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Asymmetrical Bimodal Distributions with Screw Modes

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The Boltzmann equation for the model of hard spheres is considered. Approximate bimodal solutions for the Boltzmann equation are built for the case when the Maxwellian modes are screws with different degrees of infinitesimality of angular velocities. Some sufficient conditions for the minimization of the uniform-integral remainder between the sides of the Boltzmann equation are obtained.

Key words: hard spheres, Boltzman equation, Maxwellian, screws, remainder, bimodal distribution.

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1. Introduction

To describe the interaction between flows of a gas of hard spheres, the kinetic integro-differential Boltzmann equation [1 – 3] is used

$$D(f) = Q(f, f), \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}, \tag{2}$$

$$\begin{aligned} Q(f, f) = \frac{d^2}{2} \int\limits_{R^3} dv_1 \int\limits_{\Sigma} d\alpha B(v - v_1, \alpha)[f(t, v_1', x)f(t, v', x) \\ - f(t, v_1, x)f(t, v, x)], \end{aligned} \tag{3}$$

where $f(t, v, x)$ is the distribution to be found, $\partial f/\partial x$ is its spatial gradient, $t \in R^1$ is the time, $x = (x^1, x^2, x^3) \in R^3$ is the position; $v = (v^1, v^2, v^3) \in R^3$ is the velocity of a molecule, $d > 0$ is its diameter, v and v_1 are the velocities of the molecules before the collision, v' and v_1' are the velocities of particles after the collision, $\alpha \in \Sigma$, where Σ is the unit sphere in R^3. The velocities of the particles after the collision are defined by the formulae

$$v' = v - \alpha(v - v_1, \alpha), \quad v_1' = v + \alpha(v - v_1, \alpha). \tag{4}$$

There are the well-known exact solutions of (1)–(4) in the form of global and local Maxwellians [1–3]. Much progress has been made but only in the special case of Maxwell molecules and some generalizations.

In [4–8], the bimodal distributions, i.e. the linear combinations of two Maxwellians, in particular, the local Maxwellians of special form describing the screw-shaped stationary equilibrium states of a gas (in short-screws or spirals), were considered. They have the form [1,8]:

$$M(v, x) = \rho_0 e^{\beta\omega^2 r^2} \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v - \overline{v} - [\omega \times x])^2}. \tag{5}$$

Physically, distribution (5) corresponds to the situation when the gas has an inverse temperature $\beta = 1/2T$ and rotates in whole as a solid body with the angular velocity $\omega \in R^3$ around its axis on which the point $x_0 \in R^3$ lies,

$$x_0 = \frac{[\omega \times \overline{v}]}{\omega^2}. \tag{6}$$

The square of the distance from the axis of rotation is

$$r^2 = \frac{1}{\omega^2}[\omega \times (x - x_0)]^2, \tag{7}$$

and the density of the gas has the form

$$\rho = \rho_0 e^{\beta\omega^2 r^2} \tag{8}$$

(ρ_0 is the density on the axis, that is with $r = 0$), $\overline{v} \in R^3$ is the linear mass velocity for x for which $x||\omega$, and $\overline{v} + [\omega \times x]$ is the mass velocity in the arbitrary point x. It is easy to see that formula (5) gives not only a rotational, but also a translational movement along the axis with the linear velocity

$$\frac{(\omega, \overline{v})}{\omega^2}\omega.$$

Thus, it really describes a spiral movement of the gas in general, moreover, (5) is stationary (independent of t) but inhomogeneous.

As, for instance, in [8], we will also consider an inhomogeneous, non-stationary linear combination of two Maxwellians, i.e. a distribution

$$f = \varphi_1 M_1 + \varphi_2 M_2 = \sum_{k=1}^{2} \varphi_i(t,x) M_i(v,x), \tag{9}$$

$$M_i(v,x) = \rho_i e^{\beta_i \omega_i^2 r_i^2} \left(\frac{\beta_i}{\pi}\right)^{\frac{3}{2}} e^{-\beta_i (v-\widetilde{v_i})^2}. \tag{10}$$

$$\widetilde{v_i} = \widetilde{v_i}(x) = \overline{v_i} + [\omega_i \times x],\ i = 1,2. \tag{11}$$

It is assumed that the coefficient functions $\varphi_i, i = 1,2$, are non-negative and belong to $C^1(R^4)$. It is required to find φ_i and the behavior of all parameters so that the mixed remainder [4–8], i.e. the functional of the form

$$\Delta = \sup_{(t,x)\in R^4} \int\limits_{R^3} |D(f) - Q(f,f)| dv, \tag{12}$$

becomes vanishingly small.

