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Small Oscillations of Magnetizable Ideal Fluid

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The small oscillations of a magnetizable ideal fluid in partially filled vessel are considered. Solvability of the initial-boundary problem is proved and the generic properties of the frequencies spectrum of normal free oscillations of fluid are determined. The principle of minimum of the potential energy in the problem on the stability of the fluid equilibrium states is proved.

Key words: magnetizable capillary fluid, equilibrium state, stability of equilibrium, small oscillations, spectrum of eigenfrequencies.

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Introduction

The magnetization of fluid in sufficiently strong magnetic field produces a variety of interesting physical effects. These effects are clearly observed in magnetic fluids (dispersions of ferro- or ferrimagnetic nanoparticles)and often called ferrofluids. The unique combination of ferromagnetic properties and fluidity generates a variety of forms of the free surface of ferrofluid in an external magnetic field.

Many authors considered stability and bifurcation of equilibrium forms of the free surface and the motion of magnetic fluids near the equilibrium state. The main results were obtained while studying the plane horizontal layers of ferrofluid in a homogeneous magnetic field, the axisymmetric forms of free surface in an azimuthal field and some other cases of consistent symmetry of the magnetic field and equilibrium configurations of ferrofluids. A complete review of these results can be found in [1–11] and bibliography therein.

In this paper we present a general statement of the problem of small fluctuations in magnetizable ideal (nonviscous) capillary fluid based on the ferrohydrodynamics equations [1]. We also prove the solvability of the initial-boundary problem and study the spectral problem of normal oscillations of fluid near equilibrium state.

1. Problem Statement

Let us consider a closed vessel at rest in a uniform gravitational field filled with homogeneous incompressible capillary ideal fluid and gas, which are also placed in a magnetic field. Assume that the fluid and gas are nonconductive and their ponderomotive interaction with the magnetic field is caused by magnetization of mediums. We neglect any motion of gas as well as the fluid viscosity. The oscillations of magnetizable viscous fluid were considered earlier [12].

We denote Ω_1 and Ω_2 to be the volumes filled with fluid and gas at equilibrium state, $\Omega_3 = R^3 \backslash \overline{\Omega}$ to be the unbounded volume outside the vessel, $\Omega := \Omega_1 \cup \Gamma \cup \Omega_2$, where Γ is the fluid equilibrium free surface. Let S be the closed surface of the vessel Ω; S_1 and S_2 be the surfaces of contact of the fluid and gas with the vessel wall.

For definiteness, we assume that the magnetic field is generated by currents distributed in the domain Ω_3, and the current density $\vec{j}(\vec{x})$, $\vec{x} \in \Omega_3$ remains invariant during fluid oscillations. For simplicity, the medium in domain Ω_3 is considered to be homogeneously magnetizable. Relation between the induction $\vec{B}^{(k)}$ and the field strength $\vec{H}^{(k)}$ in each of the domains Ω_k may be written in the form:

$$\vec{B}^{(k)} = \mu_0 \mu_k(H^{(k)}) \vec{H}^{(k)} = \mu_0(\vec{H}^{(k)} + \vec{M}^{(k)}), \qquad k = \overline{1,3},$$

where μ_0 is the magnetic constant, μ_k is the relative magnetic permeability of the k-medium, $\vec{M}^{(k)}$ is the magnetization of the k-medium. The functions $\mu_k : H \to \mu_k(H)$, $k = \overline{1,3}$, are considered to be given ones.

Let $\vec{x} = \vec{X}(\xi^1, \xi^2)$ be the equation of equilibrium free surface of the fluid Γ, ξ^1 and ξ^2 be the coordinate parameters of this surface. In the neighborhood of Γ we define the curvilinear coordinates $O\xi^1\xi^2\xi^3$ connected to the Cartesian coordinates by relation: $\vec{x} = \vec{X}(\xi^1, \xi^2) + \xi^3 \vec{n}(\xi^1, \xi^2)$, where $\vec{n}$ is the unit normal to the surface Γ. We agree the normals to the surface interface to be directed to the domain with a larger index.

We write the equation of oscillations of free surface of fluid $\Gamma(t)$ in the form: $\xi^3 = \zeta(t, \xi^1, \xi^2)$. Let $\vec{v}(t, \vec{x})$ and $\psi(t, \vec{x})$ be respectively the field of fluid velocities and the potential of perturbation of intensity of magnetic field generated by the fluid oscillations. The functions $\zeta(t, \xi^1, \xi^2)$, $\vec{v}(t, \vec{x})$, $\psi(t, \vec{x})$ are thought to be small quantities of first order.

