\varepsilon(\xi, \psi)$ at ξ, there exists an interval $(\eta_1(\varepsilon), \eta_2(\varepsilon))$ in which $q_\varepsilon(\xi, \psi) < 0$. Now we construct the function $g(t)$. Let $\hat{g}(\xi)$ be an arbitrary twice continuously differentiable function in R whose support is contained in the interval $(\eta_1(\varepsilon), \eta_2(\varepsilon))$. Then from equality (15) we get

$$\mathrm{E}(g(t)\psi) = \int_{\eta_1(\varepsilon)}^{\eta_2(\varepsilon)} q_\varepsilon(\xi, \psi) \left|\hat{g}(\xi)\right|^2 d\xi < 0,$$

and

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{\eta_1(\varepsilon)}^{\eta_2(\varepsilon)} \hat{g}(\xi) e^{i\xi t} d\xi.$$

The theorem is proved.

R e m a r k 1. We note that, in general, $S(u)$ is not a positive number for all $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H)$. Indeed, let $u(t) = e^{-\lambda_0 t}\varphi_0$, and φ_0 be some eigenvector of the operator A corresponding to λ_0, $\|\varphi_0\| = 1$. Then,

$$S(u) = \|u''\|^2_{L_2(R_+;H)} + \alpha\beta \left\|A^2 u\right\|^2_{L_2(R_+;H)} - 2\beta Re \left(u'', A^2 u\right)_{L_2(R_+;H)}$$

$$= \left\|\lambda_0^2 e^{-\lambda_0 t}\varphi_0\right\|^2_{L_2(R_+;H)} + \alpha\beta \left\|\lambda_0^2 e^{-\lambda_0 t}\varphi_0\right\|^2_{L_2(R_+;H)}$$

$$-2\beta Re \left(\lambda_0^2 e^{-\lambda_0 t}\varphi_0, \lambda_0^2 e^{-\lambda_0 t}\varphi_0\right)_{L_2(R_+;H)}$$

$$= (1 + \alpha\beta)\lambda_0^4 \left\|e^{-\lambda_0 t}\varphi_0\right\|^2_{L_2(R_+;H)} - 2\beta\lambda_0^4 \left\|e^{-\lambda_0 t}\varphi_0\right\|^2_{L_2(R_+;H)}$$

$$= (1 + \alpha\beta - 2\beta)\lambda_0^4 \left\|e^{-\lambda_0 t}\varphi_0\right\|^2_{L_2(R_+;H)} = (1 + \alpha\beta - 2\beta)\frac{\lambda_0^3}{2}.$$

Obviously, we can require $T = -A^{-1}$, i.e., the condition $Tu'(0) = u(0)$ to be satisfied. Then it is clear that for $2\beta - \alpha\beta > 1$ the expression $S(u)$ is negative, and for $2\beta - \alpha\beta = 1$ $S(u) = 0$, $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H)$. Thus, for $S(u)$ in the space $\overset{o}{W}{}^2_{2,T}(R_+; H)$ to be equivalent to the initial norm $\|u\|_{W_2^2(R_+;H)}$, additional conditions should be imposed on the operator T.

Lemma 4. *Let $x \in H_{\frac{1}{2}}$. If $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} \langle R(0)x, x\rangle > 0$, then*

$$S(u) \geq const\, \|u\|^2_{W_2^2(R_+;H)}$$

for any $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+;H)$.

P r o o f. From (10), for $\gamma = 0$ we get

$$S(u) \geq \langle R(0)x, x\rangle .$$

Since from the condition of the lemma $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} \langle R(0)x, x\rangle = c > 0$, it is obvious that for $\|u\|_{W_2^2(R_+;H)} = 1$ the inequality

$$S(u) \geq c$$

holds.

Since $S(u) = \|u\|^2_{W_2^2(R_+;H)} S\left(\frac{u}{\|u\|_{W_2^2(R_+;H)}}\right)$, then for all $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+;H)$

$$S(u) \geq c\, \|u\|^2_{W_2^2(R_+;H)}$$

is valid. The lemma is proved.

From Lemma 4 it follows that

$$S_T = \sup_{0\neq u \in \overset{o}{W}{}^2_{2,T}(R_+;H)} \left\|Au'\right\|_{L_2(R_+;H)} S^{-\frac{1}{2}}(u)$$

$$\leq \frac{1}{c^{\frac{1}{2}}} \sup_{0\neq u(t) \in \overset{o}{W}{}^2_{2,T}(R_+;H)} \left\|Au'\right\|_{L_2(R_+;H)} \|u\|^{-1}_{W_2^2(R_+;H)} = d,$$

and $d < \infty$ by the theorem on intermediate derivatives [2]. Consequently, $S_T < \infty$.

Since $\overset{o}{W}{}^2_{2,T}(R_+;H) \supset \overset{o}{W}{}^2_2(R_+;H)$, then $S_T \geq S_0 = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}}$.

