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Inverse Wave Spectral Problem with Discontinuous Wave Speed

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The inverse problem for the Sturm–Liouville operator with complex periodic potential and positive discontinuous coefficients on the axis is studied. The main characteristics of the fundamental solutions and the spectrum of the operator are studied. The formulation of the inverse problem and a constructive procedure for its solution are given. The uniqueness theorem of the inverse problem is proven.

Key words: spectral singularities, inverse spectral problem, continuous spectrum.

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Introduction

We consider the differential equation

$$-y''(x) + q(x)\, y(x) = \lambda^2 \rho(x)\, y(x) \tag{1}$$

in the space $L_2(-\infty, +\infty)$, where the prime denotes the derivative with respect to the space coordinate. We assume that the potential $q(x)$ is of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \tag{2}$$

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the condition $\sum_{n=1}^{\infty}|q_n|^2 = q < \infty$ is satisfied, λ is a complex number, and

$$\rho(x) = \begin{cases} 1 & \text{for} \quad x \geq 0, \\ \beta^2 & \text{for} \quad x < 0,\ \beta \neq 1, \beta > 0\,. \end{cases} \tag{3}$$

In the frequency domain this equation describes the wave propagation in an inhomogeneous medium, where $q(x)$ is the restoring force and ${}^{1}\!/\!_{\rho(x)}$ is the wave speed. The discontinuity in $\rho(x)$ usually expresses an abrupt change in the propagation medium of a wave, for example, the tension or the mass density of string, or the refractive index in a wave propagation medium.

In regard to the problems with discontinuous coefficients, we remark that Sabatier and his co-writers [1–4] studied the scattering for the impedance-potential equation and the similar problems were intensively studied by many authors in different statements [5, 6], but for the periodic complex potential they are considered for the first time.

Firstly potential (2) was considered by M.G. Gasymov [7]. Later, in 1990, the results obtained in [7] were extended by L.A. Pastur, V.A. Tkachenko [8]. As a final remark we mention some works of V. Guillemin, A. Uribe [9] and [10–12].

In the paper, our primary aim is to study the spectrum and to solve the inverse problem for singular nonself-adjoint operator by transmitting the coefficient and normalizing the numbers corresponding to quasieigenfunctions of the Sturm–Liouville operator with complex periodic potential and positive discontinuous coefficients on the axis. As the coefficient allows a bounded analytic continuation to the upper half-plane of the complex plane $z = x + it$, we can analyze the problem (1)–(3) in detail.

The paper consists of three sections. In Section 1 we study the properties of fundamental system of solutions of equation (1). The spectrum of problem (1)–(3) is studied in Section 2. In Section 3 we give a formulation of the inverse problem, prove the uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

1. Representation of Fundamental Solutions

Here we study the solutions of the main equation

$$-y''(x) + q(x)\,y(x) = \lambda^2 \rho(x)\,y(x)$$

that will be needed later.

We can prove the existence of these solutions if the condition $\sum_{n=1}^{\infty}|q_n|^2 = q < \infty$ is fulfilled for the potential. This will be unique restriction on the potential and later on we will consider it to be fulfilled.

Theorem 1. *Let $q(x)$ be of the form* (2) *and $\rho(x)$ satisfy condition* (3)*. Then equation* (1) *has special solutions of the form*

$$f_1^+(x,\lambda) = e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n+2\lambda}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right) \qquad for\ x \geq 0, \tag{4}$$

$$f_2^+(x,\lambda) = e^{-i\lambda\beta x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n-2\lambda\beta}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right) \qquad for\ x < 0, \tag{5}$$

where the numbers $V_{n\alpha}$ are determined from the following recurrent relations:

$$\alpha(\alpha-n)V_{n\alpha}+\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}=0, \qquad 1\leq n<\alpha, \tag{6}$$

$$\alpha\sum_{n=1}^{\alpha}V_{n\alpha}+q_\alpha=0, \tag{7}$$

and the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha\,|V_{n\alpha}| \tag{8}$$

converge.

The proof of the theorem is similar to that of [8] and therefore we do not cite it here.

