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Antipodal Polygons and Half-Circulant Hadamard Matrices

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As known, the question on the existence of Hadamard matrices of order $m = 4n$, where n is an arbitrary natural number, is equivalent to the question on the possibility to inscribe a regular hypersimplex into the $(4n-1)$-dimensional cube. We introduced a class of Hadamard matrices of order $4n$ of half-circulant type in 1997 and a class of antipodal n-gons inscribed into a regular (2n-1)-gon. In 2004 we proved that a half-circulant Hadamard matrix of order $4n$ exists if and only if there exist antipodal n-gons inscribed into a regular (2n-1)-gon. On this background there was developed the method of constructing of the Hadamard matrices of order $4n$, which is universal, i.e., it can be applied to any arbitrary natural number n, including a prime number case, that usually requires the individual approach to the construction of the Hadamard matrix of corresponding order. In the paper, there are obtained the necessary and sufficient algebraic-geometric conditions for the existence of antipodal polygons allowing to justify the inductive approach to be used to the proof of existence theorems for Hadamard matrices of arbitrary order $4n$, $n \geq 3$.

Key words: multidimensional cube, regular hypersimplex, Hadamard matrix, circulant matrix, antipodal polygons, necessary and sufficient conditions, mathematical induction, existence theorem.

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1. Introduction

The two convex n-gons inscribed into a regular $(2n-1)$-gon are said to be *antipodal* if the total number of their diagonals and sides of the same length is n for all admissible lengths [1, p. 48]. A square matrix H, whose every entry is $+1$ or -1, is called the *Hadamard matrix* of order m if $HH' = mI$, where

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H' is a transpose matrix and I is an identity one [2, p. 283]. As known, the question on the existence of Hadamard matrices of order $m = 4n$, where n is an arbitrary natural number, is equivalent to the question on the possibility to inscribe a regular hypersimplex into the $(4n-1)$-dimensional cube so that its every vertex coincides with one of the vertices of the cube. This was established by H. Coxeter in 1933 (see [3, p. 319]). We introduced a class of Hadamard matrices of *half-circulant type* [4, p. 459] in the form

$$H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

where A and B are matrices of order $2n$ (with bordering entries equal to 1) containing submatrices of order $2n-1$ which are the right and, respectively, left circulants (see [5, p. 27]). These matrices appeared to be closely connected with the class of antipodal polygons. Namely, it is proved that a *half-circulant Hadamard matrix of order* $4n$ *exists if and only if there exist antipodal* n*-gons inscribed into a regular* $(2n-1)$*-gon* (see [1, Th. 4]). Thus, the solution of the question either on the existence of Hadamard matrices of arbitrary order $4n$ or on the possibility of inscribing a regular hypersimplex into $(4n-1)$-dimensional cube depends on the solution of the question on the existence of antipodal n-gons. Moreover, it is possible now to set the proof problem of the existence theorem without constructing the Hadamard matrices of smallest order the existence of which is not proved. Besides, the Hadamard matrix of order 428 was constructed recently in [6] (twenty years later after its predecessor of order 268 [7]).

In the paper, there are obtained the necessary and sufficient algebraic-geometric conditions for the existence of antipodal polygons (Th. 2), which, by virtue of their universality, allow to justify the inductive approach to be used to the proof of existence theorems for Hadamard matrices of arbitrary order $4n$, $n \geq 3$ (Sect. 3).

2. Existence Conditions of Antipodal Polygons

Let $z = e^{\frac{2\pi i}{2n-1}}$, $n \geq 3$. Then the vertices of a regular $(2n-1)$-gon inscribed into a unit circle with the center in the origin of coordinates are given by the monomials z^k, $k = 0, 1, 2, \ldots, 2n-2$. Let every convex n-gon P inscribed into the regular $(2n-1)$-gon correspond to a generating polynomial $p(z) = \sum_{k=0}^{2n-2} x_k z^k$, where $x_k = 1$ if the vertex with number k belongs to P, and $x_k = 0$ otherwise. Since P is an n-gon, we have $\sum_{k=0}^{2n-2} x_k = n$. If d_k is the number of equal diagonals and sides of the n-gon P visible from the origin at the angle $\frac{2\pi k}{2n-1}$, then for the polynomial $p(z)$ the equality

$$|p|^2 = n + 2\sum_{k=1}^{n-1} d_k \cos\frac{2\pi k}{2n-1},$$

is valid, where $\sum_{k=1}^{n-1} d_k = n(n-1)/2$ (see [1, Lem. 1]). The similar formula is true for the generating polynomial $p'(z) = \sum_{k=0}^{2n-2} x'_k z^k$ of any other n-gon P' inscribed into the given regular $(2n-1)$-gon. If P and P' are antipodal, then $d_k + d'_k = n$ for every $k = 1, 2, \ldots, n-1$ by definition. Therefore, the generating polynomials of antipodal n-gons satisfy the relation $|p|^2 + |p'|^2 = n$ (see [1, Th. 3]). Indeed it holds a more general equality where the rotation group of regular $(2n-1)$-gon is taken into account. In this connection, there arises a question on the existence of an inscribed n-gon P with the number of diagonals d_k, $k = 1, 2, \ldots, n-1$, visible from the origin at the angle $\frac{2\pi k}{2n-1}$. It can be reduced to the solving of the following equation system for the coefficients x_i of its generating polynomial:

$$\begin{cases} \sum\limits_{i=0}^{2n-2} x_i & = & n; \\ \sum\limits_{i=0}^{2n-2} x_i^2 & = & n; \\ \sum\limits_{i=0}^{2n-2} x_i x_{|i+k|} & = & d_k, \\ k & = & 1, 2, \ldots, n-1, \end{cases} \tag{1}$$

