Journal of Mathematical Physics, Analysis, Geometry
2012, vol. 8, No. 2, pp. 190–206

A Symmetric Model of Viscous Relaxing Fluid. An Evolution Problem

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Received April 12, 2011

An evolution problem on small motions of the viscous rotating relaxing fluid in a bounded domain is studied. The problem is reduced to the Cauchy problem for the first-order integro-differential equation in a Hilbert space. Using this equation, we prove a strong unique solvability theorem for the corresponding initial-boundary value problem.

Key words: viscous fluid, compressible fluid, existence, uniqueness, integro-differential equation.

Mathematics Subject Classification 2000: 45K05, 58C40, 76R99.

Introduction

As it is known, a barotropic fluid is modelled by an equation of the state relating fluid pressure and density. In particular, it is supposed that this relationship is linear. The problem of motions of an ideal rotating fluid with compressibility being taken into account was first studied in V.N. Maslennikova's works. In the paper [1] the spectral problem on normal oscillations of the fluid (viscous or ideal) filling a rotating bounded domain was studied.

In [2, p. 390–410] (see also [3]) the problem on small motions of an ideal relaxing fluid in a bounded region, excluding rotation, gravity force, and at some model restrictions on the boundary conditions for dynamical density, was studied. The model of the relaxing fluid is a generalization of the barotropic fluid in the sense that the fluid pressure and the fluid density are coupled through the integral Volterra operator. In the monograph [2] the theorem on the strong solvability of the appropriate initial-boundary value problem was proved, and a spectral problem on normal oscillations was studied.

In [4, 5], the problem on small motions of an ideal relaxing fluid filling a bounded region and being influenced by a gravitational field was studied. In this case the fluid density in the state of relative equilibrium was assumed to be constant. It turns out that the neglect of a change of the stationary density causes both the breaking of a symmetry of the problem and the noncompact perturbation of the operator pencil corresponding to the spectral problem (even in the case of a barotropic model). In [6] there was proposed a modified rheological relation leading to the symmetric model of the ideal relaxing fluid with consideration for the exact steady state of a fluid. For this model, the evolution and the spectral problems were studied.

In the present paper the rheological relation from [6] is applied to construct a model of the viscous relaxing fluid. We reduce the initial-boundary value problem describing this model to the Cauchy problem for an integro-differential equation in some Hilbert space. It should be noted that there are some versions of the initial-boundary value problem in an abstract form. Here the way leading to the equation well-adapted for further studying of the spectral problem is chosen.

1. Small Motions of a Rotating Viscous Relaxing Fluid

1.1. Statement of the problem. Consider a container that uniformly rotates around the axis parallel to the gravity force and is completely filled by a viscous inhomogeneous fluid. The fluid is said to occupy a bounded region $\Omega \subset \mathbb{R}^3$. Let $\vec{n}$ be a unit vector perpendicular to the boundary $S := \partial\Omega$ and directed out of the region Ω. We introduce a system of the coordinates $Ox_1x_2x_3$ which is toughly connected with the container so that the Ox_3 axis coincides with the rotating axis and is directed opposite the gravity force, and the origin of coordinates is in the region Ω. In this case, the uniform velocity of rotation of the container takes the form $\vec{\omega}_0 := \omega_0\vec{e}_3$, where $\vec{e}_3$ is the unit vector along the rotating axis Ox_3, and $\omega_0 > 0$ for definiteness. The external stationary field of forces $\vec{F}_0$ is considered to be a gravitational field acting along the rotating axis, i.e., $\vec{F}_0 = -g\vec{e}_3$, $g > 0$.

Let us consider the relative equilibrium state of the fluid. From the Navier–Stokes equation of motion of a viscous fluid, written in the moving system of coordinates, we obtain the formula for the stationary pressure gradient

$$\nabla P_0 = \rho_0(-\vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) - g\vec{e}_3) = \rho_0\nabla(2^{-1}|\vec{\omega}_0 \times \vec{r}|^2 - gx_3), \qquad (1.1)$$

where $\vec{r}$ is the radius-vector of a moving point of the region Ω, and ρ_0 is the stationary density of the fluid.

In the state of relative equilibrium, the dynamic components of pressure and density responsible for relaxing effects in the fluid are leaking. Therefore, we will consider that the fluid is barotropic in the relative equilibrium state and satisfies the equation of the state $\nabla P_0 = a_\infty^2\nabla\rho_0$, where a_∞ is the sound velocity in the fluid. This equation and relation (1.1) allow us to conclude that ρ_0 and a_∞^2 can be functions of the parameter $z := 2^{-1}\omega_0^2(x_1^2 + x_2^2) - gx_3$. Further we will consider that the sound velocity a_∞ is defined for the fluid and it is constant. Then the stationary density can be determined as a function of the parameter z. In this case, the stationary density ρ_0 is constant only when there is neither rotation nor gravitational field in the system. The function $\rho_0(z)$ satisfies the conditions $0 < \alpha_1 \le \rho_0(z) \le \alpha_2$.

We now represent the total pressure and the density of the fluid in the form $\widehat{P}(t,x) = P_0(z) + p(t,x)$, $\widehat{\rho}(t,x) = \rho_0(z) + \widetilde{\rho}(t,x)$, where $p(t,x)$ and $\widetilde{\rho}(t,x)$ are the dynamical pressure and the density, respectively, arising at small motions of the fluid relative to its steady state. Assume that the dynamic components satisfy the following rheological relation:

$$P_m\Big(\frac{\partial}{\partial t}\Big)\nabla p(t,x) = a_\infty^2\Big(P_m\Big(\frac{\partial}{\partial t}\Big) + \rho_0(z)Q_{m-1}\Big(\frac{\partial}{\partial t}\Big)\Big)\nabla\widetilde{\rho}(t,x), \tag{1.2}$$

where $P_m(\lambda)$ and $Q_{m-1}(\lambda)$ are polynomials with the degrees m and $m-1$, respectively. In this case we can obviously consider that the coefficient of the highest degree in the polynomial $P_m(\lambda)$ is equal to 1. Following the reasoning and ideas from [7], we will assume that all roots of the polynomial $P_m(\lambda)$ are real, different from one another, and negative (we denote them by $-b_l$ $(l = \overline{1,m})$) while the roots of the polynomial $Q_{m-1}(\lambda)$ are real, negative and alternate with the roots of $P_m(\lambda)$. Thus from (1.2) we obtain the equation of the state

$$\nabla p(t,x) = a_\infty^2\Big(\nabla\widetilde{\rho}(t,x) - \rho_0(z)\int_0^t \nabla\widetilde{K}(t-s)\widetilde{\rho}(s,x)\,ds\Big), \tag{1.3}$$

where $\widetilde{K}(t) := \sum_{l=1}^m k_l \exp(-b_l t)$. The numbers b_l^{-1} are used as the times of relaxation in the system, and $k_l > 0$ $(l = \overline{1,m})$ are some structural constants. As a mathematical generalization of the presented constructions, we assume that $\widetilde{K} = \widetilde{K}(t,x)$ is a sufficiently smooth positive kernel in the evolution problem.