In the linear approximation the functions $\zeta(t,\xi^1,\xi^2), \vec{v}(t,\vec{x}), \psi(t,\vec{x})$ should satisfy the following system of equations, boundary and initial conditions:

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\nabla(p + \mu_0 \vec{M}\cdot \bigtriangledown\psi) + \vec{f}, \quad \operatorname{div}\vec{v} = 0 \qquad \text{in } \Omega_1; \tag{1}$$

$$\frac{\partial \zeta}{\partial t} = v_n \ (:= \vec{n}\cdot\vec{v}) \qquad \text{on } \Gamma; \tag{2}$$

$$p^{(1)} + \mu_0 \vec{M}^{(1)}\cdot\nabla\psi^{(1)} = \sigma(-\triangle_\Gamma + a)\zeta + \left\{B_n(\vec{n}\cdot \overset{\wedge}{\nabla}\psi) - \vec{B}_\tau\cdot\nabla_\Gamma\psi\right\}_\Gamma \text{ on } \Gamma; \tag{3}$$

$$\frac{\partial\zeta}{\partial\nu} + \kappa\zeta = 0 \qquad \text{on } \partial\Gamma; \tag{4}$$

$$v_n = 0 \qquad \text{on } S_1; \tag{5}$$

$$\operatorname{div}(\mu^{(k)} \overset{\wedge}{\nabla}{}^{(k)}\psi^{(k)}) = 0 \qquad \text{in } \Omega_k, \quad k = \overline{1,3}; \tag{6}$$

$$\{\psi\}_\Gamma = \{H_n\}_\Gamma\,\zeta, \quad \left\{\mu_0\mu\ \vec{n}\cdot\overset{\wedge}{\nabla}\psi\right\}_\Gamma = -\left\{\operatorname{div}_\Gamma\zeta\vec{B}_\tau\right\}_\Gamma \text{ on } \Gamma; \tag{7}$$

$$\{\psi\}_S = 0, \quad \left\{\mu\vec{n}\cdot\overset{\wedge}{\nabla}\psi\right\}_S = 0 \qquad \text{on } S; \tag{8}$$

$$\psi(t,\vec{x}) \to 0 \quad \text{at} \quad |\vec{x}| \to \infty; \tag{9}$$

$$\vec{v}(0,\vec{x}) = \vec{v}_0(\vec{x}), \quad \zeta(0,\xi^1,\xi^2) = \zeta_0(\xi^1,\xi^2); \tag{10}$$

$$\kappa := \frac{k_\Gamma\cos\alpha + k_S}{\sin\alpha}, \qquad \overset{\wedge}{\nabla}{}^{(k)}(\cdot) := \left(\nabla + \frac{\mu_H^{(k)}\vec{H}^{(k)}}{\mu^{(k)}H^{(k)}}(\vec{H}^{(k)}\cdot\nabla)\right)(\cdot),$$

$$\mu^{(k)}(\vec{x}) := \mu_k(H^{(k)}(\vec{x})), \qquad \mu_H^{(k)}(\vec{x}) := \frac{d\mu_k(H^{(k)}(\vec{x}))}{dH}$$

$$a := -\frac{\rho}{\sigma}\vec{g}\cdot\vec{n} - (k_1^2 + k_2^2) + \frac{1}{\sigma}\left\{\sum_{\alpha,\beta=1}^{2} b^{\alpha\beta}\left(\frac{\partial\vec{X}}{\partial\xi^\alpha}\cdot\vec{H}\right)\left(\frac{\partial\vec{X}}{\partial\xi^\beta}\cdot\vec{B}\right) - (k_1+k_2)H_nB_n\right\}_\Gamma.$$

Here ρ is the fluid density; $\vec{g}$ is the uniform acceleration of gravitational force; $\vec{f}$ is the volume density perturbations of the external field of mass forces; σ is the coefficient of surface tension on Γ; H_n and $\vec{H}_\tau$ are respectively the projection on the normal and the tangent component of the magnetic field strength on Γ; k_1, k_2 are the main curvatures of surface Γ; $b^{\alpha\beta}$ are the components of the second fundamental form of surface Γ; k_Γ and k_S are the curvatures calculated on $\partial\Gamma$ of the sections of surfaces Γ and S by the plane perpendicular to $\partial\Gamma$; $\partial/\partial\nu$ is the derivative along the $\vec{\nu}$ to $\partial\Gamma$ in the tangent plane to Γ; α is the contact angle (dihedral angle formed by fluid at the contour points $\partial\Gamma$); $\nabla_\Gamma(\cdot)$ is the gradient of the scalar functions given on Γ; $\operatorname{div}_\Gamma(\cdot)$ is the surface divergence of the tangent field of vectors on Γ; $\Delta_\Gamma = \operatorname{div}_\Gamma\nabla_\Gamma$ is the Laplace–Beltrami operator on Γ.

The curvatures of the surfaces Γ and S are considered to be positive if the corresponding normal sections are convex in the direction of Ω_1. Curly brackets in (3), (7) and everywhere below denote the jump of the respective enclosed expression on the interface of the two mediums, $\{A\}_\Gamma := (A^{(2)} - A^{(1)})|_\Gamma$. Top index numbers are used to denote the mediums to which the quantity relates.

The first equation in (1) is a linearized equation of the oscillations of magnetizable fluid [1, 2]. The kinematic condition (2) and the condition of preservation of the contact angle (4) have the same form as in the case of ordinary capillary fluid (see, e.g., [13]). The condition (3) is obtained by linearization of the dynamic conditions for the jump of normal stresses on the fluid free surface caused by surface tension and polarization of magnetic forces field.