Theorem 4. *Let the conditions of Lemma 4 be satisfied and $ReAT \geq 0$ in $H_{\frac{1}{2}}$. Then*

$$S_T = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})}}.$$

P r o o f. If the conditions of the theorem are satisfied, then $\langle R(\gamma)x, x\rangle > 0$ for any $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$. Consequently, by Lemma 3 it follows that for any $u(t) \in \overset{o}{W}{}^2_{2,T}(R_+; H)$ and $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ the inequality

$$S(u) \geq \gamma \left\|Au'\right\|^2_{L_2(R_+;H)}$$

is true. Passing in the last inequality to the limit as $\gamma \to 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})$, we have

$$S_T \leq \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})}}.$$

Thereby,

$$S_T = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})}}.$$

The theorem is proved.

Basing on the obtained results, estimating the norm of the intermediate derivative operator $A\frac{d}{dt} : \overset{o}{W}{}^2_{2,T}(R_+; H) \to L_2(R_+; H)$ with respect to the norm $\|P_0 u\|_{L_2(R_+;H)}$ and taking into account Lemma 1, we give the exact formulation of the conditional Theorem 2 in the following form.

Theorem 5. *Suppose that the conditions of Lemma 4 are satisfied,* $ReAT \geq 0$ *in* $H_{\frac{1}{2}}$ *and the operator* $B = A_1 A^{-1}$ *is bounded in* H, *moreover,* $\|B\| < \sqrt{2\alpha\left(1 + \frac{\alpha^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right)}$. *Then problem* (1), (2) *is regularly solvable.*

R e m a r k 2. In Theorem 5, the condition $ReAT \geq 0$ in $H_{\frac{1}{2}}$ provides invertibility of the operator $T_{\alpha,\beta}$ in the space $H_{\frac{3}{2}}$.

Now we will specify the value of S_T under the condition $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} Re(ATx, x)_{\frac{1}{2}} < 0$.

The following theorem holds.

Theorem 6. *Suppose that* $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} Re(ATx, x)_{\frac{1}{2}} < 0$ *and the conditions of Lemma 4 are satisfied. Then*

$$S_T = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})}} \left(1 - 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) \left| \min_{\|x\|_{H_{\frac{1}{2}}}=1} \frac{Re(ATx, x)_{\frac{1}{2}}}{1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|ATx\|^2_{H_{\frac{1}{2}}}} \right|^2 \right)^{-\frac{1}{2}}.$$

P r o o f. First we note that since the pencil $F(\lambda;\gamma;A)$ for $\gamma \in [0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ is represented in the form

$$F(\lambda;\gamma;A) = (\lambda E - \omega_1(\gamma)A)(\lambda E - \omega_2(\gamma)A),$$

where $Re\omega_k(\gamma) < 0$, $k = 1, 2$, then the Cauchy problem

$$F\left(\frac{d}{dt};\gamma;A\right) u = 0, \tag{16}$$

$$u(0) = Tx, \ \ u'(0) = x, \ \ x \in H_{\frac{1}{2}}, \tag{17}$$

has the unique solution $u(t;\gamma;x) \in W_2^2(R_+;H)$ represented in the form

$$u(t;\gamma;x)$$
$$= \frac{1}{\omega_2(\gamma) - \omega_1(\gamma)} \left[e^{\omega_1(\gamma)tA}(\omega_2(\gamma)Tx - A^{-1}x) + e^{\omega_2(\gamma)tA}(A^{-1}x - \omega_1(\gamma)Tx) \right].$$

It follows easily that

$$\|u(t;\gamma;x)\|_{W_2^2(R_+;H)} \leq d_1(\gamma) \|x\|_{H_{\frac{1}{2}}}.$$

Further, using the uniqueness of the solution of Cauchy problem (16), (17) and taking into account the Banach theorem on the inverse operator, we have

$$\|u(t;\gamma;x)\|_{W_2^2(R_+;H)} \geq d_2(\gamma) \|x\|_{H_{\frac{1}{2}}}, \ \ d_2(\gamma) > 0. \tag{18}$$

Now we note that from Theorem 4 it follows that

$$S_T > \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}})}}.$$

Consequently, $S_T^{-2} \in (0, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$. Then, in equality (10), taking for $u(t)$ the solution $u(t;\gamma;x)$ of Cauchy problem (16), (17), for $\|x\|_{H_{\frac{1}{2}}} = 1$, we obtain

$$\langle R(\gamma)x, x\rangle = S(u(t;\gamma;x)) - \gamma \left\|Au'(t;\gamma;x)\right\|^2_{L_2(R_+;H)} = S(u(t;\gamma;x))(1 - \gamma S_T^2) > 0 \tag{19}$$

for $\gamma \in [0, S_T^{-2})$. Taking into account Lemma 4 and inequality (18), we obtain

$$\langle R(\gamma)x, x\rangle \geq c d_2^2(\gamma)(1 - \gamma S_T^2) > 0.$$

By that, $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} \langle R(\gamma)x, x\rangle > 0$ for $\gamma \in [0, S_T^{-2})$. The same argument can be used for the case $\omega_1(\gamma) = \omega_2(\gamma)$.