R e m a r k 1. If $\lambda \neq -\frac{n}{2}$, $\lambda \neq \frac{n}{2\beta}$ and $\mathrm{Im}\lambda > 0$, then $f_1^+(x,\lambda) \in L_2(0,+\infty)$, $f_2^+(x,\lambda) \in L_2(-\infty,0)$.

By analogy to [7], it is easy to see that equation (1) has fundamental solutions $f_1^+(x,\lambda)$,$f_1^-(x,\lambda)$ ($f_2^+(x,\lambda)$,$f_2^-(x,\lambda)$) for which

$$W\left[f_1^+(x,\lambda),f_1^-(x,\lambda)\right]=2i\lambda,$$

$$W\left[f_2^+(x,\lambda),f_2^-(x,\lambda)\right]=-2i\lambda\beta,$$

(where $W[f,g]=f'g-fg'$) is satisfied

Then each solution of equation (1) may be represented as a linear combination of these solutions

$$f_2^+(x,\lambda)=C_{11}(\lambda)\,f_1^+(x,\lambda)+C_{12}(\lambda)\,f_1^-(x,\lambda) \quad for\ \ x\geq 0, \tag{9}$$

$$f_1^+(x,\lambda)=C_{22}(\lambda)\,f_2^+(x,\lambda)+C_{21}(\lambda)\,f_2^-(x,\lambda) \quad for\ \ x<0, \tag{10}$$

where

$$f_{1,2}^{-}(x,\lambda)=f_{1,2}^{+}(x,-\lambda),$$

$$C_{11}(\lambda)=\frac{W[f_2^{+}(0,\lambda),f_1^{-}(0,\lambda)]}{2i\lambda}, \tag{11}$$

$$C_{12}(\lambda)=\frac{W[f_1^{+}(0,\lambda),f_2^{+}(0,\lambda)]}{2i\lambda},$$

$$C_{22}(\lambda)=-\frac{1}{\beta}C_{11}(-\lambda),C_{21}(\lambda)=\frac{1}{\beta}C_{12}(\lambda). \tag{12}$$

According to physical sense of the solutions $f_1^{\pm}(x,\lambda)$,$f_2^{\pm}(x,\lambda)$, it is natural to say that $\frac{1}{\mathrm{C}_{12}(\lambda)}$ is a transmission coefficient to the left and $\frac{\mathrm{C}_{11}(\lambda)}{\mathrm{C}_{12}(\lambda)}$ is a reflection coefficient from the left for equation (1).
Let

$$f_n^{\pm}(x)=\lim_{\lambda\to\mp\frac{n}{2}}(n\pm 2\lambda)f_1^{\pm}(x,\lambda)=\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}e^{-i\frac{n}{2}x}. \tag{13}$$

It follows from relation (6) that $f_n^{\pm}(x)\not\equiv 0$ is valid for $V_{nn}\neq 0$. From this we obtained that $W[f_n^{\pm}(x),f_1^{\mp}\left(x,\mp\frac{n}{2}\right)]=0$, and consequently the functions $f_n^{\pm}(x),f_1^{\mp}\left(x,\mp\frac{n}{2}\right)$ are linear dependent.
Therefore,

$$f_n^{\pm}(x)=V_{nn}f_1^{\mp}\left(x,\mp\frac{n}{2}\right). \tag{14}$$

2. Spectrum of Operator L

Let L be an operator generated by the differential expression $\frac{1}{\rho(x)}\left\{-\frac{d^2}{dx^2}+q(x)\right\}$ in the space $L_2(-\infty,+\infty,\rho(x))$.