In the mathematical model used in [1], the magnetic field is determined by the following equations:

$$\nabla \times \vec{H}^{(k)}(\vec{x}, t) = \vec{j}(\vec{x}), \quad \nabla \cdot \vec{B}^{(k)}(\vec{x}, t) = 0, \quad \text{in} \quad \Omega_k(t), \quad k = \overline{1,3}.$$

Linearization of these equations under the assumption of invariance of current density $\vec{j}(\vec{x})$ during the fluid oscillations leads to equations (6). The equalities (7) and (8) in the linear approximation express the conditions of continuity of tangential components of tension and normal component of the magnetic field on the interface surfaces of the mediums Γ and S.

The function $\zeta(t, \xi^1, \xi^2)$ should satisfy the condition

$$\int\limits_\Gamma \zeta d\Gamma = 0 \qquad \forall t \geq 0, \tag{11}$$

which follows easily from the second equation (1) (expressing the condition of incompressibility of the fluid) and conditions of nonpenetrability of the vessel wall (5). Note also that the vector-function $\vec{v}_0(\vec{x})$ in the initial condition (10) should satisfy the second equations of (1) and condition (5).

2. Operator-Differential Formulation of Evolutionary Problem

Suppose that the fluid completely wets the vessel wall such that $\partial\Omega_1 = \Gamma \bigcup S$ and $\partial\Omega_2 = \Gamma$ (in this the case condition (4) is omitted). Assume that the surfaces Γ and S are homeomorphic to the sphere. Such an equilibrium state of ferrofluid can be easily seen during the experiments.

For brevity, the case when the surfaces Γ and S intersect is not considered here. However, the obtained below results on the solvability of evolution problem (1)–(10) and the structure of spectrum of eigenfrequencies of fluid oscillations can be extended on this case as well.

Further, the surfaces Γ, S and the vector-function $\vec{H}^{(k)}(\vec{x})$, $k = \overline{1,3}$, are assumed to be sufficiently smooth. We also assume that for equations (6) the conditions of uniform ellipticity are satisfied

$$m_0|\vec{\xi}|^2 \le \mu^{(k)}(\vec{x})|\vec{\xi}|^2 + \frac{\mu_H^{(k)}(\vec{x})}{H^{(k)}(\vec{x})}(\vec{\xi}\cdot\vec{H}^{(k)}(\vec{x}))^2 \le m^0|\vec{\xi}|^2, \tag{12}$$
$$\forall\vec{x}\in\Omega_k, \quad \forall\vec{\xi}\in R^3, \quad k=\overline{1,3},$$

where m_0, m^0 are some positive constants.

The vector-functions in (1) for $\forall t \ge 0$ are the elements of the Hilbert space $\vec{L}_2(\Omega_1)$. As known from [13], there is the orthogonal decomposition

$$\vec{L}_2(\Omega_1) = \vec{J}_0(\Omega_1) \oplus \vec{G}_{h,S}(\Omega_1) \oplus \vec{G}_{0,\Gamma}(\Omega_1),$$

where $\vec{J}_0(\Omega_1)$ is the subspace of the solenoidal vector fields with zero normal component on $\partial\Omega_1$, $\vec{G}_{h,S}(\Omega_1)$ is the subspace of the potential harmonic fields with zero normal component on S, and $\vec{G}_{0,\Gamma}(\Omega_1)$ is the subspace of the potential harmonic fields whose potential vanishes on the surface Γ.

By (1), (5), the velocity fields $\vec{v}$ belong to the subspace $\vec{J}_{0,S}(\Omega_1) := \vec{J}_0(\Omega_1) \oplus \vec{G}_{h,S}(\Omega_1)$ such that

$$\vec{v} = \vec{u} + \bigtriangledown\varphi, \quad \vec{u}\in\vec{J}_0(\Omega_1), \quad \bigtriangledown\varphi\in\vec{G}_{h,S}(\Omega_1). \tag{13}$$

Let P_0 be the operator of orthogonal projection on the medium $\vec{J}_0(\Omega_1)$ in $\vec{L}_2(\Omega_1)$. Applying P_0 to both sides of equation (1) and to initial condition (10), we get

$$\frac{d}{dt}\vec{u} = P_0\vec{f} \qquad (t>0), \qquad \vec{u}(0) = P_0\vec{v}_0. \tag{14}$$

Hence there can be found a solenoidal component of $\vec{u}$ of the fluid field velocities

$$\vec{u} := P_0\vec{v} = P_0\vec{v}_0 + \int_0^t (P_0\vec{f})(\tau)d\tau. \tag{15}$$