We will find the coefficient functions φ_i,

$$\varphi_i(t,x) = \psi_i(t,x) e^{-\beta_i \omega_i^2 r_i^2},\ i = 1,2, \tag{13}$$

and suppose that the angular velocities have the following form:

$$\omega_i = \frac{\omega_{0i} s_i}{\beta_i^{k_i}},\ \ i = 1,2, \tag{14}$$

where $s_i > 0$ are any constants, ω_{0i} are arbitrary fixed vectors, $k_i > 0, i = 1,2$ (the other parameters are also arbitrary and fixed so far).

Some approximate solutions of a given kind, for which the Maxwellians with $i = 1$ and $i = 2$ behave in the same way, were obtained in [8]. The two angular velocities ω_1 and ω_2 were supposed to tend to zero equally fast with β_1, $\beta_2 \to +\infty$ (the speeds of tending to zero are different and are equal to the selected degrees of β_i in (14), namely 1, $\frac{1}{2}$ or $\frac{1}{4}$).

The aim of the present paper is to consider other possible values of k_i, $i = 1,2$, and asymmetrical (i.e. for different degrees with different i) behavior of the angular velocities.

2. Main Results

Before formulating and proving the results, we recall some denotations introduced in [8] to be used below:

$$A_i(u,t,x) = \psi_i\psi_j\rho_j \frac{d^2}{\sqrt{\pi}} \int\limits_{R^3} dw e^{-w^2} \Bigg| \frac{u}{\sqrt{\beta_i}} + (\overline{v_i} - \overline{v_j}) + [(\omega_i - \omega_j) \times x] - \frac{w}{\sqrt{\beta_j}} \Bigg|, \tag{15}$$

$$B_i(u,t,x) = \frac{\partial \psi_i}{\partial x}\left(\frac{u}{\sqrt{\beta_i}} + \overline{v_i} - [\omega_i \times x]\right) + 2\psi_i\sqrt{\beta_i}\{(u, [\omega_i \times \overline{v_i}]) - [\omega_i \times u][\omega_i \times x]\}. \tag{16}$$

Due to the above it is possible to formulate the first theorem.

Theorem 1. *Let conditions* (13), (14) *be valid. Let the functions*

$$\psi_i, \frac{\partial \psi_i}{\partial t}, \left|\frac{\partial \psi_i}{\partial x}\right|, |[\omega_{0i} \times x]|\psi_i, \left([\omega_{0i} \times x], \frac{\partial \psi_i}{\partial x}\right), \quad i = 1,2 \tag{17}$$

be bounded with respect to t, x on R^4.

Then the remainder Δ in (12) *is correctly defined, and there is a value Δ' that*

$$\Delta \le \Delta'. \tag{18}$$

If

$$\frac{1}{2} < k_i \le 1, \quad i = 1,2, \tag{19}$$

or

$$\frac{1}{4} < k_i \le \frac{1}{2}, \quad i = 1,2, \tag{20}$$

and

$$[\omega_{0i} \times \overline{v_i}] = 0, \quad i = 1,2, \tag{21}$$

then there is the finite limit

$$L = \lim_{\substack{\beta_i \to +\infty \\ i=1,2}} \Delta' = \sum_{\substack{i,j=1 \\ i \ne j}}^{2} \rho_i \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} + \overline{v_i}\frac{\partial \psi_i}{\partial x} + \rho_j \pi d^2 \psi_1\psi_2 |\overline{v_i} - \overline{v_j}| \right| + 2\pi d^2 \rho_1\rho_2 |\overline{v_1} - \overline{v_2}| \sup_{(t,x)\in R^4} (\psi_1\psi_2). \tag{22}$$

To prove Theorem 1 we need the following lemma [8], which gives a sufficient condition for the continuity of supremum of the function of special kind of many variables. The supremum is taken respectively to a part of variables.