Divide the plane λ into sectors

$$S_k=\{k\pi<\arg\lambda<(k+1)\pi\},\quad k=0,1.$$

By means of general method for the kernel of the resolvent of operator $\left(L-\lambda^2 I\right)$ we get

$$R_{11}(x,t,\lambda)=\frac{1}{C_{12}(\lambda)}\begin{cases} f_1^{+}(x,\lambda)f_2^{+}(t,\lambda) & for\ t<x\\ f_1^{+}(t,\lambda)f_2^{+}(x,\lambda) & for\ t>x\end{cases}\quad \lambda\in S_0 \tag{15}$$

and

$$R_{12}(x,t,\lambda) = \frac{1}{C_{12}(-\lambda)} \begin{cases} f_1^-(x,\lambda)\, f_2^-(t,\lambda) & for\ t < x \\ f_1^-(t,\lambda)\, f_2^-(x,\lambda) & for\ t > x \end{cases} \lambda \in S_1. \qquad (16)$$

R e m a r k 2. $R_{\lambda^2} = (L - \lambda^2 I)^{-1}$ exists and is bounded for all λ^2 out of positive half-line and $C_{12}(\pm\lambda) \neq 0$.

Lemma 2. *The coefficient $C_{12}(\lambda)$ is an analytic function in the $Im\lambda > 0$ and has a finite number of zeros, moreover, if $C_{12}(\lambda_n) = 0$, then*

$$\frac{d}{d\lambda} C_{12}(\lambda)|_{\lambda=\lambda_n} = -i \int_{-\infty}^{+\infty} \rho(x)\, f_1^+(x,\lambda_n)\, f_2^+(x,\lambda_n)\, dx.$$

The proof of Lemma 2 is similar to that of [14, p. 173] and therefore we do not cite it here.

For the solutions $f_1^\pm(x,\lambda)$ and $f_2^\pm(x,\lambda)$ we can obtain the asymptotic equalities:

$$f_1^{\pm(j)}(0,\lambda) = \pm(i\lambda)^j + o(1) \qquad \text{for } |\lambda| \to \infty,\ j = 0,1,$$
$$f_2^{\pm(j)}(0,\lambda) = \mp(i\lambda\beta)^j + o(1) \quad \text{for } |\lambda| \to \infty,\ j = 0,1.$$

For simplicity we prove the first equality.

Since

$$f_1^\pm(0,\lambda) = 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{n \pm 2\lambda},$$

then

$$\left| f_1^\pm(0,\lambda) \right| \le 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{|V_{n\alpha}|}{|n+2\lambda|} \le 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{|V_{n\alpha}|}{\sqrt{(n+2Re\lambda)^2 + 4Im^2\lambda}}$$

$$\le 1 + \frac{1}{|Im\lambda|} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{\alpha\, |V_{n\alpha}|}{n}.$$

Therefore, as $|\lambda| \to \infty$, we obtain $f_1^\pm(0,\lambda) = 1 + o(1)$.

Analogously, we can prove the rest of asymptotic equalities as $|\lambda| \to \infty$ for the solutions $f_2^\pm(x,\lambda)$.

Then for the coefficients $C_{12}(\lambda)$, $C_{12}(-\lambda)$, we get the following asymptotic equalities:

$$C_{12}(\lambda) = \frac{1}{2i\lambda}(i\lambda\beta + i\lambda) + o(1) = \frac{\beta+1}{2} + o(1), \tag{17}$$

$$C_{12}(-\lambda) = \frac{\beta+1}{2} + o(1).$$

These asymptotic equalities and analytical properties of the coefficients $C_{12}(\lambda)$, $C_{12}(-\lambda)$ make valid the following statement.

Lemma 3. *The eigenvalues of the operator L are finite and coincide with the square of zeros of the functions $C_{12}(\lambda)$, $C_{12}(-\lambda)$ from the sectors $S_k, k = 0, 1$, respectively.*

R e m a r k 3. Taking into account (17), we can obtain the useful relation

$$\beta = 2 \lim_{Im\lambda\to\infty} C_{12}(\lambda) - 1. \tag{18}$$

Theorem 2. *The continuous spectrum of the operator L fills out the semi axis $[0,\infty)$ and on the continuous spectrum may be spectral singularities at the points of the form $\left(\frac{n}{2}\right)^2$ and $\left(\frac{n}{2\beta}\right)^2$, $n \in N$.*

P r o o f. First, we will prove that the operator L has no positive eigenvalues. We recall that equation (1) has fundamental solutions $f_1^+(x,\lambda), f_1^-(x,\lambda)$.