Let I be the unit operator in $\vec{L}_2(\Omega_1)$, and $I - P_0$ be the orthogonal projector on the subspace of the potential fields $\vec{G}(\Omega_1) := \vec{G}_{h,S}(\Omega_1) \oplus \vec{G}_{0,\Gamma}(\Omega_1)$. Extract the potential component $\nabla\chi := (I-P_0)\vec{f}$ of the perturbations $\vec{f}$ of external force field. Applying operator $(I - P_0)$ to (1), we get the Cauchy–Lagrange integral for small potential movements of magnetizable fluid

$$\rho\frac{\partial\varphi}{\partial t} + p + \mu_0\vec{M}\cdot\bigtriangledown\psi - \rho\chi = c(t), \tag{16}$$

where $c(t)$ is an arbitrary function of time. By the second equation in (1), the kinematic condition (2) and the nonpenetrability condition of the solid wall (5), the velocity potential φ satisfies the following equation and the boundary conditions:

$$\triangle\varphi = 0 \quad \text{in } \Omega_1, \quad \frac{\partial\varphi}{\partial n} = \frac{\partial\zeta}{\partial t} \quad \text{on } \Gamma, \quad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } S, \quad \int \varphi d\Gamma = 0. \tag{17}$$

Let $L_2(\Gamma)$ be the Hilbert space of the square-summable scalar functions defined on Γ, and $\mathcal{H}_0(\Gamma) := L_2(\Gamma) \ominus \{1\}$ be the subspace of the functions orthogonal to constants in $L_2(\Gamma)$. Note that $\zeta \in \mathcal{H}_0(\Gamma)\ \forall t \geq 0$ (11). Denote by $\mathcal{L}$ the elliptic differential operator of the second order defined by equality: $\mathcal{L}\zeta := (-\triangle_\Gamma + a((X))\zeta$. Define the operator $\mathcal{B}_0$

$$\mathcal{B}_0\zeta := \mathcal{P}_{\mathcal{H}}\mathcal{L}\zeta = \mathcal{L}\zeta - (\operatorname{mes}\Gamma)^{-1}\int\limits_\Gamma (\mathcal{L}\zeta)d\Gamma, \tag{18}$$
$$D(\mathcal{B}_0) = \mathcal{H}_0^2(\Gamma) := \mathcal{H}^2(\Gamma) \cap \mathcal{H}_0(\Gamma),$$

where $\mathcal{P}_{\mathcal{H}}$ is the orthogonal projector on the subspace $\mathcal{H}_0(\Gamma)$ in $L_2(\Gamma)$, $\mathcal{H}^2(\Gamma)$ is the Sobolev space of the scalar functions belonging to $L_2(\Gamma)$ together with the generalized derivatives up to the second order inclusively. As known [13], $\mathcal{B}_0$ is the selfadjoint bounded from below operator in $\mathcal{H}_0(\Gamma)$.

Denote by $\gamma_\Gamma^{(k)}$ the trace operator assigning to an arbitrary function $\psi^{(k)} \in \mathcal{H}^s(\Omega_k)$ its value on the surface Γ

$$\gamma_\Gamma^{(k)}\psi^{(k)} := \psi^{(k)}\big|_\Gamma \in \mathcal{H}^{s-1/2}(\Gamma), \quad k = \overline{1,2},$$

where $\mathcal{H}^s(\Omega_k)$, $\mathcal{H}^s(\Gamma)$, $s \geq 0$ are the Sobolev–Slobodetsky spaces [15]. The operator $\gamma_\Gamma^{(k)}$ is bounded from $\mathcal{H}^s(\Omega_k)$ to $\mathcal{H}^{s-1/2}(\Gamma)$ [15, 16].

Define the operators $\partial_\Gamma^{(k)}, \hat{\partial}_\Gamma^{(k)}$ on a set of the smooth functions by the equalities

$$\partial_\Gamma^{(k)}\psi^{(k)} := \vec{n}\cdot\nabla\psi^{(k)}\big|_\Gamma, \quad \hat{\partial}_\Gamma^{(k)} := \vec{n}\cdot\hat{\nabla}^{(k)}\psi^{(k)}\big|_\Gamma, \quad k = \overline{1,2}.$$

The operators $\partial_\Gamma^{(k)}, \hat{\partial}_\Gamma^{(k)}$ are extended by continuity to the bounded ones from $\mathcal{H}^s(\Omega_k)$ to $\mathcal{H}^{s-3/2}(\Gamma)$ [15, 16].

Define the Hilbert space $\mathcal{H}_0^s(\Gamma) := \mathcal{H}^s(\Gamma) \cap \mathcal{H}_0(\Gamma), s > 0$, and the space $(\mathcal{H}_0^s(\Gamma))^* := \mathcal{H}_0^{-s}(\Gamma)$ adjoint to $\mathcal{H}_0^s(\Gamma)$. Note that the equations (6) with the conditions (7)–(9) define uniquely the perturbation of the magnetic field potential $\psi := (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})$ for the given function $\zeta \in \mathcal{H}_0^{3/2}(\Gamma)$ and

$$\psi^{(k)} \in \mathcal{H}^2(\Omega_k), \quad k = 1,2, \quad \psi^{(3)} \in \mathcal{H}^2_{loc}(\Omega_3) \cap \mathcal{D}(\Omega_3),$$

where $\mathcal{D}(\Omega_3)$ is the closure of smooth functions by the Dirichlet norm $\|u\|_{\mathcal{D}(\Omega_3)} := \|\nabla u\|_{\vec{L}_2(\Omega_3)}$. The operator $\mathcal{M}$ assigning the function ζ to the function ψ denote by $\psi = \mathcal{M}\zeta$.