where $x_{|i+k|}$ is the least nonnegative residue of $i + k$ modulo $2n - 1$ [1, p. 59]. The realization of the natural condition $\sum_{k=1}^{n-1} d_k = C_n^2$ is also assumed. The solution of system (1) was obtained in "parametric" form (see [1, p. 60]):

$$\begin{aligned} x_0 &= \frac{1}{\sqrt{2n-1}}\Big(y_0 + \sqrt{2}\sum_{j=1}^{n-1} y_j\Big), \\ x_m &= \frac{1}{\sqrt{2n-1}}\Big[y_0 + \sqrt{2}\sum_{j=1}^{n-1}\Big(y_j \cos\frac{2\pi mj}{2n-1} + y_{N-j}\sin\frac{2\pi mj}{2n-1}\Big)\Big], \\ x_{N-m} &= \frac{1}{\sqrt{2n-1}}\Big[y_0 + \sqrt{2}\sum_{j=1}^{n-1}\Big(y_j \cos\frac{2\pi mj}{2n-1} - y_{N-j}\sin\frac{2\pi mj}{2n-1}\Big)\Big], \end{aligned} \tag{2}$$

where $N = 2n - 1$, and

$$y_0 = \frac{n}{\sqrt{2n-1}}, \quad y_j^2 + y_{N-j}^2 = \frac{2}{2n-1}\Big(n + 2\sum_{k=1}^{n-1} d_k \cos\frac{2\pi kj}{2n-1}\Big).$$

The similar representation is valid for the coefficients x'_i of the generating polynomial $p'(z)$ of P'. The antipodal n-gons P and P' generated by polynomials $p(z) = \sum_{k=0}^{2n-2} x_k z^k$ and $p'(z) = \sum_{k=0}^{2n-2} x'_k z^k$, for which the coefficients are given by (2) and by the similar formula with $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ substituted by $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$, exist if and only if the conditions

$$y_j^2 + y_{N-j}^2 + {y'_j}^2 + {y'}_{N-j}^2 = \frac{2n}{2n-1} \tag{3}$$

are true for all $j = 1, 2, \ldots, n-1$ (see [1, Lem. 3]).

Let $\bar{w}_0, \bar{w}_m$ and $\bar{w}_{N-m}$ denote the right sides of equations (2), and let $\bar{w} = \bar{w}_0^3 + \sum_{m=1}^{n-1}(\bar{w}_m^3 + \bar{w}_{N-m}^3)$. Consider now a system of equations with respect to the coordinates of vector y

$$\frac{\partial \bar{w}}{\partial y_i} = 3y_i, \quad i = 0, 1, 2, \ldots, 2n-2. \tag{4}$$

Thus, valid is the following statement: *antipodal n-gons P and P' and, consequently, a corresponding to them Hadamard matrix of order $4n$ exist if and only if system (4) has two solutions $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ and $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$ such that $y_0 = y'_0 = \frac{n}{\sqrt{2n-1}}$, and the rest of the coordinates of vectors y and y' satisfy antipodal conditions (3)* (see [1, Th. 5]). Moreover, the derivatives of (4) have the form

$$\begin{aligned}
\frac{\partial \bar{w}}{\partial y_0} =& \frac{3}{\sqrt{2n-1}} \sum_{i=0}^{2n-2} y_i^2, \\
\frac{\partial \bar{w}}{\partial y_k} =& \frac{3\sqrt{2}}{\sqrt{2n-1}} [\sqrt{2} y_0 y_k + \sum_{j=1}^{n-k-1} (y_j y_{j+k} + y_{N-j} y_{N-j-k}) \\
&+ \frac{1}{2} \sum_{r=1-k}^{k-1} (y_{[(k+r)/2]} y_{[(k-r)/2]} - y_{N-[(k+r)/2]} y_{N-[(k-r)/2]})],
\end{aligned} \tag{5}$$

$$\frac{\partial \bar{w}}{\partial y_{N-k}} = \frac{3\sqrt{2}}{\sqrt{2n-1}} [(\sqrt{2}-1) y_0 y_{N-k} + \sum_{s=n}^{2n-2} (y_{|s+k|'} - y_{|s-k|'}) y_s],$$

where $k = 1, 2, \ldots, n-1$, $|s \pm k|' = \min(|s \pm k|,\ N - |s \pm k|)$, and $[(k \pm r)/2]$ equals $\frac{k \pm r}{2}$ if it is integer, or $\frac{N-(k \pm r)}{2}$ otherwise.

Since $\bar{w}$ is a homogeneous polynomial of third degree by definition, then $\frac{\partial \bar{w}}{y_0}, \frac{\partial \bar{w}}{y_k}, \frac{\partial \bar{w}}{y_{N-k}}$ are homogeneous polynomials of second degree, moreover, the last ones are homogeneous polynomials of first degree relatively to the variables $y_1, y_2, \ldots, y_{n-1}$ and $y_n, y_{n+1}, \ldots, y_{2n-2}$. As for the expressions for $\frac{\partial \bar{w}}{y_k}$, they contain no summands with $y_j y_s$, $0 < j < n$, $s \geq n$, but they contain quadratic differences $\frac{1}{2}(y^2_{[\frac{k}{2}]} - y^2_{N-[\frac{k}{2}]})$ for $r = 0$.