Let us linearize the Navier–Stokes equation written in the moving system of coordinates with respect to the relative equilibrium state. Using the equation of the state (1.3), we obtain the problem of small motions of a viscous relaxing fluid filling a uniformly rotating solid body

$$\begin{aligned}\frac{\partial\vec{u}(t,x)}{\partial t} &- 2\omega_0\big(\vec{u}(t,x)\times\vec{e}_3\big) = \rho_0^{-1}(z)\big(\mu\Delta\vec{u}(t,x) + (\eta + 3^{-1}\mu)\nabla\mathrm{div}\vec{u}(t,x)\big)\\ &- \nabla\big(a_\infty^2\rho_0^{-1}(z)\widetilde{\rho}(t,x)\big) + \int_0^t \nabla\big(a_\infty^2\widetilde{K}(t-s,x)\widetilde{\rho}(s,x)\big)\,ds + \vec{f}(t,x) \ \ (\text{in } \Omega),\\ \frac{\partial\widetilde{\rho}(t,x)}{\partial t} &+ \mathrm{div}\big(\rho_0(z)\vec{u}(t,x)\big) = 0 \ \ (\text{in } \Omega), \quad \vec{u}(t,x) = \vec{0} \ \ (\text{on } S),\end{aligned}$$

where $\vec{u}(t,x)$ is the field of velocities in the fluid, $\widetilde{\rho}(t,x)$ is the dynamic density of the fluid, μ and η are the dynamic and the second viscosity of the fluid, and $\vec{f}(t,x)$ is a weak field of external forces imposed on the gravitational field.

To symmetrize the system, let us replace $a_\infty\rho_0^{-1/2}(z)\widetilde{\rho}(t,x) = \rho(t,x)$. As a result, we arrive at the basic problem

$$\frac{\partial\vec{u}(t,x)}{\partial t} - 2\omega_0\big(\vec{u}(t,x)\times\vec{e}_3\big) = \rho_0^{-1}(z)\big(\mu\Delta\vec{u}(t,x) + (\eta+3^{-1}\mu)\nabla\mathrm{div}\vec{u}(t,x)\big)$$
$$-\nabla\big(a_\infty\rho_0^{-1/2}(z)\rho(t,x)\big) + \int_0^t \nabla\big(K(t-s,x)\rho(s,x)\big)\,ds + \vec{f}(t,x) \ \ (\text{in } \Omega), \quad (1.4)$$
$$\frac{\partial\rho(t,x)}{\partial t} + a_\infty\rho_0^{-1/2}(z)\mathrm{div}\big(\rho_0(z)\vec{u}(t,x)\big) = 0 \ \ (\text{in } \Omega), \quad \vec{u}(t,x) = \vec{0} \ \ (\text{on } S), \quad (1.5)$$

where $K(t,x) := a_\infty\rho_0^{1/2}(z)\widetilde{K}(t,x)$.

For the completeness of the statement of the problem, we set the initial conditions

$$\vec{u}(0,x) = \vec{u}^0(x), \quad \rho(0,x) = \rho^0(x). \quad (1.6)$$

1.2. Auxiliary operators and their properties. Let us introduce a vector Hilbert space $\vec{L}_2(\Omega,\rho_0)$ with the scalar product and the norm

$$(\vec{u},\vec{v})_{\vec{L}_2(\Omega,\rho_0)} := \int_\Omega \rho_0(z)\vec{u}(x)\cdot\overline{\vec{v}(x)}\,d\Omega, \quad \|\vec{u}\|^2_{\vec{L}_2(\Omega,\rho_0)} = \int_\Omega \rho_0(z)|\vec{u}(x)|^2\,d\Omega.$$

We introduce a scalar Hilbert space $L_2(\Omega)$ of functions square summable in the region Ω, and also its subspace $L_{2,\rho_0}(\Omega) := \{f \in L_2(\Omega) \mid (f,\rho_0^{1/2})_{L_2(\Omega)} = 0\}$. We define an orthogonal projector

$$\Pi f := f - (f,\rho_0^{1/2})_{L_2(\Omega)}\|\rho_0^{1/2}\|^{-2}_{L_2(\Omega)}\rho_0^{1/2}(z).$$

Obviously, the formula $\Pi L_2(\Omega) = L_{2,\rho_0}(\Omega)$ is valid.

To pass to the operator formulation of problem (1.4)–(1.6), we introduce a number of operators and study their properties.

We introduce an operator $S\vec{u}(t,x) := i\big(\vec{u}(t,x)\times\vec{e}_3\big)$, $\mathcal{D}(S) = \vec{L}_2(\Omega,\rho_0)$. The following lemma, whose proof is similar to that of an analogous lemma on properties of the Coriolis operator from [8], is valid.

Lemma 1.1. *The operator $\mathcal{S}$ is self-adjoint and bounded in $\vec{L}_2(\Omega,\rho_0)$: $\mathcal{S} = \mathcal{S}^*$, $\mathcal{S} \in \mathcal{L}(\vec{L}_2(\Omega,\rho_0))$; moreover, $\|\mathcal{S}\| = 1$.*

In what follows, we will consider that the function $\widetilde{K}(t,x)$ is continuously differentiable with respect to spatial variables and twice continuously differentiable with respect to time, and the boundary S of the region Ω belongs to the class C^2.

Lemma 1.2. *We introduce the space* $H_A := \{\vec{u} \in \vec{W}_2^1(\Omega) \big| \, \vec{u} = \vec{0} \ (on\ S)\}$ *with the following scalar product and norm:*

$$(\vec{u},\vec{v})_A := \int_\Omega E(\vec{u},\overline{\vec{v}})\,d\Omega, \quad \|\vec{u}\|_A^2 := \int_\Omega E(\vec{u},\vec{u})\,d\Omega,$$
$$E(\vec{u},\vec{v}) := \big(\eta - \frac{2}{3}\mu\big)\mathrm{div}\vec{u}\,\mathrm{div}\vec{v} + \frac{1}{2}\mu\sum_{j,k=1}^{3}\Big(\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\Big)\Big(\frac{\partial v_j}{\partial x_k}+\frac{\partial v_k}{\partial x_j}\Big).$$

The space H_A is a Hilbert space; it is compactly embedded in the space $\vec{L}_2(\Omega,\rho_0)$: $H_A \hookrightarrow\hookrightarrow \vec{L}_2(\Omega,\rho_0)$. A generating operator A of a Hilbert pair $(H_A;\vec{L}_2(\Omega,\rho_0))$, which is self-adjoint and positive definite in $\vec{L}_2(\Omega,\rho_0)$, possesses a discrete spectrum and is defined on $\mathcal{D}(A) = \vec{W}_2^2(\Omega)\cap H_A$. For every field $\vec{w} \in \vec{L}_2(\Omega,\rho_0)$, the generalized solution of the problem

$$-\rho_0^{-1}(z)\big(\mu\Delta\vec{u} + (\eta + 3^{-1}\mu)\nabla\mathrm{div}\vec{u}\big) = \vec{w} \quad (in\ \Omega), \quad \vec{u} = \vec{0} \quad (on\ S),$$

given by the formula $\vec{u} = A^{-1}\vec{w}$, exists and it is unique.

P r o o f. We now show that H_A is a Hilbert space. Let us introduce a new equivalent norm in the space $\vec{W}_2^1(\Omega)$ by the formula

$$\|\vec{u}\|^2_{\vec{W}^1_{2,S}(\Omega)} := \int_\Omega \sum_{k=1}^{3}|\nabla u_k|^2\,d\Omega + \Bigg(\int_S \vec{u}\,dS\Bigg)^2.$$

Let $\vec{W}^1_{2,S}(\Omega)$ denote the space $\vec{W}_2^1(\Omega)$ with a new norm.

For any field $\vec{u} \in \vec{W}^1_{2,S}(\Omega)$ with the condition $\vec{u} = \vec{0}$ on the boundary S, the following Korn inequality is valid (see [9, p. 23, Theorem 2.7]):

$$\int_\Omega E(\vec{u},\vec{u})\,d\Omega \ge c_1 \int_\Omega \sum_{k=1}^{3}|\nabla\vec{u}_k|^2\,d\Omega = c_1\|\vec{u}\|^2_{\vec{W}^1_{2,S}(\Omega)},$$

where c_1 is positive constant depending only on the domain Ω.