Then for the case $\lambda^2 > 0$ the solution of equation (1) can be written in the form

$$y(x,\lambda) = C_1 e^{i|\lambda|x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2|\lambda|} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right)$$
$$+ C_2 e^{-i|\lambda|x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n-2|\lambda|} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right).$$

So, $y(x,\lambda) \notin L_2(-\infty,+\infty)$ since the principle parts of the solutions are periodic.

Taking into account Remark 2, we will study the function

$$R(x,t,\lambda) = \begin{cases} R_{11}(x,t,\lambda), & \lambda \in S_0 \\ R_{12}(x,t,\lambda), & \lambda \in S_1 \end{cases}$$

in the neighborhood of poles λ_0 from $[0,\infty)$. Then the number λ_0 coincides with one of the numbers $\frac{n}{2\alpha}, n, \alpha \in N$. From (15)–(16) it follows that the limit $\lim\limits_{\lambda\to\lambda_0}(\lambda - \lambda_0)R(x,t,\lambda) = R_0(x,t)$ exists and $R_0(x,t)$ is a bounded function with respect to all the variables. Let $\theta(x)$ be an arbitrary finite function. Then $\varphi(x) = \int\limits_{-\infty}^{+\infty} R_0(x,t)\theta(t)\,dt$ is a bounded solution of equation (1) for $\lambda = \lambda_0$. Therefore,

$\varphi(x) = C_0 f_1^+(x, \lambda_0)$. Comparison of the last relation with formulae (15)–(16) shows that if $\lambda_0 \neq \frac{n}{2}, n \in N$, then $C_0 = 0$ and so the kernel of the resolvent has removable singularity at the point λ_0. Thus, there is a case $\lambda_0 = \frac{n}{2}, n \in N$, where the kernel of resolvent has the poles of first order. Since $f_1^+(x, \lambda_0) \notin L_2(-\infty, +\infty)$, then $\lambda_0^2 = (\frac{n}{2})^2$ is a spectral singularity of the operator L in the sense of M.N. Naimark [13]. (Analogously, we can show that the resolvent has the poles of first order at the points $\lambda_0 = \frac{n}{2\beta}$, $n \in N$ and $(\frac{n}{2\beta})^2$ are spectral singularities of the operator L.)

In order all numbers $\lambda^2 > 0$ belong to the continuous spectra, it suffices to show that the domain of value $R_{L-\lambda^2 I}$ of the operator $(L - \lambda^2 I)$ is dense in $L_2(-\infty, +\infty)$, so that the orthogonal complement of the set $R_{L-\lambda^2 I}$ consists of only zero element. However, the orthogonal complement of the set $R_{L-\lambda^2 I}$ coincide with the space of the solutions of equation $L^* f = \lambda^2 f$. It is easy to see that the operator L^* is adjoint to the operator L.

Let $\psi(x) \in L_2(-\infty, +\infty)$, $\psi(x) \neq 0$, and

$$\int_{-\infty}^{+\infty} \left(Lf - \lambda^2 f\right) \overline{\psi(x)} dx = 0 \tag{19}$$

be satisfied for any $f(x) \in D(L)$.

From (19) it follows that $\psi(x) \in D(L^*)$, and $\psi(x)$ is an eigenfunction of the operator L^* corresponding to eigenvalues λ. More exactly, $\overline{\psi(x)}$ is the solution of the equation

$$-z'' + q(x) z = \lambda^2 \rho(x) z \tag{20}$$

belonging to $L_2(-\infty, +\infty)$. We obtained that $\psi(x) = 0$, since the operator generated by the expression in the left-hand side of (20) is an operator of type L. This contradiction shows that the domain of value $R_{L-\lambda^2 I}$ of the operator $(L - \lambda^2 I)$ is everywhere dense in $L_2(-\infty, +\infty)$. The theorem is proved.