By analogy with the homogeneous polynomial $\bar{w} = \bar{w}(y)$, we introduce a polynomial $\bar{w}' = \bar{w}_0'^3 + \sum_{m=1}^{n-1}(\bar{w}_m'^3 + \bar{w}_{N-m}'^3)$, where $\bar{w}'$ are right sides of equality (2) after substitution of the coordinates of vector $y = \{y_0, y_1, \ldots, y_{2n-2}\}$ by the corresponding coordinates of vector $y' = \{y'_0, y'_1, \ldots, y'_{2n-2}\}$. Then *the equations, valid for the coordinates of vectors y and y' such that antipodal n-gons as well as a half-circulant Hadamard matrix of order $4n$ exist, can be represented* *in the form*

$$\begin{cases} \bar{W}_i = \frac{\partial \bar{w}}{\partial y_i} - 3y_i = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \bar{W}_i' = \frac{\partial \bar{w}'}{\partial y_i'} - 3y_i' = 0, \qquad i = 0, 1, 2, \ldots, 2n-2, \\ \bar{W}_{2n-1} = y_0 - \frac{n}{\sqrt{2n-1}} = 0, \quad \bar{W}_{2n-1}' = y_0' - \frac{n}{\sqrt{2n-1}} = 0, \\ Y_j = y_j^2 + y_{N-j}^2 + y_j'^2 + y_{N-j}'^2 - \frac{2n}{2n-1} = 0, \\ j = 1, 2, \ldots, n-1. \end{cases} \tag{6}$$

Thus, by Theorem 5 [8] for $n \geq 3$ there directly follows

Theorem 1. *Antipodal convex n-gons and a half-circulant Hadamard matrix of order $4n$ exist if and only if there do not exist the polynomials A_i, A_i', A_{2n-1}, A_{2n-1}', B_j depending on the variables $y_0, y_1, \ldots, y_{2n-2}, y_0', \ldots, y_{2n-2}'$ such that the left sides of equations (6) satisfy the identity*

$$\sum_{i=0}^{2n-2} (A_i \bar{W}_i + A_i' \bar{W}_i') + A_{2n-1} \bar{W}_{2n-1} + A_{2n-1}' \bar{W}_{2n-1}' + \sum_{j=1}^{n-1} B_j Y_j \equiv 1.$$

The obtained results submitted in this section, as shown further, will be used as a background for applying the inductive approach to the proof of the existence theorems for antipodal polygons.

3. Basis of Inductive Approach

First, we have to simplify the system of equations (6). For this we find an expression for the homogeneous polynomial of third degree $\bar{w}(y_0, y_1, \ldots, y_{2n-2})$. By definition, $\bar{w}(y)$ equals the sum of right sides of (2) raised to the third power for all $m = 1, 2, \ldots, n-1$, where $\cos \frac{2\pi mj}{2n-1}$ and $\sin \frac{2\pi mj}{2n-1}$, $j = 1, 2, \ldots, n-1$, are used as coefficients. Let us transform its expression by applying the formulas $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$ and $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$, where α and β are expressions of the form $\frac{2\pi mj}{2n-1}$, and also an identity

$$\frac{1}{2} + \sum_{j=1}^{n-1} \cos \frac{2\pi mj}{2n-1} \equiv 0,$$

which is valid for all $m \not\equiv 0 \pmod{2n-1}$ (it follows directly from the more general identity (see problem N16 in [9, p. 88])). Substituting twice the products of *cosines* and *sines* for the sum and the difference, respectively, and changing the summation order, finally we obtain

$$\begin{aligned}\bar{w} =& \frac{1}{\sqrt{2n-1}}[y_0^3 + 3y_0 \sum_{i=1}^{2n-2} y_i^2 + \frac{1}{\sqrt{2}} \sum_{3j=N} (y_j^3 - 3y_j y_{N-j}^2) \\ &+ \frac{3}{\sqrt{2}} \sum_{s=|2j|'}^{s\neq j} (y_j^2 - y_{N-j}^2) y_s + 3\sqrt{2} \sum_{j\neq N/3}^{n-1} (-1)^{|2j|'} y_j y_{N-j} y_{N-|2j|'} \\ &+ 3\sqrt{2} \sum_{j<s}^{s<N-j-s} ((y_j y_s - y_{N-j} y_{N-s}) y_{|j+s|'} \\ &+ (-1)^{j+s+|j+s|'} (y_j y_{N-s} + y_s y_{N-j}) y_{N-|j+s|'})],\end{aligned}$$

where the value of index j varies from 1 to $n-1$. Moreover, $N = 2n-1$ and the sign of absolute value, in particular $|2j|'$, denotes $\min(2j, N-2j)$. Let us represent this polynomial as a polynomial of third degree with respect to y_0

$$\bar{w} = \frac{1}{\sqrt{2n-1}}(y_0^3 + 3y_0 \sum_{i=1}^{2n-2} y_i^2 + 3\sqrt{2}w),$$

where w is a kernel of $\bar{w}$ not containing variable y_0 and having the form

$$\begin{aligned}w =& \frac{1}{2} \sum_{3j=N} (y_j^3/3 - y_j y_{N-j}^2) + \sum_{j\neq N/3}^{n-1} [\frac{1}{2}(y_j^2 - y_{N-j}^2) y_{|2j|'} \\ &+ (-1)^{|2j|'} y_j y_{N-j} y_{N-|2j|'}] + \sum_{j<s}^{s<N-j-s} [(y_j y_s - y_{N-j} y_{N-s}) y_{|j+s|'} \\ &+ (-1)^{j+s+|j+s|'} (y_j y_{N-s} + y_s y_{N-j}) y_{N-|j+s|'}].\end{aligned} \tag{7}$$

Notice that if N can not be divided by 3, then the first sum in (7) is absent and therefore the condition $j \neq N/3$ in the second sum drops out. It should be added that if $j = s$ in the last sum, then it practically coincides with the second sum, and if $j = s = |j+s|'$, then it coincides also with the first sum.