L e m m a. *Suppose the following conditions:*
1) $\forall z \in Z$, *the function* $g(y,z)$ *is bounded on* Y*;*
2) $g(y,z)$ *is continuous by* z *uniformly with respect to* y*,i.e.,*

$$\forall z_0 \in Z, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in Y, \forall z \in Z,$$

$$|z - z_0| < \delta \Rightarrow |g(y,z) - g(y,z_0)| < \varepsilon$$

are valid for the function $g(y,z) : Y \times Z \to R^1; Y \in R^p; Z \in R^q$.

Then the function

$$l(z) = \sup_{y \in Y} |g(y,z)|$$

is continuous on the variable $z \in Z$.

P r o o f of Theorem 1. According to (15) and (16), we write the inequality obtained in [8]

$$\int_{R^3} |D(f) - Q(f,f)| dv \le \sum_{i=1}^{2} \int_{R^3} \left[\left| \frac{\partial \psi_i}{\partial t} + B_i(u,t,x) + A_i(u,t,x) \right| \right. \\ \left. + A_i(u,t,x) \right] \cdot \frac{\rho_i}{\pi^{3/2}} e^{-u^2} du. \tag{23}$$

From (12), (15)–(17), (23) and the properties of supremum, there follows the existence of the remainder Δ, and the following holds:

$$\Delta \le \Delta' = \sum_{i=1}^{2} \frac{\rho_i}{\pi^{3/2}} \int_{R^3} \left[\sup_{(t,x) \in R^4} \left| \frac{\partial \psi_i}{\partial t} + B_i(u,t,x) + A_i(u,t,x) \right| \right. \\ \left. + \sup_{(t,x) \in R^4} A_i(u,t,x) \right] e^{-u^2} du. \tag{24}$$

If we substitute (14) into (15), (16) and introduce the new denotation

$$\gamma = (\gamma_1, \gamma_2) = \left(\frac{1}{\sqrt{\beta_1}}, \frac{1}{\sqrt{\beta_2}} \right), \tag{25}$$

then

$$A_i(u,t,x) = \psi_i \psi_j \rho_j \frac{d^2}{\sqrt{\pi}} \int_{R^3} dw e^{-w^2} |\gamma_i u + (\overline{v_i} - \overline{v_j}) + s_i \gamma_i^{2k_i} [\omega_{0i} \times x] \\ - s_j \gamma_j^{2k_j} [\omega_{0j} \times x] - \gamma_j w|, \tag{26}$$

$$
\begin{aligned}
B_i(u,t,x) = \frac{\partial \psi_i}{\partial x}\Big(\gamma_i u + \overline{v_i} + s_i \gamma_i^{2k_i}[\omega_{0i} \times x]\Big) \\
+ 2\psi_i s_i \gamma_i^{2k_i-1}\Big\{(u,[\omega_{0i}\times \overline{v_i}]) - s_i\gamma_i^{2k_i}[\omega_{0i}\times u][\omega_{0i}\times x]\Big\},
\end{aligned}
\tag{27}
$$

where $i = 1, 2$, $i \neq j$.

Apply the aforesaid lemma to every supremum contained in (24), where $y = (t,x), Y = R^4, z = (u,\gamma), Z = R^3 \times R^2_+$. Fulfillment of the first condition follows from (17), (26), (27), and the second condition of the lemma can be verified with the help of (17), polynomial structure of (27) with respect to variables u and γ, and uniform convergence of the integral (26) by u and γ at any compact, and with respect to t, x — on the whole space R^4.

Then we can see that every integral in (24) is taken from the function which is continuous by u, γ and converges uniformly with respect to the variables γ at any compact, because there is an integrating majorant. Hence, in general, the value Δ' is continuous by γ on R^2_+. So, in (24) we can pass to the limit with $\beta_1, \beta_2 \to +\infty$, that is equivalent to the tending of γ_i, $i = 1, 2$ to zero in (26), (27).