By using the Korn inequality, for any field $\vec{u} \in H_A$, we can get the following inequalities:

$$c_1\|\vec{u}\|^2_{\vec{W}^1_{2,S}(\Omega)} \le \|\vec{u}\|_A^2 \le \max\{3\eta, 2\mu\}\|\vec{u}\|^2_{\vec{W}^1_{2,S}(\Omega)}. \tag{1.7}$$

The inequalities imply that H_A is a Hilbert space.

The Hilbert space H_A is dense in $\vec{L}_2(\Omega, \rho_0)$. Taking into account that the inequality $\|\vec{u}\|_{\vec{L}_2(\Omega,\rho_0)} \le c_2 \|\vec{u}\|_{\vec{W}^1_{2,S}(\Omega)}$ is valid for any field $\vec{u} \in \vec{W}^1_{2,S}(\Omega)$, from the left-hand inequality in (1.7) we obtain that H_A and $\vec{L}_2(\Omega, \rho_0)$ form a Hilbert pair $(H_A; \vec{L}_2(\Omega, \rho_0))$.

To find the generating operator A of the Hilbert pair $(H_A; \vec{L}_2(\Omega, \rho_0))$, we use the identity (see [8, p. 33])

$$(A\vec{u}, \vec{v})_{\vec{L}_2(\Omega,\rho_0)} = (\vec{u}, \vec{v})_A, \quad \vec{u} \in \mathcal{D}(A), \quad \vec{v} \in H_A = \mathcal{D}(A^{1/2}). \tag{1.8}$$

The identity of Betty is valid for any $\vec{u} \in \vec{W}^2_2(\Omega)$, $\vec{v} \in \vec{W}^1_{2,S}(\Omega)$:

$$\int_\Omega L\vec{u} \cdot \vec{v}\, d\Omega = \int_\Omega E(\vec{u}, \vec{v})\, d\Omega - \int_S \sigma(\vec{u})\vec{n} \cdot \vec{v}\, dS,$$

$$L\vec{u} := -\big(\mu\Delta\vec{u} + (\eta + 3^{-1}\mu)\nabla \mathrm{div}\vec{u}\big), \quad \sigma(\vec{u}) := \{\sigma_{j,k}(\vec{u})\}^3_{j,k=1},$$

$$\sigma_{j,k}(\vec{u}) := \big(\eta - \frac{2}{3}\mu\big)\delta_{jk}\mathrm{div}\vec{u} + \mu\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big), \quad j, k = 1, 2, 3.$$

By using the identities above, for the twice differentiable field $\vec{u}$ we can transform identity (1.8) as follows:

$$(A\vec{u}, \vec{v})_{\vec{L}_2(\Omega,\rho_0)} = \int_\Omega E(\vec{u}, \vec{v})\, d\Omega = \int_\Omega L\vec{u} \cdot \vec{v}\, d\Omega + \int_S \sigma(\vec{u})\vec{n} \cdot \vec{v}\, dS$$

$$= \int_\Omega L\vec{u} \cdot \vec{v}\, d\Omega = (-\rho_0^{-1}(z)\big(\mu\Delta\vec{u} + (\eta + 3^{-1}\mu)\nabla \mathrm{div}\vec{u}\big), \vec{v})_{\vec{L}_2(\Omega,\rho_0)}.$$

This implies that any twice differentiable solution $\vec{u}$ of the equation $A\vec{u} = \vec{w}$ is a solution of the problem $-\rho_0^{-1}(z)\big(\mu\Delta\vec{u} + (\eta + 3^{-1}\mu)\nabla \mathrm{div}\vec{u}\big) = \vec{w}$ (in Ω), $\vec{u} = \vec{0}$ (on S). This problem has a unique generalized solution $\vec{u} = A^{-1}\vec{w}$ for any field $\vec{w} \in \vec{L}_2(\Omega, \rho_0)$. It should be noticed that the field $\vec{u}$ is called a generalized solution of the described problem if the identity

$$\int_\Omega E(\vec{u}, \vec{v})\, d\Omega = \int_\Omega \rho_0 \vec{w} \cdot \vec{v}\, d\Omega \left[= (\vec{w}, \vec{v})_{\vec{L}_2(\Omega,\rho_0)}\right] \quad \forall \vec{v} \in H_A$$

holds.

It follows from the smoothness of the boundary S that $\mathcal{D}(A) = \vec{W}^2_2(\Omega) \cap H_A$.

It follows from the left-hand inequality in (1.7) and the compactness of the embedding of the space $\vec{W}^1_{2,S}(\Omega)$ in $\vec{L}_2(\Omega, \rho_0)$ that H_A is compactly embedded in the space $\vec{L}_2(\Omega, \rho_0)$: $H_A \hookrightarrow\hookrightarrow \vec{L}_2(\Omega, \rho_0)$. This implies the compactness of the operator A^{-1}; hence the spectrum of the operator A is discrete. ∎

Define an operator $B\vec{u}(t,x) := a_\infty \rho_0^{-1/2}(z)\mathrm{div}(\rho_0(z)\vec{u}(t,x))$, $\mathcal{D}(B) := H_A$.

Lemma 1.3. *The adjoint operator* $B^*\rho = -\nabla(a_\infty \rho_0^{-1/2}\rho)$, $\mathcal{D}(B^*) = W^1_{2,\rho_0}(\Omega)$, $W^1_{2,\rho_0}(\Omega) := W^1_2(\Omega) \cap L_{2,\rho_0}(\Omega)$. *The following inequality takes place:*

$$\exists\, c > 0 : \quad \|B\vec{u}\|_{W^1_{2,\rho_0}(\Omega)} \le c\|A\vec{u}\|_{\vec{L}_2(\Omega,\rho_0)} \quad \forall \vec{u} \in \mathcal{D}(A).$$

P r o o f. Let $\vec{u} \in \mathcal{D}(B) := \mathcal{D}(A^{1/2}) = H_A$. Then we have

$$(B\vec{u},\rho)_{L_{2,\rho_0}(\Omega)} = \int\limits_\Omega a_\infty \rho_0^{-1/2}\mathrm{div}(\rho_0\vec{u})\overline{\rho}\, d\Omega = -\int\limits_\Omega \rho_0 \vec{u}\cdot \overline{\nabla\big(a_\infty\rho_0^{-1/2}\rho\big)}\, d\Omega$$

$$+\int\limits_S a_\infty \rho_0^{1/2}\overline{\rho}\vec{u}\cdot\vec{n}\, dS = -\int\limits_\Omega \rho_0\vec{u}\cdot\overline{\nabla\big(a_\infty\rho_0^{-1/2}\rho\big)}\, d\Omega = \big(\vec{u}, -\nabla\big(a_\infty\rho_0^{-1/2}\rho\big)\big)_{\vec{L}_2(\Omega,\rho_0)}.$$

The above implies the formula for the operator B^*.