As follows from (5), the first equation in (6) under $i = 0$ has the form

$$\frac{3}{\sqrt{2n-1}} \sum_{j=0}^{2n-2} y_j^2 - 3y_0 = 0.$$

The first equation of the second group of (6) has a similar form for $i = 0$. Substituting into them the known value $y_0 = y_0' = \frac{n}{\sqrt{2n-1}}$ (one of the necessary conditions for the existence of antipodal n-gons), we obtain

$$\sum_{i=1}^{2n-2} y_i^2 - \frac{n(n-1)}{2n-1} = 0, \qquad \sum_{i=1}^{2n-2} y_i'^2 - \frac{n(n-1)}{2n-1} = 0.$$

It is obvious that these two equations and the last $n-1$ equations of system (6) are linearly dependent, since

$$\sum_{j=1}^{n-1}(y_j^2 + y_{N-j}^2 + {y_j'}^2 + y_{N-j}'^2 - \frac{2n}{2n-1}) = \sum_{i=1}^{2n-2}(y_i^2 + y_i'^2) - \frac{2n(n-1)}{2n-1}.$$

Thus, one of the first two equations may not be included into transformed system of equations (6). Without loss of generality, we eliminate the second of them, containing y_i', as well as $y_0 = \frac{n}{\sqrt{2n-1}}$ and $y_0' = \frac{n}{\sqrt{2n-1}}$ from the rest of the equations $\bar{W}_i$, $\bar{W}_i'$, $i = 1, 2, \ldots, 2n-2$, by using the expressions of (5) for the derivatives of polynomial $\bar{w}$. Then (6) takes the form

$$\begin{cases} Y_0 &= -\frac{n(n-1)}{2n-1} + \sum\limits_{i=1}^{2n-2} y_i^2 = 0; \\ Y_j &= -\frac{2n}{2n-1} + \quad y_j^2 + y_{N-j}^2 + y_j'^2 + y_{N-j}'^2 = 0, \\ j &= \quad 1, 2, \ldots, n-1; \\ W_i &= \frac{y_i}{\sqrt{4n-2}} + \frac{\partial w}{\partial y_i} = 0, \quad i = 1, 2, \ldots, 2n-2; \\ W_i' &= \frac{y_i'}{\sqrt{4n-2}} + \frac{\partial w'}{\partial y_i'} = 0, \quad i = 1, 2, \ldots, 2n-2, \end{cases} \tag{8}$$

where w' is obtained from w by substituting y_j', y_{N-j}' for y_j, y_{N-j} and so on. Evidently that the existence of solution of the system of algebraic equations (6) leads to the existence of solution of system (8), and conversely, adding the relations $y_0 = y_0' = \frac{n}{\sqrt{2n-1}}$ to the solution of system (8), we obtain the solution of system (6). Thereby, the following theorem is valid.

Theorem 2. *Antipodal convex n-gons exist if and only if there do not exist polynomials $B^j(y, y')$, $j = 0, 1, 2, \ldots, n-1$, $C^i(y, y')$ and $C'^i(y, y')$, $i = 1, 2, \ldots, 2n-2$, such that for the left sides of non-homogeneous algebraic equations (8) the following identity is valid:*

$$L = \sum_{j=0}^{n-1} B^j Y_j + \sum_{i=1}^{2n-2}(C^i W_i + C'^i W_i') \equiv 1. \tag{9}$$

This relation means that after a similar reduction in polynomial L all coefficients, except a constant term, turn into 0. Thus, a question on the proof of the existence of antipodal polygons is reduced to a question on the nonexistence of such polynomials of any finite degree m, for which $L(y, y')$ satisfies identity (9). In this connection, we denote a polynomial in the left side of (9) by L^m, where m points out its degree. Since the degree of $Y_j(y, y')$, $W_i(y), W_i'(y')$ is two, then the degree of polynomials $B^j(y, y'), C^i(y, y')$ and $C'^i(y, y')$ is $m - 2$. Let us denote the coefficients of polynomial $L^m(y, y')$: K_0 is a constant term, K_i is a coefficient at y_i, $i = 1, 2, \ldots, 2n-2$, $K_{i^2j'}$ is a coefficient at $y_i^2 y_j'$, etc. By L_w we denote the following linear combination of the coefficients $L^m(y, y')$:

$$\begin{aligned} L_w = \sum_{3j=N} (K_{j^3} - K_{j(N-j)^2}) + \sum_{j \neq N/3}^{n-1} [K_{j^2|2j|'} - K_{(N-j)^2|2j|'} \\ + (-1)^{|2j|'} K_{j(N-j)(N-|2j|')}] + \sum_{j<s}^{s<N-j-s} [K_{js|j+s|'} - K_{(N-j)(N-s)|j+s|'} \\ + (-1)^{j+s+|j+s|'} (K_{j(N-s)(N-|j+s|')} + K_{s(N-j)(N-|j+s|')}]. \end{aligned} \tag{10}$$

By $L_{w'}$ we denote the corresponding linear combination of the coefficients with primes. It is easy to see that L_w includes only coefficients of polynomial L^m found at its terms coinciding with the monomials of polynomial w and, moreover, having the same signs as w. Since it is a homogeneous polynomial of third degree, then, by Euler rule, $\sum_{i=1}^{2n-2} y_j \frac{\partial w}{\partial y_i} = 3w$. Hence, in a homogeneous polynomial of second degree $\frac{\partial w}{\partial y_i}$ the coefficient sign of the monomial $y_p y_q$ coincides with the one of $y_p y_q y_i$ in polynomial w, and therefore, the coefficient sign of $y_p y_q y_i$ (some of these numbers can be equal each other) of polynomial L_m, $m \geq 3$, included in the combination L_w, coincides with the coefficient sign of $y_p y_q$ in polynomial $\frac{\partial w}{\partial y_i}$ as well as with the coefficient sign of $y_q y_i$ in polynomial $\frac{\partial w}{\partial y_p}$ and the one of $y_p y_i$ in polynomial $\frac{\partial w}{\partial y_q}$. The same can said about the linear combination $L_{w'}$. Let us denote by L_w^m the expression L_w of (10) written via the coefficients of polynomial L^m.