Thus,

$$
\begin{aligned}
\lim_{\substack{\beta_i\to+\infty\\ i=1,2}} \Delta' = \sum_{\substack{i,j=1\\ i\neq j}}^{2} \frac{\rho_i}{\pi^{3/2}} \int_{R^3} \Bigg[\sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} + \overline{v_i}\frac{\partial \psi_i}{\partial x} + \psi_i\psi_j\rho_j \frac{d^2}{\sqrt{\pi}}\pi^{3/2}|\overline{v_i}-\overline{v_j}| \right| \\
+ \sup_{(t,x)\in R^4} \psi_i\psi_j\rho_j \pi d^2 |\overline{v_i}-\overline{v_j}| \Bigg] e^{-u^2} du.
\end{aligned}
\tag{28}
$$

From (28), as a result of integration by u, there follows (22). The theorem is proved.

In this theorem the behavior of the angular velocity with $i = 1$ is identical to that with $i = 2$. Now we will obtain possible results for the case of asymmetrical behavior of ω_1 and ω_2.

Theorem 2. *Let condition* (14) *be valid with*

$$
k_1 = 1, \qquad k_2 = \frac{1}{2}. \tag{29}
$$

Thereby, if conditions (17) *are valid, then inequality* (18) *holds true. Moreover,*

$$
\lim_{\substack{\beta_i\to+\infty\\ i=1,2}} \Delta' = L + \frac{4}{\sqrt{\pi}}\rho_2 s_2 |[\omega_{02}\times\overline{v_2}]| \sup_{(t,x)\in R^4} \psi_2. \tag{30}
$$

P r o o f. Let us use estimation (24) without introducing the denotation for its right-hand side yet. Using again the denotations of (25), we will substitute (14) into (15) and (16).

Then, instead of (26), (27) we will obtain

$$A_1(u,t,x) = \psi_1\psi_2\rho_2 \frac{d^2}{\sqrt{\pi}} \int\limits_{R^3} dw e^{-w^2} |\gamma_1 u + (\overline{v_1} - \overline{v_2}) + s_1\gamma_1^2[\omega_{01}\times x] - s_2\gamma_2^2[\omega_{02}\times x] - \gamma_2 w|, \tag{31}$$

$$B_1(u,t,x) = \frac{\partial\psi_1}{\partial x}\left(\gamma_1 u + \overline{v_1} + s_1\gamma_1^2[\omega_{01}\times x]\right) + 2\psi_1 s_1\gamma_1\left\{(u,[\omega_{01}\times\overline{v_1}]) - s_1\gamma_1^2[\omega_{01}\times u][\omega_{01}\times x]\right\}, \tag{32}$$

$$A_2(u,t,x) = \psi_1\psi_2\rho_1 \frac{d^2}{\sqrt{\pi}} \int\limits_{R^3} dw e^{-w^2} |\gamma_2 u + (\overline{v_2} - \overline{v_1}) + s_2\gamma_2[\omega_{02}\times x] - s_1\gamma_1[\omega_{01}\times x] - \gamma_1 w|, \tag{33}$$

$$B_2(u,t,x) = \frac{\partial\psi_2}{\partial x}\left(\gamma_2 u + \overline{v_2} + s_2\gamma_2[\omega_{02}\times x]\right) + 2\psi_2 s_2\left\{(u,[\omega_{02}\times\overline{v_2}]) - s_2\gamma_2[\omega_{02}\times u][\omega_{02}\times x]\right\}. \tag{34}$$