Consider the problem

$$L\vec{u} := -\mu\Delta\vec{u} - (\eta + 3^{-1}\mu)\nabla\mathrm{div}\vec{u} = \vec{f} \ (\text{in } \Omega), \quad B_L\vec{u} := \vec{u} = \vec{g} \ (\text{on } S). \tag{1.9}$$

It can be shown in the usual way that the matrix differential expression L is nonsingular and properly elliptic, and the boundary condition B_L covers it (see [10]). It follows from the theorem of normal solvability (see [10, p. 241]) that there exist constants $c_3 > 0$, $c_4 > 0$ (not depending on the field $\vec{u}$) such that the following inequalities are valid:

$$c_3\|\vec{u}\|^2_{\vec{W}^2_2(\Omega)} \le \|L\vec{u}\|^2_{\vec{L}_2(\Omega)} \le c_4\|\vec{u}\|^2_{\vec{W}^2_2(\Omega)} \quad \forall \vec{u} \in \vec{W}^2_2(\Omega, B_L), \tag{1.10}$$

$$\vec{W}^2_2(\Omega, B_L) := \{\vec{u} \in \vec{W}^2_2(\Omega) \big| B_L\vec{u} = \vec{u} = \vec{0} \ (\text{on } S)\} = \mathcal{D}(A),$$

$$\|\vec{u}\|^2_{\vec{W}^2_2(\Omega)} := \sum_{k=1}^3 \Big[\|u_k\|^2_{L_2(\Omega)} + \sum_{|\alpha|=2}\|D^\alpha u_k\|^2_{L_2(\Omega)}\Big].$$

Further, from the Erling–Nirenberg inequality (see [11, p. 33]) it follows that there exists a constant $c_5 > 0$ (not depending on the field $\vec{u}$) such that the equality

$$\Big\|\frac{\partial u_k}{\partial x_j}\Big\|^2_{L_2(\Omega)} \le c_5\|\vec{u}\|^2_{\vec{W}^2_2(\Omega)} \quad \forall \vec{u} \in \vec{W}^2_2(\Omega), \quad k, j = 1, 2, 3 \tag{1.11}$$

is valid.

Let now $\vec{u} \in \mathcal{D}(A) = \vec{W}^2_2(\Omega, B_L)$. By using inequalities (1.10), (1.11), we get

$$\|B\vec{u}\|^2_{W^1_{2,\rho_0}(\Omega)} = a^2_\infty \int\limits_\Omega \big(|\nabla\rho_0^{-1/2}\mathrm{div}(\rho_0\vec{u})|^2 + |\rho_0^{-1/2}\mathrm{div}(\rho_0\vec{u})|^2\big)\, d\Omega \le c_6\|\vec{u}\|^2_{\vec{W}^2_2(\Omega)}$$

$$\le c_6 c_3^{-1}\|L\vec{u}\|^2_{\vec{L}_2(\Omega)} \le c_6 c_3^{-1}\max_{x\in\overline{\Omega}}\rho_0 \int\limits_\Omega \rho_0|\rho_0^{-1}L\vec{u}|^2\, d\Omega = c\|A\vec{u}\|^2_{\vec{L}_2(\Omega,\rho_0)},$$

where $c = c(c_3, c_5, \rho_0, a_\infty, \Omega) > 0$ is some absolute constant. ■

We define the operators $Q := BA^{-1/2}$, $Q^+ := A^{-1/2}B^*$.

Lemma 1.4. *The operator Q is bounded $Q \in \mathcal{L}(\vec{L}_2(\Omega,\rho_0),\ L_{2,\rho_0}(\Omega))$. The operator Q^* admits an extension in a continuity to the bounded operator Q^+, $\overline{Q^+} = Q^*$, $Q^+ = Q^*|_{\mathcal{D}(B^*)}$.*

P r o o f. Let $\vec{u} \in \mathcal{D}(B) := \mathcal{D}(A^{1/2}) = H_A$. We have

$$\|B\vec{u}\|^2_{L_{2,\rho_0}(\Omega)} = \int_\Omega a_\infty^2 \rho_0^{-1} |\mathrm{div}(\rho_0 \vec{u})|^2 \, d\Omega \le 2a_\infty^2 \int_\Omega \Big(\rho_0^{-1} |\nabla \rho_0|^2 |\vec{u}|^2 + \rho_0 |\mathrm{div}\vec{u}|^2 \Big) \, d\Omega \le c_7 \int_\Omega E(\vec{u},\vec{u}) \, d\Omega = c_7 \|A^{1/2}\vec{u}\|^2_{\vec{L}_2(\Omega,\rho_0)},$$

where $c_7 = c_7(a_\infty, \rho_0, \Omega) > 0$ is some absolute constant. After replacement $A^{1/2}\vec{u} = \vec{v}$, $\vec{v} \in \vec{L}_2(\Omega,\rho_0)$, it follows from the last inequality that the operator Q is bounded. This implies the boundedness of the operator Q^* and the simply checked relation $Q^+ = Q^*|_{\mathcal{D}(B^*)}$. ■

We define an operator function $M(t)\rho(t,x) := \Pi\rho_0(z)\widetilde{K}(t,x)\Pi\rho(t,x)$. Obviously, the operator function $M(t)$ is bounded self-adjoint and positive definite in $L_{2,\rho_0}(\Omega)$.

1.3. Reduction to the first-order integro-differential equation. Solvability of the initial-boundary value problem. Using the operators introduced above, we can write problem (1.4)–(1.6) as a system of two equations with initial-value conditions in the Hilbert space $H = \vec{L}_2(\Omega,\rho_0) \oplus L_{2,\rho_0}(\Omega)$,

$$\begin{cases} \dfrac{d\vec{u}}{dt} + (2\omega_0 iS + A)\vec{u} - B^*\rho + \displaystyle\int_0^t B^* M(t-s)\rho(s)\, ds = \vec{f}(t), \\ \dfrac{d\rho}{dt} + B\vec{u} = 0, \quad (\vec{u};\rho)^\tau(0) := (\vec{u}^0;\rho^0)^\tau. \end{cases} \tag{1.12}$$

Notice that the kernel $\widetilde{K}(t,x)$ is continuously differentiable with respect to spatial variables, and twice continuously differentiable with respect to time. This implies that the operator function $M(t)$ is twice continuously differentiable with values in the space $\mathcal{L}(W^1_{2,\rho_0}(\Omega))$.

Let us change the variables in system (1.12) according to the formula

$$\widehat{\rho}(t) := \int_0^t M(t-s)\rho(s)\, ds, \quad \widehat{\rho}(0) = 0.$$

Considering this differentiated relation as a differential equation associated with system (1.12), we arrive at a Cauchy problem for a first-order integro-differential equation in the Hilbert space $\mathcal{H} := \vec{L}_2(\Omega, \rho_0) \oplus L_{2,\rho_0}(\Omega) \oplus L_{2,\rho_0}(\Omega)$,

$$\frac{d\zeta}{dt} + \mathcal{B}\zeta = \int_0^t \mathcal{M}(t-s)\zeta(s)\,ds + \mathcal{F}(t), \quad \zeta(0) = \zeta^0, \tag{1.13}$$

where $\zeta(t) := (\vec{u}(t); \rho(t); \widehat{\rho}(t))^\tau$, $\zeta^0 := (\vec{u}^0; \rho^0; 0)^\tau$, $\mathcal{F}(t) := (\vec{f}(t); 0; 0)^\tau$. The nonzero components of the operator blocks $\mathcal{B}$ and $\mathcal{M}(t)$ have the following representation: $\mathcal{M}_{3,2}(t) := M'(t)$, $\mathcal{B}_{1,1} := 2\omega_0 iS + A$, $\mathcal{B}_{1,2} := -B^*$, $\mathcal{B}_{1,3} := B^*$, $\mathcal{B}_{2,1} := B$, $\mathcal{B}_{3,2} := -M(0)$.

Thus, if $\vec{u}$ and ρ are a solution of problem (1.4)–(1.6) (the problem on small motions of the viscose rotating relaxing fluid in a bounded domain) such that all the reasonings above are applicable, then the function ζ is a solution of the Cauchy problem for the first-order integro-differential equation (1.13).