Continuing, by definition S_T, for all $\gamma \in (S_T^{-2}, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ there exists a function $v(t;\gamma) \in \overset{o}{W}{}_{2,T}^{2}(R_+;H)$ such that

$$S(v(t;\gamma)) < \gamma \left\| Av'(t;\gamma) \right\|_{L_2(R_+;H)}^2 .$$

Consequently, for $\gamma \in (S_T^{-2}, 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}))$ from equality (10) it follows that

$$\langle R(\gamma)x_\gamma, x_\gamma \rangle + \left\| F\left(\frac{d}{dt};\gamma;A\right) v(t;\gamma) \right\|_{L_2(R_+;H)}^2 < 0, \;\; x_\gamma = v'(0;\gamma),$$

i.e.,

$$\min_{\|x\|_{H_{\frac{1}{2}}}=1} \langle R(\gamma)x, x \rangle \le \langle R(\gamma)x_\gamma, x_\gamma \rangle < 0.$$

Thus, since $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} \langle R(\gamma)x, x \rangle$ is a continuous function and it changes its sign at the point $\gamma = S_T^{-2}$, then $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} \left\langle R(S_T^{-2})x, x \right\rangle = 0.$

To complete the proof, we consider the functional of the form

$$Q(\gamma;x) = \sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) - \gamma} + 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}) \frac{Re(ATx,x)_{\frac{1}{2}}}{1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|ATx\|_{H_{\frac{1}{2}}}^2}, \;\; \|x\|_{H_{\frac{1}{2}}} = 1.$$

Since

$$Q(\gamma;x) = \frac{\langle R(\gamma)x, x \rangle}{1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|ATx\|_{H_{\frac{1}{2}}}^2} \le \langle R(\gamma)x, x \rangle ,$$

then

$$\min_{\|x\|_{H_{\frac{1}{2}}}=1} Q(S_T^{-2};x) \le \min_{\|x\|_{H_{\frac{1}{2}}}=1} \left\langle R(S_T^{-2})x, x \right\rangle = 0.$$

On the other hand,

$$\begin{aligned} \left\langle R(S_T^{-2})x, x \right\rangle &= Q(S_T^{-2};x) \left(1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|ATx\|_{H_{\frac{1}{2}}}^2\right) \\ &\le Q(S_T^{-2};x) \left(1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|AT\|_{H_{\frac{1}{2}} \to H_{\frac{1}{2}}}^2\right) \end{aligned}$$

and, consequently,

$$\min_{\|x\|_{H_{\frac{1}{2}}}=1} Q(S_T^{-2};x) \ge \left(1 + \alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} \|AT\|_{H_{\frac{1}{2}} \to H_{\frac{1}{2}}}^2\right)^{-1} \min_{\|x\|_{H_{\frac{1}{2}}}=1} \left\langle R(S_T^{-2})x, x \right\rangle = 0.$$

Thus we have

$$\min_{\|x\|_{H_{\frac{1}{2}}}=1} Q(S_T^{-2}; x) = 0.$$

So,

$$\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}) - S_T^{-2}} = -2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}) \min_{\|x\|_{H_{\frac{1}{2}}}=1} \frac{Re(ATx,x)_{\frac{1}{2}}}{1+\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\,\|ATx\|^2_{H_{\frac{1}{2}}}}.$$

Then, taking into account $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} Re(ATx,x)_{\frac{1}{2}} < 0$, from the above we get

$$S_T = \frac{1}{\sqrt{2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})}} \left(1 - 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}) \left| \min_{\|x\|_{H_{\frac{1}{2}}}=1} \frac{Re(ATx,x)_{\frac{1}{2}}}{1+\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\,\|ATx\|^2_{H_{\frac{1}{2}}}} \right|^2 \right)^{\frac{1}{2}}.$$

The theorem is proved.

In the case when $\min\limits_{\|x\|_{H_{\frac{1}{2}}}=1} Re(ATx,x)_{\frac{1}{2}} < 0$, Theorem 2 is formulated as follows.

Theorem 7. *Suppose that the conditions of Theorem 6 are satisfied, the operator $T_{\alpha,\beta}$ is bounded invertible in the space $H_{\frac{3}{2}}$ and the operator $B = A_1 A^{-1}$ is bounded in H, moreover*

$$\|B\| < \sqrt{2\alpha\left(1+\frac{\alpha^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right)} \left(1 - 2\beta^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}) \left| \min_{\|x\|_{H_{\frac{1}{2}}}=1} \frac{Re(ATx,x)_{\frac{1}{2}}}{1+\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\,\|ATx\|^2_{H_{\frac{1}{2}}}} \right|^2 \right)^{\frac{1}{2}}.$$

Then problem (1), (2) is regularly solvable.

Despite the fact that equation (1) was presented in a more general form in the paper of A.R.Aliev [18], in this paper only the case when $ReAT \geq 0$ is studied. Moreover, our results improve the results obtained in [18].

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