Lemma 1. *If in relation (9) the degree of polynomial L is three, then L_w^3 is expressed via the coefficients of polynomials C^i, $i = 1, 2, \ldots, 2n-2$, in the following way:*

$$L_w^3 = (2n-3) \sum_{i=1}^{2n-2} C_i^i, \tag{11}$$

where C_i^i is a coefficient of y_i in C^i.

P r o o f. Since in all equations of system (8) the degree is two, then the required expression for L_w^3 a priori may contain only the coefficients of polynomials B^j, $j = 0, 1, \ldots, n-1$, and C^i, $i = 1, 2, \ldots, 2n-2$, with the first degree terms but actually it contains only the last coefficients. Non of the coefficients of polynomial B^j with the first degree terms is contained in L^3_w, as the term y_j^2 is contained in the equality Y_j together with the term $y_{(N-j)^2}$ and by (10), the summands $K_{j^2|2j|'}$ and $K_{(N-j)^2|2j|'}$ are contained in L_w with opposite signs.

As follows from (7), the derivative $\frac{\partial w}{\partial y_i}$ for $i = |2j|'$ contains the semidifference $\frac{y_j^2 - y_{N-j}^2}{2}$ and the terms of the form $\pm y_\alpha y_\beta$ $(\alpha \neq \beta)$. If the degree of L is three, then from polynomial C^i, $i = 1, 2, \ldots, 2n-2$, the summands $C^i_i = \frac{1}{2}\mathrm{sign}^2(K_{j^2 i})C^i_i + \frac{1}{2}\mathrm{sign}^2(K_{(N-j)^2 i})C^i_i$ and $C^i_i = \mathrm{sign}^2(K_{\alpha\beta i})C^i_i$ (for each admissible pair of indices α, β) enter into L^3_w. Since in the derivative $\frac{\partial w}{\partial y_i}$ under $i < n$ there are $2n-4$ summands with coefficients equal to 1 (as seen from (5)) for $\frac{\partial \bar{w}}{\partial y_k} : 2(n-k-\ 1) + 2(k-1) = 2n-4$, one summand with the coefficient $+\frac{1}{2}$ and one with the coefficient $-\frac{1}{2}$, then C^i_i is contained in L^3_w with the coefficient $2n-4+2\cdot\frac{1}{2} = 2n-3$. The same coefficient is for $i = N - |2j|' \geq n$. In this case, as follows from (5), $\frac{\partial \bar{w}}{\partial y_i}$ contains $2n-3$ summands with coefficients equal to ± 1). Moreover, when j runs the values from 1 to $n-1$, then $i = |2j|'$ runs the same values but in other order, and $i = N - |2j|'$ runs all values from n to $2n-2$. Thus, L^3_w contains every C^i_i, $i = 1, 2, \ldots, 2n-2$, with the coefficient $2n-3$, as shown in (11).

The conclusion of the lemma will be proved completely if L^3_w, after substituting the coefficients of polynomials C^i from (9), does not contain any other terms of the form C^t_i, $t \neq i$. Complement terms can appear only when in some equation W_t from (8) there is at least one summand $y_p y_q$ with the coefficient $\mathrm{sign}(K^w_{pqt})$, where K^w_{pqt} is a coefficient at $y_p y_q y_t$ in the polynomial w, which occurs in another equation W_i with the coefficient $\mathrm{sign}(K^w_{pqi})$, where K^w_{pqi} is a coefficient at $y_p y_q y_i$ in w. Furthermore, $p \neq q$ and $p + q \neq N$. Otherwise, the term $y_p y_q$ could not be contained in two different equations of system (8) and, as follows from the form of polynomial w of (7), both equations W_t and W_i have the term $y_{N-p} y_{N-q}$ but with the sign $\mathrm{sign}(K^w_{(N-p)(N-q)i})$ in the equation W_i and with the sign $\mathrm{sign}(K^w_{(N-p)(N-q)t})$ in W_t.

In the last sum of the polynomial w of (7) replace index s by $s - j$, and $|j + s|'$ by $|j + (s-j)|' = s$. Taking into account that $j + (s - j) = s < N - s$, we may conclude that this sum contains also the summands $(y_j y_{s-j} - y_{N-j} y_{N-(s-j)}) y_s + (y_j y_{N-(s-j)} + y_{N-j} y_{s-j}) y_{N-s} = (y_j y_s + y_{N-j} y_{N-s}) y_{s-j} + (y_j y_{N-s} - y_{N-j} y_{N-(s-j)}) y_{N-(s-j)}$. Hence, the signs of the monomials $y_p y_q$ and $y_{N-p} y_{N-q}$ are identical in one of the equations W_t or W_i, and they are opposite in another one. Thus, in L^3_w the term C^t_i $(t \neq i)$ as well as C^i_t is with the coefficient equal to $\mathrm{sign}(K^w_{pqt}) \cdot \mathrm{sign}(K^w_{pqi}) + \mathrm{sign}(K^w_{(N-p)(N-q)t}) \cdot \mathrm{sign}(K^w_{(N-p)(N-q)} i) = 0$, i.e., L^3_w actually does not contain C^t_i. If a product of the first two co-factors is 1, then, by virtue of the proved above, a product of the last two coefficients is -1, and conversely. Lemma 1 is proved.