Now, taking (9)–(10) and (14) into account, we calculate

$$\begin{aligned}\lim_{\lambda \to n/2} (n - 2\lambda) R_{11}(x, t, \lambda) &= \lim_{\lambda \to n/2} (n - 2\lambda) \tfrac{1}{2i\lambda} [f_1^+(x, \lambda) f_1^+(t, \lambda) \tfrac{W[f_2^+, f_1^-]}{W[f_1^+, f_2^+]} \\ &+ f_1^+(x, \lambda) f_1^-(t, \lambda)] = \tfrac{1}{in} [V_{nn} f_1^+\left(x, \tfrac{n}{2}\right) f_1^+\left(t, \tfrac{n}{2}\right) \\ &+ V_{nn} f_1^+\left(x, \tfrac{n}{2}\right) f_1^+\left(t, \tfrac{n}{2}\right)] = \tfrac{2}{in} V_{nn} f_1^+\left(x, \tfrac{n}{2}\right) f_1^+\left(t, \tfrac{n}{2}\right).\end{aligned} \tag{21}$$

Analogously, taking into account that $\tilde{f}_2^+(x, \lambda) = f_2^+(x, \lambda)(n - 2\lambda\beta)$ has no poles at the points $\lambda = \frac{n}{2\beta}$, $n \in N$, we get

$$\lim_{\lambda \to {}^{n}\!/_{2\beta}} (n - 2\lambda\beta) R_{11}(x,t,\lambda) = [-\frac{1}{in} \frac{W\left[f_2^-\left(0,{}^{n}\!/_{2\beta}\right), f_1^+\left(0,{}^{n}\!/_{2\beta}\right)\right]}{W\left[f_1^+\left(0,{}^{n}\!/_{2\beta}\right), \tilde{f}_2^+\left(0,{}^{n}\!/_{2\beta}\right)\right]} \tilde{f}_2^+\left(x,{}^{n}\!/_{2\beta}\right)$$
$$+ \frac{W\left[f_1^+\left(0,{}^{n}\!/_{2\beta}\right), \tilde{f}_2^+\left(0,{}^{n}\!/_{2\beta}\right)\right]}{in} f_2^-\left(x,{}^{n}\!/_{2\beta}\right)] \tilde{f}_2^+\left(t,{}^{n}\!/_{2\beta}\right) = F\left(x,t,{}^{n}\!/_{2\beta}\right)$$

3. Eigenfunction Expansions

To define the natural spectral data of the operator L, it is necessary to obtain the eigenfunction expansion for the same operator. For this we consider the following lemma.

Lemma 4. *Let $\psi(x)$ be an arbitrary twice continuously differentiable function belonging to $L_2(-\infty,+\infty,\rho(x))$. Then*

$$\int_{-\infty}^{+\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\infty}^{+\infty} R(x,t,\lambda) g(t)\,dt,$$

where

$$g(t) = -\psi''(x) + q(x)\psi(x) \in L_2(-\infty,+\infty).$$

Integrating the both hand sides along the circle $|\lambda| = R$ and passing to the limit as $R \to \infty$, we get

$$\psi(x) = -\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} 2\lambda d\lambda \int_{-\infty}^{+\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt.$$

The function $\int_{-\infty}^{+\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt$ is analytical inside the contour with respect to λ excepting the points $\lambda = \lambda_n, n = 1,2,\ldots,$ $\lambda = \frac{n}{2}$, $\lambda = \frac{n}{2\beta}$, $n = 1,2,\ldots.$ Denote by $\Gamma_0^+\left(\Gamma_0^-\right)$ the contour formed by segments $[0, \frac{1}{2\beta} - \delta], [\frac{1}{2\beta} + \delta, \frac{1}{2} - \delta], \ldots [\frac{n}{2\beta} + \delta, \frac{n}{2} - \delta]$ and semicircles of radius δ with the centers at points $\frac{n}{2}$, $\frac{n}{2\beta}$ $n = 1,2,\ldots,$ located in the upper (lower) half plane. Then

$$\psi(x) = -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} 2\lambda\rho(t)\psi(t)\,[\int_{\Gamma_0^+} R_{11}(x,t,\lambda)\,d\lambda - \int_{\Gamma_0^-} R_{12}(x,t,\lambda)\,d\lambda]dt$$

$$= -\frac{1}{2i\pi}\int\limits_{-\infty}^{+\infty} 2\lambda\rho(t)\,\psi(t)\int\limits_{\Gamma_0^-}[R_{11}(x,t,\lambda) - R_{12}(x,t,\lambda)]d\lambda dt$$

$$+\underset{\lambda=\lambda_n}{Res}\,R_{11}(x,t,\lambda) + \underset{\lambda=\frac{n}{2\beta}}{Res}\,R_{11}(x,t,\lambda) + \underset{\lambda=\frac{n}{2}}{Res}\,R_{11}(x,t,\lambda).$$