By substituting (31)–(34) into (24), we can obtain the estimation

$$\Delta \le \Delta' = \frac{\rho_1}{\pi^{3/2}} \int\limits_{R^3} \Bigg[\sup_{(t,x)\in R^4} \Bigg| \frac{\partial\psi_1}{\partial t} + A_1(u,t,x) + \frac{\partial\psi_1}{\partial x}(\gamma_1 u + \overline{v_1} + s_1\gamma_1^2[\omega_{01}\times x]) + 2\psi_1\gamma_1 s_1\left\{(u,[\omega_{01}\times\overline{v_1}]) - s_1\gamma_1^2[\omega_{01}\times u][\omega_{01}\times x]\right\}\Bigg| + \sup_{(t,x)\in R^4} A_1(u,t,x)\Bigg] e^{-u^2}du + \frac{\rho_2}{\pi^{3/2}} \int\limits_{R^3} \Bigg[\sup_{(t,x)\in R^4} \Bigg| \frac{\partial\psi_2}{\partial t} + A_2(u,t,x) + \frac{\partial\psi_2}{\partial x}(\gamma_2 u + \overline{v_2} + s_2\gamma_2[\omega_{02}\times x]) - 2\psi_2 s_2^2\gamma_2[\omega_{02}\times u][\omega_{02}\times x]\Bigg| + \sup_{(t,x)\in R^4} A_2(u,t,x) + \sup_{(t,x)\in R^4} 2\psi_2 s_2|u|\,|[\omega_{02}\times\overline{v_2}]|\Bigg] e^{-u^2}du. \tag{35}$$

The application of the lemma to each of the supremums in (35) (similarly as in the proof of Theorem 1, where the last summand under integral in (35) does not depend on β_2, namely γ_2 at all) allows to pass to the limit in (35) with $\beta_i \to +\infty$, $i = 1,2$. Then the result will differ from (22) by the only mentioned last addendum calculation of which leads to the following integral:

$$I = \int\limits_{R^3} |u| e^{-u^2} du.$$

The integral can be easily calculated by passing to spherical system of coordinates.
All the above leads to (30). The theorem is proved.

Theorem 3. *Let conditions* (14) *and* (17) *of Theorem 1 be valid with*

$$k_1 = \frac{1}{2}, \qquad k_2 = \frac{1}{4}. \tag{36}$$

Moreover,

$$[\omega_{02} \times \overline{v_2}] = 0. \tag{37}$$

Then the analogue of (22) below holds true

$$\begin{aligned}\lim_{\substack{\beta_i \to +\infty \\ i=1,2}} \Delta' = L + \frac{4}{\sqrt{\pi}} \rho_1 s_1 |[\omega_{01} \times \overline{v_1}]| \sup_{(t,x) \in R^4} \psi_1 \\ + \frac{4}{\sqrt{\pi}} \rho_2 s_2^2 |\omega_{02}| \sup_{(t,x)\in R^4} (|[\omega_{02} \times x]| \psi_2).\end{aligned} \tag{38}$$

P r o o f. Let us again use estimation (24) without introducing the denotation Δ' for its right-hand side, and in (15), (16), using the denotations from (25), substitute (14) with $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{4}$.

Then we obtain:

$$\begin{aligned} A_1(u,t,x) = \psi_1 \psi_2 \rho_2 \frac{d^2}{\sqrt{\pi}} \int_{R^3} dw e^{-w^2} |\gamma_1 u + (\overline{v_1} - \overline{v_2}) + s_1 \gamma_1 [\omega_{01} \times x] \\ - s_2 \gamma_2 [\omega_{02} \times x] - \gamma_2 w|, \end{aligned} \tag{39}$$

$$\begin{aligned} B_1(u,t,x) = \frac{\partial \psi_1}{\partial x} (\gamma_1 u + \overline{v_1} + s_1 \gamma_1 [\omega_{01} \times x]) \\ + 2\psi_1 s_1 \{(u, [\omega_{01} \times \overline{v_1}]) - s_1 \gamma_1 [\omega_{01} \times u][\omega_{01} \times x]\}, \end{aligned} \tag{40}$$

$$\begin{aligned} A_2(u,t,x) = \psi_1 \psi_2 \rho_1 \frac{d^2}{\sqrt{\pi}} \int_{R^3} dw e^{-w^2} |\gamma_2 u + (\overline{v_2} - \overline{v_1}) + s_2 \sqrt{\gamma_2} [\omega_{02} \times x] \\ - s_1 \sqrt{\gamma_1} [\omega_{01} \times x] - \gamma_1 w|, \end{aligned} \tag{41}$$

$$\begin{aligned} B_2(u,t,x) = \frac{\partial \psi_2}{\partial x} (\gamma_2 u + \overline{v_2} + s_2 \sqrt{\gamma_2} [\omega_{02} \times x]) \\ - 2\psi_2 s_2^2 [\omega_{02} \times u][\omega_{02} \times x]. \end{aligned} \tag{42}$$