Define the operator $\mathcal{B}_1$

$$\mathcal{B}_1\zeta := \mathcal{P}_{\mathcal{H}}\left\{B_n(\hat{\partial}_\Gamma\, \mathcal{M}\zeta) - \vec{B}_\tau \cdot \nabla_\Gamma(\gamma_\Gamma \mathcal{M}\zeta)\right\}_\Gamma$$
$$= \left\{B_n\, \hat{\partial}_\Gamma\, \psi - \vec{B}_\tau \cdot \nabla_\Gamma \psi\right\}_\Gamma - (\operatorname{mes}\Gamma)^{-1} \int\limits_\Gamma \left\{B_n\, \hat{\partial}_\Gamma\, \psi - \vec{B}_\tau \cdot \nabla_\Gamma \psi\right\}_\Gamma d\Gamma. \quad (19)$$

The operator $\mathcal{B}_1$ is the bounded operator from $\mathcal{H}_0^{3/2}(\Gamma)$ to $\mathcal{H}_0^{1/2}(\Gamma)$ [12]. $\mathcal{B}_1$ is the unbounded symmetric operator in $\mathcal{H}_0(\Gamma)$ with the domain $D(\mathcal{B}_1) := \mathcal{H}_0^{3/2}(\Gamma)$ [12]. Define the operator $\mathcal{B}$ in $\mathcal{H}_0(\Gamma)$

$$\mathcal{B} := \mathcal{B}_0 + \mathcal{B}_1, \qquad D(\mathcal{B}) = \mathcal{H}^2(\Gamma) \cap \mathcal{H}_0(\Gamma). \quad (20)$$

Note that operator $\mathcal{B}$ can be Friedrichs extended to the selfadjoint semibounded operator $\mathcal{H}_0(\Gamma)$ with the domain $\mathcal{H}_0^1(\Gamma)$, [12]. Below this extension will be again denoted by $\mathcal{B}$.

Let $\mathcal{H}_0^s(\Omega_1)$ be the subspace of the functions $\varphi \in \mathcal{H}^s(\Omega_1)$ such that $\gamma_\Gamma^{(1)}\varphi \in \mathcal{H}_0^{s-1/2}(\Gamma)$. The unique solution $\varphi \in \mathcal{H}_0^1(\Omega_1)$ to the Neumann problems (17) for $\forall(d\zeta/dt|_\Gamma) \in \mathcal{H}_0^{-1/2}(\Gamma)$ exists [13]. Let T be the operator that assigns the value of normal derivative of Γ to the solution of problem (17). Following [13], we define the operator $\mathcal{C} := \gamma_\Gamma^{(1)} T$,

$$\varphi|_\Gamma = \gamma_\Gamma^{(1)} T \left(\left.\frac{\partial\varphi}{\partial n}\right|_\Gamma\right) = \mathcal{C}\frac{d\zeta}{dt}. \quad (21)$$

The operator $\mathcal{C}$ is a bounded operator from $\mathcal{H}_0^{-1/2}(\Gamma)$ to $\mathcal{H}_0^{1/2}(\Gamma)$; the restriction of $\mathcal{C}$ on the space $\mathcal{H}_0(\Gamma)$ is the selfadjoint positive compact operator in $\mathcal{H}_0(\Gamma)$ [13].

Writing the Cauchy–Lagrange integral (16) on the free surface Γ by using the dynamic condition (3), equalities (18)–(20) and the initial conditions (10), we get the Cauchy problem in the Hilbert space $\mathcal{H}_0(\Gamma)$ which describes small potential fluid motions near the equilibrium state

$$\rho\mathcal{C}\frac{d^2\zeta}{dt^2} + \sigma\mathcal{B}\zeta = \rho\chi_0(t) \quad (t > 0); \quad (22)$$

$$\zeta(0) = \zeta_0, \quad \frac{d\zeta(0)}{dt} = \zeta_0' := ((I - P_0)\vec{v}_0 \cdot \vec{n})|_\Gamma; \quad (23)$$

$$\chi_0(t) := \chi(t) + c(t), \quad c(t) := -\frac{1}{mes\,\Gamma}\int\limits_\Gamma \chi d\Gamma.$$

Thus, the general evolutionary problem on small motions of the magnetizable capillary fluid (1)–(10) reduces to two independent problems (14) and (22), (23).

As known [3], the equilibrium state of the system corresponds to the stationary value of the functional of potential energy Π such that $\delta\Pi(\Gamma;\zeta) = 0\ \forall\zeta \in \mathcal{H}_0^1(\Gamma)$, where $\delta\Pi$ is the first variation of the functional Π. It is easy to show that the second variation of the potential energy $\delta^2\Pi(\Gamma;\zeta)$ coincides (up to the factor) with the quadratic form of the operator $\mathcal{B}$. Thus, if the deviations of the free surface of fluid from its equilibrium state are small, there can be applied the equality

$$\Pi = \frac{1}{2}\delta^2\Pi(\Gamma;\zeta) = \frac{\sigma}{2}\left(\mathcal{B}\zeta,\zeta\right)_0. \tag{24}$$

Here and below by $(\cdot,\cdot)_0$ we denote the continuation of the scalar product in $L_2(\Gamma)$ to the adjoint spaces $\mathcal{H}_0^{-s}(\Gamma)$ and $\mathcal{H}_0^s(\Gamma)$.