Notice that due to the symmetry of the equations W_i and W'_i in (8), with respect to their variables, a similar relation is valid for $L^3_{w'}$:

$$L^3_{w'} = (2n-3)\sum_{i=1}^{2n-2} C'^i_{i'}. \tag{12}$$

Lemma 2. *A constant term K_0 of the polynomial $L^3(y,y')$ is a linear combination of its other coefficients, namely,*

$$K_0 = -\frac{n}{4n-2}\sum_{i=1}^{2n-2}(K_{i^2}+K_{i'^2}) + \frac{n(L_w+L_{w'})}{(2n-3)(4n-2)^{3/2}}. \tag{13}$$

P r o o f. Since the left side of relation (9) is a polynomial of third degree, and the left sides of equations (8) are polynomials of second degree by the condition of the lemma, then the polynomials B^j, $j=0,1,2,\ldots,n-1$, and C^i, C'^i, $i=1,2,\ldots,2n-2$, are of first degree. Therefore,

$$\begin{aligned} B^j &= B^j_0 + \sum_{k=1}^{2n-2}(B^j_k y_k + B^j_{k'} y'_k), \\ C^i &= C^i_0 + \sum_{s=1}^{2n-2}(C^i_s y_s + C^i_{s'} y'_s), \\ C'^i &= C'^i_0 + \sum_{s=1}^{2n-2}(C'^i_s y_s + C'^i_{s'} y'_s). \end{aligned} \tag{14}$$

Substituting these expressions into (9) and taking into account expressions for Y_j, $j=0,1,\ldots,n-1$, W_i and W'_i, $i=1,2,\ldots,2n-2$, from (8), we can find

$$K_0 = -\frac{n(n-1)}{2n-1}B^0_0 - \frac{2n}{2n-1}\sum_{j=1}^{n-1}B^j_0. \tag{15}$$

Calculate now the coefficients of the polynomial L^3 of the second powers y_i^2 and $y_i'^2$. Let $j=1,2,\ldots,n-1$. Then

$$\begin{aligned} K_{j^2} &= B^0_0 + B^j_0 + \tfrac{1}{2}C^{|2j|'}_0 + \frac{C^j_j}{\sqrt{4n-2}}, \\ K_{(N-j)^2} &= B^0_0 + B^j_0 - \tfrac{1}{2}C^{|2j|'}_0 + \frac{C^{N-j}_{N-j}}{\sqrt{4n-2}}, \\ K_{j'^2} &= B^j_0 + \tfrac{1}{2}C'^{|2j|'}_0 + \frac{C'^j_{j'}}{\sqrt{4n-2}}, \\ K_{(N-j)'^2} &= B^j_0 - \tfrac{1}{2}C'^{|2j|'}_0 + \frac{C'^{N-j}_{(N-j)'}}{\sqrt{4n-2}}. \end{aligned}$$

By summing up these equations termwise, we obtain

$$\sum_{i=1}^{2n-2}(K_{i^2}+K_{i'^2})=(2n-2)B_0^0+4\sum_{j=1}^{n-1}B_0^j+\frac{\sum\limits_{i=1}^{2n-2}(C_i^i+C_{i'}^{\prime i})}{\sqrt{4n-2}}.$$

Eliminating B_0^0 and B_0^j from (15) and applying the last equality, we find

$$K_0=-\frac{n}{4n-2}\sum_{i=1}^{2n-2}(K_{i^2}+K_{i'^2})+\frac{n}{(4n-2)^{3/2}}\sum_{i=1}^{2n-2}(C_i^i+C_{i'}^{\prime i}).$$

Using Lemma 1, for K_0 we obtain the required linear expression via the coefficients of polynomial $L^3(y,y')$. Lemma 2 is proved.

From the lemma above it follows that in the case when the polynomial L in relation (9) is of third degree, then its constant term is a linear combination of its other coefficients turned into 0 by identity (9), and therefore it must turn into 0 too, what contradicts to the same identity (9). To prove that K_0 equals null at any degree of polynomial L, it is possible to use the mathematical induction principle. For this it is sufficient to verify a possibility of the inductive passage at least in the case of the third degree polynomial L when its degree becomes increased by one.

Let the degree of polynomial L of (9) be four. Then the polynomials B^j, C^i and $C^{\prime i}$ of (14) are added the summands of second degree denoted $B^j_{\{2\}}, C^i_{\{2\}}$ and $C^i_{\{2\}}$, respectively. Consequently, the right side of equality (13) obtains the difference $\Delta^2_{K_0}$ induced by the new summands

$$\Delta^2K_0=-\frac{n}{4n-2}\sum_{i=1}^{2n-2}(\Delta^2K_{i^2}+\Delta^2K_{i'^2})+\frac{n(\Delta^2L_w+\Delta^2L_{w'})}{(2n-3)(4n-2)^{3/2}}. \tag{16}$$

Substituting new expressions for $B^j, C^i, C^{\prime i}$ of (9), we get

$$\begin{aligned}\Delta^2K_{i^2}&=-\tfrac{n(n-1)}{2n-1}B^0_{i^2}-\tfrac{2n}{2n-1}\sum_{j=1}^{n-1}B^j_{i^2},\\ \Delta^2K_{i'^2}&=-\tfrac{n(n-1)}{2n-1}B^0_{i'^2}-\tfrac{2n}{2n-1}\sum_{j=1}^{n-1}B^j_{i'^2},\end{aligned} \tag{17}$$

where $B^0_{i^2}$ and $B^j_{i^2}$ are coefficients at y_i^2 in the polynomials B^0 and B^j, $j=1,2,\ldots,n-1$, whereas $B^0_{i'^2}, B^j_{i'^2}$ are analogous coefficients. To find Δ^2L_w we use (10) for L_w expressed via the coefficients of polynomial L

$$\Delta^2L_w=\{\sum_{i=1}^{2n-2}\frac{y_iC^i_{\{2\}}}{\sqrt{4n-2}}\}_{L_w}=\frac{\sum\limits_{i=1}^{2n-2}\sum\limits_{i\in I}^{I\subset L_w}\operatorname{sign}(K_I^w)C^i_{I/i}}{\sqrt{4n-2}}=\frac{C_w}{\sqrt{4n-2}}, \tag{18}$$

where the braces mean that in product $y_i C^i_{\{2\}}$ from all possible terms K_I of third degree we can take the coefficients only of the terms that are in the polynomial w, and with the same sign as in (10). Notice that the subscript in $C^i_{I/i}$ is two-valued obtained by eliminating index i from I. Thus, a linear combination C_w in $\Delta^2 L_w$ is obtained from L_w by raising one by one the subscripts of every summand and substituting "K" by "C"(if in any summand of L_w the index is repeated, then it can be raised only once). An expression similar to (18) can be obtained for $\Delta^2 L_{w'}$.