By substituting (39)–(42) into (24,) we will obtain the following estimation:

$$\Delta \le \Delta' = \frac{\rho_1}{\pi^{3/2}} \int\limits_{R^3} \Big[\sup_{(t,x)\in R^4} \Big| \frac{\partial \psi_1}{\partial t} + A_1(u,t,x) + \frac{\partial \psi_1}{\partial x}(\gamma_1 u + \overline{v_1} + s_1\gamma_1[\omega_{01}\times x])$$
$$- 2\psi_1 s_1^2[\omega_{01}\times u][\omega_{01}\times x]\Big| + \sup_{(t,x)\in R^4} A_1(u,t,x)$$
$$+ \sup_{(t,x)\in R^4} 2\psi_1 s_1|u|\big|[\omega_{01}\times \overline{v_1}]\big|\Big] e^{-u^2}du + \frac{\rho_2}{\pi^{3/2}} \int\limits_{R^3} \Big[\sup_{(t,x)\in R^4} \Big| \frac{\partial \psi_2}{\partial t}$$
$$+ A_2(u,t,x) + \frac{\partial \psi_2}{\partial x}(\gamma_2 u + \overline{v_2} + s_2\sqrt{\gamma_2}[\omega_{02}\times x])\Big|$$
$$+ \sup_{(t,x)\in R^4} A_2(u,t,x) + \sup_{(t,x)\in R^4} \big|2\psi_2 s_2^2[\omega_{02}\times u][\omega_{01}\times x]\big|\Big] e^{-u^2}du. \tag{43}$$

From (39) and (41) we can see that the limit of the value $A_2(u,t,x)$ with $\gamma \to 0$ is the same as the limit of the value $A_1(u,t,x)$, and the estimation for the modulus having $B_i(u,t,x)$, contained in (23), leads to the separation of two summands independent of γ (i.e of $\beta_i, i = 1,2$), in which there appear the expressions

$$2s_1 \int\limits_{R^3} \sup_{(t,x)\in R^4} \Big[\psi_1|u|\big|[\omega_{01}\times \overline{v_1}]\big|\Big] \frac{\rho_1}{\pi^{3/2}} e^{-u^2}du, \tag{44}$$

$$2s_2^2 \int\limits_{R^3} \sup_{(t,x)\in R^4} \Big[\psi_2|\omega_{02}||u|\big|[\omega_{02}\times x]\big|\Big] \frac{\rho_2}{\pi^{3/2}} e^{-u^2}du. \tag{45}$$

Further calculation of (44) and (45), analogously as in the proof of Theorem 2, leads to (38). The theorem is proved.

Theorem 4. *Let condition (14) be valid with*

$$k_1 = 1, \qquad k_2 = \frac{1}{4}. \tag{46}$$

Thereby, if conditions (17) and (37) are valid, then inequality (18) holds true, where

$$\lim_{\substack{\beta_i\to+\infty\\ i=1,2}} \Delta' = L + \frac{4}{\sqrt{\pi}}\rho_2 s_2^2|\omega_{02}| \sup_{(t,x)\in R^4} (|[\omega_{02}\times x]|\psi_2). \tag{47}$$

P r o o f is the same as that of Theorem 2. But now, owing to (14) with $k_1 = 1$, $k_2 = \frac{1}{4}$ and (37), instead of (33), (34), we get (41), (42) from (15), (16). Thus, $A_1(u,t,x), B_1(u,t,x)$ have the form of (31), (32). From (41), (42) we can see that the limit of the value $A_2(u,t,x)$ with $\gamma \to 0$ is the same as the limit in Theorem 3 and similarly to $A_1(u,t,x)$ from Theorem 2. Furthermore, the estimation for the modulus having $B_2(u,t,x)$, contained in (23), was obtained in the proof of Theorem 3. So, the result will differ from (22) only by the last addendum defined in (45). Further calculation leads to (47). The theorem is proved.