The kinetic energy K of the potential motions of fluid is represented as a quadratic form associated with the operator $\mathcal{C}$ [13]

$$K = \frac{\rho}{2}\int\limits_{\Omega_1} |\nabla\varphi|^2 d\Omega = \frac{\rho}{2}\left(\mathcal{C}\frac{d\zeta}{dt}, \frac{d\zeta}{dt}\right)_0. \tag{25}$$

The operators $\mathcal{B}$ and $\mathcal{C}$ in equalities (24) and (25) are naturally called the operators of potential and kinetic energy, respectively.

3. Eigenfrequency Oscillations of Magnetizable Fluid. Solvability of the Cauchy Problem

Consider the normal eigenoscillations of the magnetizable fluid, described by solutions of homogeneous equation (22), depending on time t according to the law

$$\zeta = e^{i\omega t}u(\xi^1,\xi^2),$$

where ω is the circular frequency of oscillations, $u(\xi^1,\xi^2)$ is the mode oscillations of the free surface of fluid. The equation (22) leads to a spectral problem

$$\mathcal{B}u = \lambda\mathcal{C}u, \qquad \lambda := \omega^2\rho/\sigma. \tag{26}$$

The operator of potential energy $\mathcal{B}$ is the semibounded from below operator with a discrete real spectrum; generically $\mathcal{B}$ has a countable set of positive eigenvalues of $\lambda_k(\mathcal{B})$ and, possibly, a finite number of negative and zero eigenvalues [13]. Let æ be the number of negative eigenvalues of the operator $\mathcal{B}$ (counting their multiplicity), æ$_0$ be the multiplicity of zero eigenvalue. The eigenvalues of the operator $\mathcal{B}$ are numbered, as usual, in the ascending order

$$\begin{gathered}\lambda_1(\mathcal{B}) \le \lambda_2(\mathcal{B}) \le \ldots \le \lambda_{æ}(\mathcal{B}) < 0, \quad \lambda_{æ+1}(\mathcal{B}) = \ldots = \lambda_{æ+æ_0}(\mathcal{B}) = 0,\\ 0 < \lambda_{æ+æ_0+1}(\mathcal{B}) \le \lambda_{æ+æ_0+2}(\mathcal{B}) \le \lambda_{æ+æ_0+3}(\mathcal{B}) \le \ldots \quad .\end{gathered} \tag{27}$$

Using the results of [13], we formulate the generic properties of the eigenvalue and eigenfunction of problem (26).

Theorem 1. *Let the eigenvalues of the operator of potential energy* $\mathcal{B}$ *satisfy (27). Then the problem (26) has a discrete spectrum* $\{\lambda_k\}_{k=1}^{\infty}$ *consisting of eigenvalues* λ_k *of finite multiplicity; all eigenvalues* λ_k *are real and* $\lambda_k := \lambda_k^- < 0$ $\forall k = \overline{1, æ}$, $\lambda_{æ+k} := \lambda_k^0 = 0$ $\forall k = \overline{1, æ_0}$, $\lambda_{æ+æ_0+k} := \lambda_k^+ > 0$ $\forall k = \overline{1, \infty}$, *and* $\lambda_k \to +\infty$ *when* $k \to \infty$. *The set of eigenfunctions* $\{u_k\}_{k=1}^{\infty} := \{u_k^-\}_{k=1}^{æ} \cup \{u_k^0\}_{k=1}^{æ_0} \cup \{u_k^+\}_{k=1}^{\infty}$ *(*$u_k^{\pm}, u_k^0$ *are the eigenfunctions corresponding to the eigenvalues* $\lambda_k^{\pm}, \lambda_k^0$, *respectively:* $u_k := u_k^-$ $\forall k = \overline{1, æ}$, $u_{æ+k} := u_k^0$ $\forall k = \overline{1, æ_0}$, $u_{æ+æ_0+k} := u_k^+$ $\forall k = \overline{1, \infty}$*) is complete in* $\mathcal{H}_0(\Gamma)$. *It also forms the Riesz basis and can be chosen to satisfy the relation*

$$(\mathcal{B}u_j, u_k)_0 = \delta_{jk}\lambda_k, \quad (\mathcal{C}u_j, u_k)_0 = \delta_{jk}. \tag{28}$$

It can be shown that for a magnetizable fluid the asymptotics of the spectrum, given in [13] for a regular capillary fluid, remains valid

$$\lambda_k = \lambda_k(\mathcal{B}, \mathcal{C}) = \left(\frac{\operatorname{mes}\Gamma}{4\pi}\right)^{-3/2} k^{3/2}(1 + o\,(1)), \qquad k \to \infty.$$

Note also that the spectrum of eigenfrequencies of fluid oscillations contains æ pairs of purely imaginary frequencies $\omega_k = \pm i|\lambda_k\sigma/\rho|^{1/2}$ $\forall k = \overline{1, æ}$ and a countable set of real frequencies $\omega_k = \pm(\lambda_k\sigma/\rho)^{1/2}$ $\forall k > (æ + æ_0)$.