Substituting expressions (17) and (18) into (16), setting

$$B = \frac{n-1}{2} \sum_{i=1}^{2n-2} (B^0_{i^2} + B^0_{i'^2}) + \sum_{i=1}^{2n-2} \sum_{j=1}^{n-1} (B^j_{i^2} + B^j_{i'^2}),$$

we find

$$\Delta^2 K_0 = \frac{n^2 B}{(2n-1)^2} + \frac{n(C_w + C_{w'})}{(2n-3)(4n-2)^2}, \tag{19}$$

where $C_{w'}$ is determined by analogy with C_w (see equality (18)).

Representation (13) for the constant term K_0 given by (15), obtained under supposition that the degree m of the polynomial $L(y, y')$ is three, when passing to $m = 4$ has the following form:

$$K_0 = -\frac{n}{4n-2} \sum_{i=1}^{2n-2} (K_{i^2} + K_{i'^2}) + \frac{n(L_w + L_{w'})}{(2n-3)(4n-2)^{3/2}} - \Delta^2 K_0, \tag{20}$$

where the first two groups of summands coincide with those of (13) in form, but now they belong to the polynomial $L^4(y, y')$, and $\Delta^2 K_0$ (19) is expressed via the coefficients of additional summands of the second degree of polynomials B^j, C^i, C'^i of (14). To find the representation $\Delta^2 K_0$ via the coefficients of the polynomial $L^4(y, y')$ we have to prove the following lemma.

Lemma 3. *If the degree of the polynomial L in relation (9) is four, then for its coefficients there are valid the following equalities:*

$$[[2 \sum_{i=1}^{2n-2} K_{i^4} + \sum_{s=1}^{n-1} (K_{s^4} + K_{(N-s)^4} + K_{s^2(N-s)^2})]]$$

$$-2 \sum_{i=1}^{2n-2} (K_{i^2 i'^2} + K_{i^2 (N-i)'^2}) + \frac{4}{n-1} \sum_{p,q=1}^{2n-2} K_{p^2 q'^2} = \frac{8B}{n-1} + C, \tag{21}$$

$$[[2 \sum_{i=1}^{2n-2} K_{i^4} + \sum_{p \leq q} K_{p^2 q^2}]] - \sum_{i=1}^{2n-2} (K_{i^2 i'^2} + K_{i^2 (N-i)'^2})$$

$$+\frac{n+1}{n-1}\sum_{p,q=1}^{2n-2} K_{p^2q'^2} = \frac{4nB}{n-1} + C_w + C_{w'}, \tag{22}$$

where $B = \frac{n-1}{2}\sum_{i=1}^{2n-2}(B^0_{i^2}+B^0_{i'^2})+\sum_{i=1}^{2n-2}\sum_{j=1}^{n-1}(B^j_{i^2}+B^j_{i'^2})$, $C = [[\sum_{s=1}^{n-1}(C^{|2s|'}_{s^2} - C^{|2s|'}_{(N-s)^2} + (-1)^{|2s|'}C^{N-|2s|'}_{s(N-s)}]]$, $C_w = \sum_{i\in I}^{I\subset L_w} \mathrm{sign}(K^w_I)C^i_{I/i}$, $C_{w'} = \sum_{i\in I}^{I\subset L_w} \mathrm{sign}(K^w_I)C'^i_{(I/i)'}$, *and the expression enclosed within double square brackets should be added the same expression with the subscripts* $i', s', (N-s)', p', q'$.

P r o o f. All coefficients of L in relations (21) and (22) are coefficients at the products of monomials of second degree from $B^j_{\{2\}}, C^i_{\{2\}}, C'^i_{\{2\}}$ and monomials of second degree in the left sides of the equations of system (8): $\{B^0_{\{2\}}\sum_{i=1}^{2n-2} y_i^2 + \sum_{j=1}^{n-1} B^j_{\{2\}}(y_j^2 + y^2_{N-j} + y'^2_j + y'^2_{N-j}) + \sum_{i=1}^{2n-2}(C^i_{\{2\}}\frac{\partial w}{\partial y_i} + C'^i_{\{2\}})\frac{\partial w'}{\partial y'_i}$. Notice that not all these products are considered but only those the first co-factors of which have the coefficients $B^j_{i^2}(B^j_{i'^2})$, or $C^i_{\alpha\beta}(C'^i_{\alpha'\beta'})$, where α and β are such that there is a monomial of the form $y_\alpha y_\beta y_i$ in w as well as in w'. We have for $s < n$:

$$\begin{aligned} K_{s^4} =& B^0_{s^2} + B^s_{s^2} + \frac{1}{2}C^{|2s|'}_{s^2}, \\ K_{(N-s)^4} =& B^0_{(N-s)^2} + B^s_{(N-s)^2} - \frac{1}{2}C^{|2s|'}_{(N-s)^2}, \\ K_{s^2(N-s)^2} =& B^0_{s^2} + B^0_{(N-s)^2} + B^s_{s^2} + B^s_{(N-s)^2} \\ & - \frac{1}{2}(C^{|2s|'}_{s^2} - C^{|2s|'}_{(N-s)^2}) + (-1)^{|2s|'}C^{N-|2s|'}_{s(N-s)}. \end{aligned} \tag{23}$$