Now, by means of the expressions obtained for the limits with $\beta_i \to +\infty$, $i = 1, 2$, we can find the sufficient conditions under which the remainder Δ tends to zero. For more convenience, the conditions are formulated in the form of corollaries of Theorems 1–4. In what follows it is assumed that the passage to the limit $\beta_1, \beta_2 \to +\infty$ is performed.

Corollary 1. *Let all the assumptions of Theorem 1 be valid. Then the statement*

$$\Delta \to 0 \tag{48}$$

holds true if at least one of the conditions fulfils the following:

1) $\overline{v_1} = \overline{v_2} = 0$,

$$\psi_i = \psi_i(x), i = 1, 2, \tag{49}$$

where $\psi_i(x)$ *are any functions satisfying the requirement* (17)*;*

2) $\overline{v_1} = \overline{v_2} \neq 0$, *condition (21) is valid, and*

$$\psi_i = C_i\left([x \times \overline{v_i}]\right), \quad i = 1, 2, \tag{50}$$

where $C_i \geq 0$ *are any smooth and finite (i.e. with finite support) or fast decreasing functions on its vector arguments;*

3) $\overline{v_1} = \overline{v_2} \neq 0$, *condition* (21) *is valid, and*

$$\psi_i = C_i\left(x - \overline{v_i}t\right), \quad i = 1, 2, \tag{51}$$

where $C_i \geq 0$ *are the same as in item* 2)*;*

4) $\overline{v_1} = 0$, *the vectors* $\overline{v_2}$, ω_{01}, ω_{02} *are collinear,*

$$\psi_1 = \psi_1(t,x) = h\left([x \times \overline{v_2}]\right)\Bigg\{\lambda + C\left([x \times \overline{v_2}]\right) \times \left[-\pi d^2|\overline{v_2}|h\left([x \times \overline{v_2}]\right)\left(\frac{x^1}{\overline{v}_2^1}\left(\frac{\rho_2}{\mu} + \frac{\rho_1}{\lambda}\right) - \frac{\rho_2}{\mu}t\right)\right]\Bigg\}^{-1}, \tag{52}$$

$$\psi_2 = \psi_2(t,x) = \frac{1}{\mu}\Big(h\left([x \times \overline{v_2}]\right) - \lambda\psi_1(t,x)\Big), \tag{53}$$

where $\lambda, \mu > 0$ *are any constants, the functions* h *and* C *have the same properties as* C_i, $i = 1, 2$ *in* (50), *and, in addition,*

$$d \to 0; \tag{54}$$

5) $\overline{v}_1 \neq 0, \overline{v}_2 \neq 0$ *are arbitrary, equality* (21) *is valid, the functions* ψ_i, $i = 1, 2$, *have the form of* (50) *or* (51), *and*

$$supp\psi_1 \cap supp\psi_2 = \varnothing; \tag{55}$$

6) $\overline{v}_1 \neq 0, \overline{v}_2 \neq 0$ *are arbitrary, the functions* ψ_i, $i = 1, 2$, *have the form of* (50) *or* (51) *under the condition of* (21), *and* (54) *is valid.*

P r o o f of all items is based on (21) and (22).

1) All summands of the expressions having $\overline{v}_i, i = 1, 2$ become equal to zero, and $\frac{\partial \psi_i}{\partial t}$ is equal to zero too as $\psi_i = \psi_i(x)$, $i = 1, 2$. Consequently, the whole expression (22) is equal to zero.

2) The summands having the difference $|\overline{v}_i - \overline{v}_j|$, $i, j = 1, 2$, $i \neq j$, become equal to zero due to the given conditions. Let us consider the addendum having $\frac{\partial \psi_i}{\partial x}$. Since $\Big(C\big([x \times a]\big)\Big)'_x = [a \times C']$, where a is an arbitrary constant vector, then

$$\Big(\overline{v}_i, \big[\overline{v}_i, C_i'\big]\Big) = 0.$$

The condition (21) means that $\omega_{0i} || \overline{v}_i$. So in (50) we can write ω_{0i} instead of $\overline{v}_i$. Thus we obtain the boundedness in (17). The same is true for all the following items of this corollary.