The function $\zeta(t)$, *which is continuous on* $t \in [0, T]$ *in the norm of* $\mathcal{H}_0^1(\Gamma)$ *with continuous first derivative on* $t \in [0, T]$ *in the norm of the space* $\mathcal{H}_0^{-1/2}(\Gamma)$,

$$\zeta(t) \in \mathcal{C}([0, T]; \mathcal{H}_0^1(\Gamma)), \quad \zeta'(t) \in \mathcal{C}([0, T]; \mathcal{H}_0^{-1/2}(\Gamma)) \quad (' := d/dt),$$

satisfies the integral identity

$$\int_0^T \big(\rho(\mathcal{C}\zeta'(t), \eta'(t))_0 - \sigma(\mathcal{B}\zeta(t), \eta(t))_0 + \rho(\chi_0(t), \eta(t))_0\big)\, dt + \rho(\mathcal{C}\zeta_0', \eta(0))_0 = 0,$$

$$\forall\, \eta(t) \in L_2(0, T; \mathcal{H}_0^1(\Gamma)), \quad \eta'(t) \in L_2(0, T; \mathcal{H}_0^{-1/2}(\Gamma)), \quad \eta(T) = 0, \tag{29}$$

and is called **the generalized solution of the Cauchy problem (22) and (23)**.

Theorem 2. *Let the conditions* $\zeta_0 \in \mathcal{H}_0^1(\Gamma)$, $\zeta_0' \in \mathcal{H}_0^{-1/2}(\Gamma)$ *and* $\chi_0(t) \in L_2(0, T; \mathcal{H}_0(\Gamma))$ *are satisfied. Then there exists a unique weak solution to the Cauchy problem (22) and (23).*

P r o o f of the theorem for the case of a positive definite potential energy operator $\mathcal{B}$ is given in [13]. For the general case, when negative and zero eigenvalues $\lambda_j(\mathcal{B})$ of the operator $\mathcal{B}$ exist, the theorem can be easily proved analogously to [16, Ch. 3].

The generalized solution to the problem (22) and (23) can be written in the form:

$$\zeta(t) = \zeta^-(t) + \zeta^0(t) + \zeta^+(t) = \sum_{j=1}^{æ} c_j^-(t)u_j^- + \sum_{j=1}^{æ_0} c_j^0(t)u_j^0 + \sum_{j=1}^{\infty} c_j^+(t)u_j^+. \quad (30)$$

The coefficients $c_j^{\pm}(t)$, $c_j^0(t)$ in (30) are defined as follows:

$$c_j^-(t) = \alpha_j^- \cosh(|\omega_j|t) + \beta_j^- \sinh(|\omega_j|t) + |\omega_j|^{-1} \int_0^t \sinh(|\omega_j|(t-\tau))\,\chi_j^-(\tau)d\tau \quad \forall j = \overline{1, æ},$$

$$c_j^+(t) = \alpha_j^+ \cos(|\omega_j|t) + \beta_j^+ \sin(|\omega_j|t) + |\omega_j|^{-1} \int_0^t \sin(|\omega_j|(t-\tau))\,\chi_j^+(\tau)d\tau \quad \forall j = \overline{(æ + æ_0 + 1), \infty}, \quad (31)$$

$$c_j^0(t) = \alpha_j^0 + \beta_j^0 t + \int_0^t d\tau \int_0^\tau \chi_j^0(s)ds \quad \forall j = \overline{(æ+1), (æ+æ_0)},$$

where

$$\alpha_j^{\pm} := (\mathcal{C}\zeta_0, u_j^{\pm})_0, \qquad \beta_j^{\pm} := (\mathcal{C}\zeta_0', u_j^{\pm})_0, \qquad \chi_j^{\pm}(t) := (\chi_0(t), u_j^{\pm})_0,$$
$$\alpha_j^0 := (C\zeta_0, u_j^0)_0, \qquad \beta_j^0 := (C\zeta_0', u_j^0)_0, \qquad \chi_j^0(t) := (\chi_0(t), u_j^0)_0.$$

The set of functions $\vec{v}(t,\vec{x})$, $\zeta(t,\xi^1,\xi^2)$, $\psi^{(k)}(t,\vec{x})$, $k = \overline{1,3}$, where the function ζ is the generalized solution to the Cauchy problem (22) and (23), the velocity field $\vec{v}$ has the form: $\vec{v} = \vec{u} + \nabla\varphi$, the vortex component $\vec{u}$ is defined in (15), and the potential component $\nabla\varphi = \nabla(T\zeta')$ and the potential $\psi := (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})$ of perturbations of the magnetic field are determined by perturbations ζ of the free surface of the fluid by equality: $\psi = \mathcal{M}\zeta$, is called **the generalized solution to the initial-boundary problem (1)–(10)**.