Definition 1.1. *If the function ζ is a strong solution of the Cauchy problem* (1.13)*, then the corresponding functions $\vec{u}$, ρ are called a strong solution of the initial-boundary value problem* (1.4)–(1.6)*. A function $\zeta(t)$ is called a strong solution of the Cauchy problem* (1.13) (*see* [12, p. 38]) *if $\zeta(t) \in \mathcal{D}(\mathcal{B})$ for any t from $\mathbb{R}_+ := [0, +\infty)$, $\mathcal{B}\zeta(t) \in C(\mathbb{R}_+; \mathcal{H})$, $\zeta(t) \in C^1(\mathbb{R}_+; \mathcal{H})$, $\zeta(0) = \zeta^0$, and the equation from* (1.13) *is satisfied for any $t \in \mathbb{R}_+$.*

Let us change the sough function in problem (1.13), $\zeta(t) = \mathrm{e}^{at}\xi(t) := \mathrm{e}^{at}(\vec{v}(t); q(t); \widehat{q}(t))^\tau$,

$$\frac{d\xi}{dt} + (\mathcal{A} + \mathcal{S})\xi = \int_0^t \mathrm{e}^{-a(t-s)}\mathcal{M}(t-s)\xi(s)\,ds + \mathrm{e}^{-at}\mathcal{F}(t), \ \ \xi(0) = \zeta^0, \tag{1.14}$$

where nonzero components of the operator blocks $\mathcal{A}$ and $\mathcal{S}$ have the form $\mathcal{A}_{1,1} := A$, $\mathcal{A}_{1,2} := -B^*$, $\mathcal{A}_{1,3} := B^*$, $\mathcal{A}_{2,1} := B$, $\mathcal{A}_{2,2} = \mathcal{A}_{3,3} := aI$, $\mathcal{S}_{3,2} := -M(0)$, $\mathcal{S}_{1,1} := 2\omega_0 iS + aI$ (here I is a unit operator in the corresponding space). The operator $\mathcal{S}$ is bounded in $\mathcal{H}$. The domain of definition of the operator $\mathcal{A}$ has the form $\mathcal{D}(\mathcal{A}) := \mathcal{D}(A) \oplus \mathcal{D}(B^*) \oplus \mathcal{D}(B^*)$. The operator $\mathcal{A}$ appears to be non closed. This fact complicates the using of theorems of solvability of abstract differential equations. In this connection we will prove the following lemma.

Lemma 1.5. *Let $a > 4^{-1}\|Q\|^2$. Then the operator $\mathcal{A}$ admits a closure to the maximal uniformly accretive operator $\overline{\mathcal{A}}$ which can be presented in the symmetric form*

$$\overline{\mathcal{A}} = \operatorname{diag}(A^{1/2}, I, I) \begin{pmatrix} I & -Q^* & Q^* \\ Q & aI & 0 \\ 0 & 0 & aI \end{pmatrix} \operatorname{diag}(A^{1/2}, I, I)$$

and in the Schur–Frobenius form $\overline{\mathcal{A}} = (\mathcal{I}+\mathcal{D}_1)\mathrm{diag}(A, aI+QQ^*, aI)(\mathcal{I}+\mathcal{D}_2)$. *Here* $\mathcal{I}$ *is a unit operator in* $\mathcal{H}$, *and the operators* $\mathcal{I}+\mathcal{D}_1$, $\mathcal{I}+\mathcal{D}_2$ *are lower and upper triangular blocks, respectively. The distinct from zero components of the operator blocks* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *have the form* $(\mathcal{D}_1)_{2,1} := QA^{-1/2}$, $(\mathcal{D}_2)_{1,2} := -A^{-1/2}Q^*$, $(\mathcal{D}_2)_{1,3} := A^{-1/2}Q^*$, $(\mathcal{D}_2)_{2,3} := -(aI+QQ^*)^{-1}QQ^*$. *The domain of the operator* $\overline{\mathcal{A}}$ *has the form* $\mathcal{D}(\overline{\mathcal{A}}) = \{(\vec{v};q;\widehat{q})^\tau \in \mathcal{H}|\ q,\widehat{q} \in L_{2,\rho_0}(\Omega),\ \vec{v} - A^{-1/2}Q^*(q-\widehat{q}) \in \mathcal{D}(A)\}$.

P r o o f. From the formulas for the operator $\overline{\mathcal{A}}$ it is seen that it is presented as a product of three closed operators. It follows from the Schur–Frobenius factorization that there exists the operator $(\overline{\mathcal{A}})^{-1}$, and the domain of the operator $(\overline{\mathcal{A}})^{-1}$ is the whole space $\mathcal{H}$. Thus, $\overline{\mathcal{A}}$ is the closed operator.

Let us check that the operator $\overline{\mathcal{A}}$ is accretive on $\mathcal{D}(\overline{\mathcal{A}})$. Let $\xi = (\vec{v};q;\widehat{q})^\tau \in \mathcal{D}(\overline{\mathcal{A}})$. We fix $\|Q\|(2a)^{-1} < \varepsilon < 2\|Q\|^{-1}$. Using the Cauchy inequality, we get

$$\begin{aligned}\mathrm{Re}(\overline{\mathcal{A}}\xi,\xi) &= \|A^{1/2}\vec{v}\|^2 + \mathrm{Re}(Q^*\widehat{q}, A^{1/2}\vec{v}) + a\|q\|^2 + a\|\widehat{q}\|^2 \\ &\geq \|A^{1/2}\vec{v}\|^2 - \|Q^*\|\|\widehat{q}\|\|A^{1/2}\vec{v}\| + a\|q\|^2 + a\|\widehat{q}\|^2 \\ &\geq \|A^{1/2}\vec{v}\|^2 - \|Q^*\|\big(\varepsilon 2^{-1}\|A^{1/2}\vec{v}\|^2 + (2\varepsilon)^{-1}\|\widehat{q}\|^2\big) + a\|q\|^2 + a\|\widehat{q}\|^2 \\ &\geq \min\{(1-\|Q^*\|\varepsilon 2^{-1})\gamma(A), a - \|Q^*\|(2\varepsilon)^{-1}\}\|\xi\|^2 =: \gamma(\overline{\mathcal{A}})\|\xi\|^2,\end{aligned}$$

where $\gamma(A) > 0$ is an exact lower boundary of the operator A. It follows from the estimations above that $\gamma(\overline{\mathcal{A}}) > 0$ if only the number a satisfies the lemma conditions. Hence the operator $\overline{\mathcal{A}}$ is uniformly accretive. The range of the values of the operator $\overline{\mathcal{A}}$ coincides with the whole space $\mathcal{H}$. Hence it is a maximal uniformly accretive operator.

It can be shown that the factorizations from this lemma are valid for the operator $\mathcal{A}$ when Q^* is replaced by Q^+. By direct computations it is also checked that the operator $\mathcal{A}$ is uniformly accretive on the domain $\mathcal{D}(\mathcal{A})$. In this case the operator $\mathcal{A}$ admits a closure which coincides with $\overline{\mathcal{A}}$. ■

Notice that in the lemma the number $a > 0$ can be chosen to be so large that $\overline{\mathcal{A}} + \mathcal{S}$ would be a maximal uniformly accretive operator. Everywhere below we will assume that it is.

Let us consider along with (1.14) a Cauchy problem with the closed operator

$$\frac{d\xi}{dt} + (\overline{\mathcal{A}} + \mathcal{S})\xi = \int_0^t \mathrm{e}^{-a(t-s)}\mathcal{M}(t-s)\xi(s)\,ds + \mathrm{e}^{-at}\mathcal{F}(t),\ \xi(0) = \zeta^0. \qquad (1.15)$$

The following solvability theorem is valid for problem (1.15).