3) In (22) all summands but one instantly become equal to zero due to the given conditions. However, this summand also disappears, because the function of the form of (51) satisfies the following system of equations:

$$\frac{\partial \psi_i}{\partial t} + \overline{v}_i \frac{\partial \psi_i}{\partial x} = 0. \tag{56}$$

4) Under the conditions mentioned above, the functions (52) and (53) are the solutions of the following system of differential equations (as shown in [6]):

$$\frac{\partial \psi_i}{\partial t} + \overline{v}_i \frac{\partial \psi_i}{\partial x} = -\rho_j \pi d^2 \psi_1 \psi_2 |\overline{v}_2|, \quad i, j = 1, 2, i \neq j. \tag{57}$$

All other summands tend to zero due to condition (54).

5) Under condition (55), every function $\psi_i \neq 0, i = 1, 2$, but the product $\psi_1 \psi_2 = 0$ identically.

The function of the form of (50) or (51) satisfies the system of equations (56). Thus, expression (22) is evidently equal to zero.

6) Conditions (54) and (55) lead to the fact that the following summands are equal or tend to zero:

$$\rho_j \pi d^2 \psi_1 \psi_2 |\overline{v}_i - \overline{v}_j|, \quad i,j = 1,2, i \neq j,$$

$$2\pi d^2 \rho_1 \rho_2 |\overline{v}_1 - \overline{v}_2| \sup_{(t,x)\in R^4} (\psi_1 \psi_2).$$

For the function ψ_i of the form of (50) or (51), the statement (56) is fulfilled. So, the equality of the expression (22) to zero is proved.

Corollary 2. *Let all the assumptions of Theorem 2 be valid. Then the statement* (48) *holds true if one of the conditions of Corollary 1 takes place and, in addition, at least one of the following requirements is fulfilled :*
1. $s_2 \to 0$*;*
2. *Condition* (21) *with* $i = 2$.

P r o o f is evident and leans on (30).

Corollary 3. *Let all the assumptions of Theorem 3 be valid. Then the statement* (48) *holds true if one of the conditions of Corollary 1 takes place and, in addition, at least one of the following requirements is fulfilled:*
1. $s_i \to 0, \quad i = 1, 2$*;*
2. $s_2 \to 0$*, condition* (21) *with* $i = 1$.

P r o o f is evident and leans on (38).

Corollary 4. *Let all the assumptions of Theorem 4 be valid. Then (48) holds true if at least one of the conditions of Corollary 1 takes place and, in addition, the first requirement of Corollary 2 is satisfied.*

P r o o f can be easily done by means of (47).

R e m a r k 1. It should be noted that the exact solutions of the Boltzman equation, which have the well-known physical sense, are the spiral Maxwell distributions (10), (11) themselves. However, the bimodal distributions (9), considered in the paper, give only an approximate description of interaction of such spirals in the sense of minimization of the remainder Δ. Nevertheless, they can be reasonably interpreted physically. In fact, the common property of all obtained distributions is that they describe the non-uniform cooling gas ($\beta_i \to +\infty$, $i = 1, 2$). Besides, the rotation of both spirals decelerates ($\omega_i \to 0$, $i = 1, 2$), although in different degrees in accordance with (14) and under the conditions of (19), (20), (29), (36), and (46) (in some cases, condition 1) of Corollary 3 is also assumed). At the same time, the distribution f itself does not tend to any of Maxwellians (i.e. to the known exact solution of the Boltzmann equations).

R e m a r k 2. Theorem 1 is devoted to the case of equal degrees of infinitesimality of both angular velocities ω_i in the most general case, namely, when $k_i \in (\frac{1}{4}, 1]$. It is worth being noted that, as we can see from Theorems 2–4, the consideration of different degrees of β_i with $i = 1$ and $i = 2$ leads to the appearance of "mixed" expressions of the limiting values for the estimation Δ'. These expressions include various summands mentioned in the paper.

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