The previous discussion leads to the following theorem.

Theorem 3. *Let the conditions* $\vec{f}(t) \in C([0,T];\vec{L}_2(\Omega_1))$, $\zeta_0 \in \mathcal{H}_0^1(\Gamma)$ *and* $\vec{v}_0 \in \vec{J}_{0,S}(\Omega_1)$ *are satisfied. Then there exists a unique solution to the initial-boundary value problem (1)–(10). The balance equation of (kinetic + potential) energy*

$$\frac{1}{2}(\rho\|\vec{v}(t)\|^2_{\vec{L}_2(\Omega_1)} + \sigma(\mathcal{B}\zeta(t),\zeta(t))_0) = \frac{1}{2}(\rho\|\vec{v}_0\|^2_{\vec{L}_2(\Omega_1)} + \sigma(\mathcal{B}\zeta_0,\zeta_0)_0) + \rho\int_0^t (\vec{f}(\tau),\vec{v}(\tau))_{\vec{L}_2(\Omega_1)}d\tau \quad (32)$$

is satisfied for the generalized solution.

We consider the stability conditions of equilibrium states of magnetizable capillary fluid. The **equilibrium state** is called stable *if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any initial perturbations* ζ_0 *and* $\vec{v}_0$*, satisfying* $\|\zeta_0\|_{\mathcal{H}_0^1(\Gamma)} < \delta$ *and* $\|\vec{v}_0\|_{\vec{L}_2(\Omega_1)} < \delta$*, the inequalities* $\|\zeta(t)\|_{\mathcal{H}_0^1(\Gamma)} < \varepsilon$ *and* $\|\vec{v}(t)\|_{\vec{L}_2(\Omega_1)} < \varepsilon$ *for* $\forall t > 0$ *hold true.*

The stability (or instability) of the equilibrium states of magnetizable fluid is determined by the sign of the smallest eigenvalue $\lambda_1(\mathcal{B})$ of the operator of potential energy $\mathcal{B}$.

The spectral criterion of stability. *If there are no perturbations of external field of mass forces* ($\vec{f} \equiv 0$)*, then the equilibrium state of the magnetizable capillary fluid is stable if the lowest eigenvalue* $\lambda_1(\mathcal{B})$ *of the operator of potential energy* $\mathcal{B}$ *is positive,* $\lambda_1(\mathcal{B}) > 0$*, and is unstable if* $\lambda_1(\mathcal{B}) < 0$.

Thus, if $\lambda_1(\mathcal{B}) > 0$, then by (32) and (15), it easy to see that under condition $\vec{f} \equiv 0$ the equilibrium state is stable. If $\lambda_1(\mathcal{B}) < 0$, then by (30) and (31), there are arbitrarily small initial perturbations (in the norm of $\mathcal{H}_0^1(\Gamma)$) of the equilibrium state such that $\|\zeta(t)\|_{\mathcal{H}_0^1(\Gamma)} \to \infty$ when $t \to \infty$.

In some cases, the spectral criterion of stability can be checked rather easily. In [3], there are given the examples of using it for constructing the boundary of stability in the space of physical parameters characterizing the equilibrium states of fluid.

By (24), the functional of the potential energy has a minimum value in the equilibrium state of magnetizable fluid if the operator of potential energy $\mathcal{B}$ is positive definite and, consequently, $\lambda_1(\mathcal{B}) > 0$. From (24) it follows that if $\lambda_1(\mathcal{B}) < 0$, then the second variation of potential energy may take negative values. Thus, we have the following theorem.

Theorem 4. *If there are no perturbations of external field of mass forces* ($\vec{f} \equiv 0$)*, then the equilibrium state of magnetizable capillary fluid is stable if it corresponds to an isolated local minimum of potential energy. If the equilibrium*

state corresponds to the stationary value of the functional of potential energy but it is not a local minimum, and the second variation of potential energy can take negative values, then the equilibrium state is unstable.

This theorem is analogous to the well-known Lagrange theorem (and its reverse) on the stability of equilibrium of the conservative systems with finite number of degrees of freedom.

Conclusion

The generic formulation of the problem on the small motions of magnetizable ideal fluid near the equilibrium state is given. The problem is reduced to the evolution problem for a system of operator-differential equations in the Hilbert space. The solvability of the evolutionary problem is proved.

The spectral problem on normal eigen-oscillations of magnetizable capillary fluid is reduced to the studying of eigenvalues and vectors of the linear bunch of operators of kinetic and potential energies. The main qualitative properties of the spectrum of natural frequencies and modes of normal vibrations of magnetizable capillary fluid are found. In particular, it is proved that the system of normal modes of oscillations forms a basis in certain functional spaces. This allows one to provide the solutions to evolution equations in the form of expansions in series of the eigenfunctions of the problem.

The principle of minimum potential energy in the problem of stability of equilibrium states of the magnetizable fluid is provided in the linear approximation. The spectral criterion of stability of equilibrium states is formulated. Efficient methods of calculating the stability of equilibrium states of magnetizable fluid can be based on the the spectral criterion.

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