Theorem 1.1. *Let $\zeta^0 \in \mathcal{D}(\overline{\mathcal{A}})$ and function $\mathcal{F}(t)$ satisfies the Gelder condition,* $\forall\, \tau \in \mathbb{R}_+ \;\exists\, K = K(\tau) > 0,\; k(\tau) \in (0,1]$,

$$\|\mathcal{F}(t) - \mathcal{F}(s)\|_{\mathcal{H}} \le K|t-s|^k \quad at \quad 0 \le s,t \le \tau.$$

Then the strong solution of the Cauchy problem (1.15) *exists and it is unique.*

P r o o f. Using the Schur–Frobenius factorization for the operator $\overline{\mathcal{A}}$, we change the sought function in problem (1.15), $z(t) = (\mathcal{I} + \mathcal{D}_2)\xi(t)$. After a series of simple transformations we get the following Cauchy problem:

$$\frac{dz}{dt} + \widehat{\mathcal{B}}z = \int_0^t \widehat{\mathcal{M}}(t-s)z(s)\,ds + \widehat{\mathcal{F}}(t), \quad z(0) = (\mathcal{I}+\mathcal{D}_2)\zeta^0 =: z^0, \tag{1.16}$$

where $\widehat{\mathcal{B}} := (\mathcal{I}+\mathcal{T})\mathcal{A}_0 + \mathcal{D}$, $\widehat{\mathcal{M}}(t) := \mathrm{e}^{-at}(\mathcal{I}+\mathcal{D}_2)\mathcal{M}(t)(\mathcal{I}+\mathcal{D}_2)^{-1}$, $\widehat{\mathcal{F}}(t) := \mathrm{e}^{-at}(\mathcal{I}+\mathcal{D}_2)\mathcal{F}(t)$. The operator $\mathcal{A}_0 := \mathrm{diag}(A, aI + QQ^*, aI)$ is the central block in the Schur–Frobenius factorization of the operator $\overline{\mathcal{A}}$. The nonzero components of the operator $\mathcal{T}$ have the form $\mathcal{T}_{1,2} := -A^{-1/2}Q^*$, $\mathcal{T}_{1,3} := A^{-1/2}Q^*$, $\mathcal{T}_{2,1} := (\mathcal{I} - A^{-1/2}Q^*)QA^{-1/2}$. The nonzero component of the operator $\mathcal{D}_3$ has the form $(\mathcal{D}_3)_{2,3} := -a(aI+QQ^*)^{-1}QQ^*$, and $\mathcal{D} := (\mathcal{I}+\mathcal{D}_2)\mathcal{S}(\mathcal{I}+\mathcal{D}_2)^{-1} + \mathcal{D}_3$.

It follows from the lemmas about auxiliary operators and the introduced notations that the operator $\mathcal{D}$ is bounded, and $\mathcal{T}$ is a compact operator in $\mathcal{H}$. The domain of the operator $\mathcal{A}_0$ is $\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \oplus L_{2,\rho_0}(\Omega) \oplus L_{2,\rho_0}(\Omega)$. The operator $\mathcal{A}_0$ is self-adjoint and positive definite, and the operator $-\mathcal{A}_0$ is a generator of a strongly continuous analytical semigroup of operators. This implies (see [12]) that the operator $-\widehat{\mathcal{B}}$ is a generator of a strongly continuous semigroup $\mathcal{U}(t) := \exp(-t\widehat{\mathcal{B}})$ in $\mathcal{H}$ analytical in some sector containing a positive semiaxis.

It follows from $M(t) \in C^2(\mathbb{R}_+; \mathcal{L}(W^1_{2,\rho_0}(\Omega)))$ and formulas for the operator function $\widehat{\mathcal{M}}(t)$ that $\widehat{\mathcal{M}}(t) \in C^1(\mathbb{R}_+; \mathcal{L}(\mathcal{H}))$.

The further proof follows the ideas from monograph [2].

By the conditions of the theorem, the function $\mathcal{F}(t)$ satisfies the Gelder condition, $\zeta^0 \in \mathcal{D}(\overline{\mathcal{A}})$, and hence $z^0 := (\mathcal{I}+\mathcal{D}_2)\zeta^0 \in \mathcal{D}(\mathcal{A}_0)$. We will assume that the Cauchy problem (1.16) has a strong solution $z(t)$. Taking into account Theorem 1.4 (see [13, p. 130]), we obtain

$$\begin{aligned} z(t) &= \mathcal{U}(t)z^0 + \int_0^t \mathcal{U}(t-s)\widehat{\mathcal{F}}(s)\,ds + \int_0^t \mathcal{U}(t-s)\left\{\int_0^s \widehat{\mathcal{M}}(s-\tau)z(\tau)\,d\tau\right\}ds \\ &= \mathcal{U}(t)z^0 + \int_0^t \mathcal{U}(t-s)\widehat{\mathcal{F}}(s)\,ds + \int_0^t d\tau \int_\tau^t \mathcal{U}(t-s)\widehat{\mathcal{M}}(s-\tau)z(\tau)\,ds. \end{aligned} \tag{1.17}$$

Consider the internal integral in (1.17). It follows from $\widehat{\mathcal{M}}(t) \in C^1(\mathbb{R}_+;\mathcal{L}(\mathcal{H}))$ that there exists the partial derivative

$$\frac{\partial}{\partial s}\Big(\mathcal{U}(t-s)\widehat{\mathcal{B}}^{-1}\widehat{\mathcal{M}}(s-\tau)z(\tau)\Big)=\mathcal{U}(t-s)\widehat{\mathcal{M}}(s-\tau)z(\tau)+\mathcal{U}(t-s)\widehat{\mathcal{B}}^{-1}\frac{\partial}{\partial s}\widehat{\mathcal{M}}(s-\tau)z(\tau).$$

Integrating this from τ to t with respect to s, we can see that

$$\int_{\tau}^{t}\mathcal{U}(t-s)\widehat{\mathcal{M}}(s-\tau)z(\tau)\,ds = \widehat{\mathcal{B}}^{-1}\Big(\widehat{\mathcal{M}}(t-\tau)z(\tau) - \mathcal{U}(t-\tau)\widehat{\mathcal{M}}(0)z(\tau)$$
$$- \int_{\tau}^{t}\mathcal{U}(t-s)\frac{\partial}{\partial s}\widehat{\mathcal{M}}(s-\tau)z(\tau)\,ds\Big) =: \widehat{\mathcal{B}}^{-1}\widehat{\mathcal{M}}_1(t,\tau)z(\tau). \quad (1.18)$$

From (1.17) and (1.18) we get that any strong solution $z(t)$ of the Cauchy problem (1.16) satisfies the following Volterra integral equation:

$$z(t) = \widehat{z}(t)+\int_0^t \widehat{\mathcal{B}}^{-1}\widehat{\mathcal{M}}_1(t,s)z(s)\,ds, \quad \widehat{z}(t) := \mathcal{U}(t)z^0+\int_0^t \mathcal{U}(t-s)\widehat{\mathcal{F}}(s)\,ds. \quad (1.19)$$

Here $\widehat{z}(t)$ is the solution of the Cauchy problem (1.16), which does not contain an integral term, and thus $\widehat{z}(t) \in C(\mathbb{R}_+;\mathcal{D}(\widehat{\mathcal{B}})) \cap C^1(\mathbb{R}_+;\mathcal{H})$.