Calculate separately every term.

$$R_{11}(x,t,\lambda) - R_{12}(x,t,\lambda) = \frac{f_1^+(x,\lambda)\,f_1^+(t,\lambda)}{2i\lambda C_{12}(\lambda)\,C_{22}(\lambda)}$$

The residues of the resolvent $R_{11}(x,t,\lambda)$ in $\lambda_1,\lambda_2,\ldots\lambda_l$ denote by $G_{11}(\lambda_n,x,t)$. Thus $G_{11}(\lambda_n,x,t)$ will be equal to

$$G_{11}(\lambda_n,x,t) = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)\,R_{11}(x,t,\lambda).$$

Then for every function $\psi(x)$ belonging to $L_2(-\infty,+\infty,\rho(x))$ we get the eigenfunction expansion in the form

$$\begin{aligned}\psi(x) = -\tfrac{1}{2i\pi}\int\limits_{-\infty}^{+\infty}\rho(t)\,\psi(t)\Big[\oint\limits_{\Gamma_0^-}\tfrac{f_1^+(x,\lambda)f_1^+(t,\lambda)}{2i\lambda C_{12}(\lambda)C_{22}(\lambda)}d\lambda \\ +G_{11}(\lambda_n,x,t) + \tfrac{2}{in}V_{nn}f_1^+\left(x,\tfrac{n}{2}\right)f_1^+\left(t,\tfrac{n}{2}\right) + F\Big(x,t,{}^{n}\!/_{2\beta}\Big)\Big]dt\end{aligned} \tag{22}$$

4. Solution of the Inverse Problem

Let us study the inverse problem for the problem (1–3). From the representation (15)–(16) it also follows that for each x and t from $(-\infty,+\infty)$ the kernel $R(x,t,\lambda)$ admits a meromorphic continuation from the sector $S = \{\lambda : 0 < arg\lambda < \pi\}$ and may have poles at the points $\left(\frac{n}{2}\right)^2$ and $\left(\frac{n}{2\beta}\right)^2$, $n \in N$, outside of S. These poles of the resolvent are called quasi-stationary states of the operator L.

Thus the quasistationary states of the operator L are the numbers $\left(\frac{n}{2}\right)^2$ and $\left(\frac{n}{2\beta}\right)^2$, $n \in N$. In spectral expansion (22) the numbers V_{nn}, $n \in N$, play a part of the normalizing numbers corresponding to quasieigenfunction of the operator L. So, it makes natural the formulation of the inverse problem about reconstruction of the potential of the equation (1) and the number β.

Inverse Problem

Given the spectral data $\{\ C_{12}(\lambda), V_{nn}\}$, construct β and the potential $q(x)$. Using the results obtained above, we arrive at the following procedure for the solution of the inverse problem:
1. Taking into account (14), we get

$$V_{n,\alpha+n} = V_{nn} \sum_{m=1}^{\alpha} \frac{V_{m\alpha}}{m+n},$$

from which all the numbers $V_{n\alpha}$, $\alpha = 1, 2, \ldots,\ n = 1, 2, \ldots, n < \alpha$, are defined.
2. From recurrent formula (6)–(8), find all numbers q_n.
3. The number β is defined by the formula

$$\beta = 2 \lim_{Im\lambda\to\infty} C_{12}(\lambda) - 1.$$

So, the inverse problem has a unique solution and the numbers β and q_n are defined constructively by spectral data.

Theorem 3. *The specification of spectral data uniquely determines β and the potential $q(x)$.*

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