Let us show that equation (1.19) has a unique solution and this solution is the strong solution of the Cauchy problem (1.16). Introduce the space $\mathcal{H}(\widehat{\mathcal{B}}) := (\mathcal{D}(\widehat{\mathcal{B}}), \|\cdot\|_{\mathcal{H}(\widehat{\mathcal{B}})})$, where $\|z\|_{\mathcal{H}(\widehat{\mathcal{B}})} := \|\widehat{\mathcal{B}}z\|$ for any $z \in \mathcal{D}(\widehat{\mathcal{B}}) = \mathcal{D}(\mathcal{A}_0)$. It is known that $\mathcal{H}(\widehat{\mathcal{B}})$ is a Banach space.

It follows from (1.18) that $\widehat{\mathcal{B}}^{-1}\widehat{\mathcal{M}}_1(t,s) \in C$, $0 \le s \le t < +\infty; \mathcal{L}(\mathcal{H}(\widehat{\mathcal{B}}))$. Therefore, equation (1.19), considered in $\mathcal{H}(\widehat{\mathcal{B}})$, is a Volterra integral equation of the second kind with continuous kernel. Therefore, taking into account the inclusion $\widehat{z}(t) \in C(\mathbb{R}_+;\mathcal{H}(\widehat{\mathcal{B}}))$, we may conclude that equation (1.19) has the unique solution $z(t) \in C(\mathbb{R}_+;\mathcal{H}(\widehat{\mathcal{B}}))$.

From the inclusion $\widehat{z}(t) \in C^1(\mathbb{R}_+;\mathcal{H})$, we obtain that $z(t)$ is a continuously differentiable function with values in the Hilbert space $\mathcal{H}$. Direct computations show that $z(t)$ satisfies definition 1.1; thus, $z(t)$ is the unique strong solution of problem (1.16). Then $\xi(t) = (\mathcal{I}+\mathcal{D}_2)^{-1}z(t)$ is the unique strong solution of problem (1.15). ■

Using Theorem 1.1, we study the strong solutions of problem (1.4)–(1.6) (the problem on small motions of the viscous rotating relaxing fluids in a bounded domain).

Theorem 1.2. *Let the field $\vec{f}(t,x)$ satisfy the Gelder condition* $\forall\, \tau \in \mathbb{R}_+ \ \exists\, K = K(\tau) > 0,\ k(\tau) \in (0,1]$*:* $\|\vec{f}(t) - \vec{f}(s)\|_{\vec{L}_2(\Omega,\rho_0)} \le K|t-s|^k,\ 0 \le s,t \le \tau$*. Then for any $\vec{u}^0 \in \mathcal{D}(A)$ and $\rho^0 \in \mathcal{D}(B^*)$ there exists a unique strong solution of the initial-boundary value problem* (1.4)–(1.6).

P r o o f. By the definition 1.1, we are to prove that problem (1.13) has a unique strong solution. By the theorem conditions, $\zeta^0 := (\vec{u}^0;\rho^0;0)^\tau \in \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\overline{\mathcal{A}})$, and the function $\mathcal{F}(t) := (\vec{f}(t);0;0)^\tau$ satisfies the Gelder condition. From Theorem 1.1 we get that problem (1.15) has the unique strong solution $\xi(t) = (\vec{v}(t);q(t);\widehat{q}(t))^\tau$. We represent the equation from (1.15) as the system

$$\begin{cases} \dfrac{d\vec{v}}{dt} + (aI + 2\omega_0 iS)\vec{v} + A(\vec{v} - A^{-1/2}Q^*q + A^{-1/2}Q^*\widehat{q}) = \mathrm{e}^{-at}\vec{f}(t), \\ \dfrac{dq}{dt} + aq + QA^{1/2}\vec{v} = 0, \\ \dfrac{d\widehat{q}}{dt} + a\widehat{q} - M(0)q = \displaystyle\int_0^t \mathrm{e}^{-a(t-s)}M'(t-s)q(s)\,ds. \end{cases} \tag{1.20}$$

The function $\xi(t)$ will be the unique strong solution of the Cauchy problem (1.14) if in the first equation of this system it is possible to remove the brackets before the operator A.

Using the formula $\xi(0) = \zeta^0 = (\vec{u}^0;\rho^0;0)^\tau$ and $Q = BA^{-1/2}$, from the second equation of system (1.20) we obtain

$$q(t) = \mathrm{e}^{-at}\rho^0 - \int_0^t \mathrm{e}^{-a(t-s)}QA^{1/2}\vec{v}(s)\,ds = \mathrm{e}^{-at}\rho^0 - \int_0^t \mathrm{e}^{-a(t-s)}B\vec{v}(s)\,ds.$$

From the above, by using the third equation from (1.20), we get

$$\widehat{q}(t) = \int_0^t M(t-s)\left[\mathrm{e}^{-as}\rho^0 - \int_0^s \mathrm{e}^{-a(s-\tau)}B\vec{v}(\tau)\,d\tau\right] ds.$$

Lemma 1.4 implies that $A^{-1/2}Q^*|_{\mathcal{D}(B^*)} = A^{-1/2}Q^+ = A^{-1}B^*$. Using the formulas for $q(t)$ and $\widehat{q}(t)$, we get that the inclusion $\vec{v}(t) - A^{-1/2}Q^*(q(t) - \widehat{q}(t)) \in$

$\mathcal{D}(A)$ is valid if and only if

$$\vec{v}(t) + A^{-1/2}Q^* \int_0^t \left[\mathrm{e}^{-a(t-s)} B\vec{v}(s) - M(t-s) \left\{ \int_0^s \mathrm{e}^{-a(s-\tau)} B\vec{v}(\tau)\, d\tau \right\} \right] ds$$

$$= \vec{v}(t) + \int_0^t A^{-1/2}Q^* \left\{ \mathrm{e}^{-a(t-s)} I - \int_s^t M(t-\tau)\mathrm{e}^{-a(\tau-s)}\, d\tau \right\} B\vec{v}(s)\, ds$$

$$= \vec{v}(t) + \int_0^t A^{-1/2}Q^* R(t,s) B\vec{v}(s)\, ds =: \vec{w}(t) \in \mathcal{D}(A), \qquad (1.21)$$

$$R(t,s) := \mathrm{e}^{-a(t-s)} I - \int_s^t \mathrm{e}^{-a(\tau-s)} M(t-\tau)\, d\tau.$$

Analogously as in Theorem 1.1, introduce the space $H(A) := (\mathcal{D}(A), \|\cdot\|_{H(A)})$, where $\|\vec{v}\|_{H(A)} := \|A\vec{v}\|_{\vec{L}_2(\Omega,\rho_0)}$. Equation (1.21), considered in $H(A)$, is the Volterra integral equation of the second kind with the right side of the equation $\vec{w}(t)$ which is continuous in $H(A)$. We show that $A^{-1/2}Q^*R(t,s)B \in C,\ 0 \le s \le t < +\infty; \mathcal{L}(H(A))$.

By Lemma 1.3, $BA^{-1} \in \mathcal{L}(\vec{L}_2(\Omega,\rho_0), W^1_{2,\rho_0}(\Omega))$. It follows from Lemma 1.4 that $A^{1/2}Q^*|_{W^1_{2,\rho_0}(\Omega)=\mathcal{D}(B^*)} = B^* \in \mathcal{L}(W^1_{2,\rho_0}(\Omega), \vec{L}_2(\Omega,\rho_0))$. Finally, it follows from the properties of the operator function $M(t)$ that the kernel $R(t,s)$ is a continuous operator function with values in $\mathcal{L}(W^1_{2,\rho_0}(\Omega))$. Now estimate the norm

$$\|A^{-1/2}Q^*R(t,s)B\vec{v}\|_{H(A)} = \|A^{1/2}Q^*R(t,s)B\vec{v}\| = \|B^*R(t,s)(BA^{-1})A\vec{v}\|$$
$$\le \|B^*\|_{\mathcal{L}(W^1_{2,\rho_0},\vec{L}_2)} \|R(t,s)\|_{\mathcal{L}(W^1_{2,\rho_0})} \|BA^{-1}\|_{\mathcal{L}(\vec{L}_2,W^1_{2,\rho_0})} \|\vec{v}\|_{H(A)} =: const\|\vec{v}\|_{H(A)}.$$

This implies that Eq. (1.21) is the Volterra integral equation of the second kind with continuous kernel. It follows from inclusion $\vec{w}(t) \in C(\mathbb{R}_+; H(A))$ that equation (1.21) has the unique solution $\vec{v}(t) \in C(\mathbb{R}_+; H(A))$. Thus, the solution component $\vec{v}(t) \in \mathcal{D}(A)$ and therefore, in system (1.20), the brackets before the operator A can be removed. As a result, $\xi(t)$ is the solution of equation (1.14) with non closed operator. Using the inverse replacement in (1.14), we obtain that $\zeta(t) = \mathrm{e}^{at}\xi(t)$ is the unique strong solution of the Cauchy problem (1.13). ■

1.4. Reduction to the first-order differential equation in the case of the kernel of a special type. Let us consider the case when $\widetilde{K}(t) = \sum_{l=1}^m k_l(x)\exp(-b_l t)$, and $k_l(x) > 0$ $(l = \overline{1,m})$ are some structural functions that are assumed to be continuously differentiable in the domain Ω. In this case the operator function $M(t)$ from (1.12) becomes $M(t)=\sum_{l=1}^{m}\exp(-b_l t)M_l$, where $M_l\rho(t,x):=\Pi\rho_0(z)k_l(x)\Pi\rho(t,x)$. Obviously, the operators M_l $(l=\overline{1,m})$ are bounded self-adjoint and positive definite in $L_{2,\rho_0}(\Omega)$.

In system (1.12) make the changes

$$\rho_l(t):=\int_0^t \exp(-b_l(t-s))M_l\rho(s)\,ds, \quad \rho_l(0)=0 \quad (\overline{l=1,m}),$$

and represent the obtained system of differential equations with initial conditions as a Cauchy problem for a first-order differential equation in the Hilbert space $\mathcal{H}:=\vec{L}_2(\Omega,\rho_0)\oplus L_{2,\rho_0}(\Omega)\oplus\widehat{H}$, where $\widehat{H}:=\oplus_{l=1}^{m}L_{2,\rho_0}(\Omega)$,

$$\frac{d}{dt}\begin{pmatrix}\vec{u}\\ \rho\\ \widehat{\rho}\end{pmatrix}+\begin{pmatrix}2\omega_0 iS+A & -B^* & \widehat{B}^*\\ B & 0 & 0\\ 0 & -\widehat{M} & \widehat{I}_b\end{pmatrix}\begin{pmatrix}\vec{u}\\ \rho\\ \widehat{\rho}\end{pmatrix}=\begin{pmatrix}\vec{f}\\ 0\\ 0\end{pmatrix}, \quad \begin{pmatrix}\vec{u}(0)\\ \rho(0)\\ \widehat{\rho}(0)\end{pmatrix}=\begin{pmatrix}\vec{u}^0\\ \rho^0\\ 0\end{pmatrix}. \quad (1.22)$$

The following notations are used here: $\widehat{\rho}:=(\rho_1;\ldots;\rho_m)^\tau$, $\widehat{B}^*:=(B^*,\ldots,B^*)$, $\widehat{M}:=(M_1,\ldots,M_m)^\tau$, $\widehat{I}_b:=\mathrm{diag}(b_1I,\ldots,b_mI)$.

We change the sough function in problem (1.22): $(\vec{u}(t);\rho(t);\widehat{\rho}(t))^\tau=\mathrm{e}^{at}(\vec{v}(t);q(t);\widehat{q}(t))^\tau$. Then this problem is transformed into the Cauchy problem

$$\frac{d\xi}{dt}+(\mathcal{A}+\mathcal{S})\xi=\mathcal{F}(t), \quad \xi(0)=\xi^0, \qquad (1.23)$$

where $\xi(t):=(\vec{v}(t);q(t);\widehat{q}(t))^\tau$, $\xi^0:=(\vec{u}^0;\rho^0;0)^\tau$, $\mathcal{F}(t):=(\mathrm{e}^{-at}\vec{f}(t);0;0)^\tau$,

$$\mathcal{A}:=\begin{pmatrix}A & -B^* & \widehat{B}^*\\ B & aI & 0\\ 0 & 0 & a\widehat{I}\end{pmatrix}, \quad \mathcal{S}:=\begin{pmatrix}2\omega_0 iS+aI & 0 & 0\\ 0 & 0 & 0\\ 0 & -\widehat{M} & \widehat{I}_b\end{pmatrix}.$$

For the operator A the lemma below is valid.

Lemma 1.6. *Let $a>4^{-1}\|Q\|^2$. Then the operator $\mathcal{A}$ admits a closure to a maximal uniformly accretive operator $\overline{\mathcal{A}}$ and can be presented in the symmetric form*

$$\overline{\mathcal{A}}=\mathrm{diag}(A^{1/2},I,\widehat{I})\begin{pmatrix}I & -Q^* & \widehat{Q}^*\\ Q & aI & 0\\ 0 & 0 & a\widehat{I}\end{pmatrix}\mathrm{diag}(A^{1/2},I,\widehat{I}), \quad \widehat{Q}^*:=(Q^*,\ldots,Q^*)$$

and in the Schur–Frobenius form $\overline{\mathcal{A}}=(\mathcal{I}+\mathcal{D}_1)\mathrm{diag}(A,aI+QQ^*,a\widehat{I})(\mathcal{I}+\mathcal{D}_2)$, where $\mathcal{I}$ is a unit operator in $\mathcal{H}$, and the operators $\mathcal{I}+\mathcal{D}_1$ and $\mathcal{I}+\mathcal{D}_2$ are *lower and upper triangular blocks, respectively. The distinct from zero components of the operator blocks* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *have the forms* $(\mathcal{D}_1)_{2,1} := QA^{-1/2}$, $(\mathcal{D}_2)_{1,2} := -A^{-1/2}Q^*$, $(\mathcal{D}_2)_{1,3} := A^{-1/2}\widehat{Q}^*$, $(\mathcal{D}_2)_{2,3} := -(aI + QQ^*)^{-1}Q\widehat{Q}^*$. *The domain of the operator* $\overline{\mathcal{A}}$ *has the form* $\mathcal{D}(\overline{\mathcal{A}}) = \{q \in L_{2,\rho_0}(\Omega), \widehat{q} \in \widehat{H} : \ \vec{v} - A^{-1/2}Q^*q + A^{-1/2}\widehat{Q}^*\widehat{q} \in \mathcal{D}(A)\}$.

The proof of Lemma 1.6 is similar to that of Lemma 1.5.

Basing on Eq. (1.23), as in the previous point, it is possible to prove Theorem 1.2 of strong solvability of initial-boundary value problem (1.4)–(1.6).

Let us consider the homogeneous equation (1.23) with the closed operator $\overline{\mathcal{A}}$. We search for its solution in the following form: $\xi(t) = \exp(-(\lambda + a)t)\xi$. As a result, we get the spectral problem $(\overline{\mathcal{A}} + \mathcal{S} - a\mathcal{I})\xi = \lambda\xi$ which we will associate with the problem on normal oscillations of the viscous rotating relaxing fluid. The last one will be studied in the next paper.

The author expresses his gratitude to Prof. N.D. Kopachevsky for fruitful discussions of the paper.

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