By summing up these equalities termwise over s from 1 to $n-1$, we have

$$\begin{aligned} &\sum_{s=1}^{n-1}(K_{s^4} + K_{(N-s)^4} + K_{s^2(N-s)^2}) \\ &= 2\sum_{i=1}^{2n-2} B^0_{i^2} + 2\sum_{s=1}^{n-1}(B^s_{s^2} + B^s_{(N-s)^2}) + \sum_{s=1}^{n-1}(-1)^{|2s|'}C^{N-|2s|'}_{s(N-s)}. \end{aligned} \tag{24}$$

Since there are no summands of the form y'^2_i in the left side of the first equation of system (8), by analogy we obtain

$$\begin{aligned} &\sum_{s=1}^{n-1}(K_{s'^4} + K_{(N-s)'^4} + K_{s'^2(N-s)'^2}) \\ &= 2\sum_{s=1}^{n-1}(B^s_{s'^2} + B^s_{(N-s)'^2}) + \sum_{s=1}^{n-1}(-1)^{|2s|'}C'^{N-|2s|'}_{s'(N-s)'}. \end{aligned} \tag{25}$$

Further we have

$$K_{s^2s'^2} = B^0_{s'^2} + B^s_{s'^2} + B^s_{s^2} + \frac{1}{2}(C^{|2s|'}_{s'^2} + C'^{|2s|'}_{s^2}),$$

$$K_{(N-s)^2(N-s)'^2} = B^0_{(N-s)'^2} + B^s_{(N-s)'^2} + B^s_{(N-s)^2} - \frac{1}{2}(C^{|2s|'}_{(N-s)'^2} + C'^{|2s|'}_{(N-s)^2}),$$

$$K_{s^2(N-s)'^2} = B^0_{(N-s)'^2} + B^s_{(N-s)'^2} + B^s_{s^2} + \frac{1}{2}(C^{|2s|'}_{(N-s)'^2} - C'^{|2s|'}_{s^2}),$$

$$K_{(N-s)^2s'^2} = B^0_{s'^2} + B^s_{s'^2} + B^s_{(N-s)^2} - \frac{1}{2}(C^{|2s|'}_{s'^2} - C'^{|2s|'}_{(N-s)^2}).$$

By summing up these equalities termwise over s from 1 to $n-1$, we get

$$\sum_{i=1}^{2n-2}(K_{i^2i'^2} + K_{i^2(N-i)'^2}) = 2\sum_{i=1}^{2n-2} B^0_{i'^2} + 2\sum_{s=1}^{n-1}(B^s_{s^2} + B^s_{s'^2} + B^s_{(N-s)^2} + B^s_{(N-s)'^2}). \tag{26}$$

Substituting the values K_{s^4} and $K_{(N-s)^4}$ from (23) into the first summand of equation (21) with regard to relations (24–26), we obtain

$$[[2\sum_{i=1}^{2n-2} K_{i^4} + \sum_{s=1}^{n-1}(K_{s^4} + K_{(N-s)^4} + K_{s^2(N-s)^2})]]$$

$$-2\sum_{i=1}^{2n-2}(K_{i^2i'^2} + K_{i^2(N-i)'^2}) = 4\sum_{i=1}^{2n-2}(B^0_{i^2} - B^0_{i'^2}) + C. \tag{27}$$

To establish the first conclusion of the lemma we have to find the sum $\sum\limits_{p<q} K_{p^2q'^2}$, taking into account that under $1 \le p < q < N$

$$K_{p^2q'^2} = B^0_{q'^2} + B^{|p|'}_{q'^2} + B^{|p|'}_{p^2} + \frac{1}{2}[(-1)^{p+|p|'}C^{|2p|'}_{q'^2} + (-1)^{q+|q|'}C^{|2q|'}_{p^2}],$$

where $|p|' = \min(p, N-p), |2p|' = \min(|2p|, N-|2p|)$, and $|2p|$ is the smallest positive residue by modulus $2n-1$

$$\sum_{p<q} K_{p^2q'^2} = (2n-2)\sum_{i=1}^{2n-2} B^0_{i'^2} + 2\sum_{i=1}^{2n-2}\sum_{j=1}^{n-1}(B^j_{i^2} + B^j_{i'^2}). \tag{28}$$

Multiplying this equality by $\frac{4}{n-1}$ and adding termwise to equality (27), we obtain relation (21) of the lemma.

To prove the second conclusion of the lemma, first we have to find $\sum\limits_{p<q}^{p+q\neq N} K_{p^2q^2}$, where $1 \le p < q < N$ (the cases of $p = q$ and $p+q = N$ are in (23)). If $p < q < n$ or $n \le p < q$, then

$$K_{p^2q^2} = B^0_{p^2} + B^0_{q^2} + B^{|p|'}_{q^2} + B^{|q|'}_{p^2} + \frac{1}{2}[(-1)^{p+|p|'}C^{|2p|'}_{q^2}$$

$$+(-1)^{q+|q|'}C_{p^2}^{|2q|'}] + \mathrm{sign}(K^w_{pq|q-p|'})C_{pq}^{|q-p|'} + \mathrm{sign}(K^w_{pq|q+p|'})C_{pq}^{|q+p|'}. \qquad (29)$$

Thus, if $p < n \le q$, then

$$K_{p^2q^2} = B^0_{p^2} + B^0_{q^2} + B^{|p|'}_{q^2} + B^{|q|'}_{p^2} + \frac{1}{2}[(-1)^{p+|p|'}C^{|2p|'}_{q^2} + (-1)^{q+|q|'}C^{|2q|'}_{p^2}]$$

$$+\mathrm{sign}(K^w_{pq(N-|q-p|')})C^{N-|q-p|'}_{pq} + \mathrm{sign}(K^w_{pq(N-|q+p|')})C^{N-|q+p|'}_{pq}. \qquad (30)$$

There are similar representations for $K_{p'^2q'^2}$. Using equalities (23, 26, 29, 30), we obtain

$$[[2\sum_{i=1}^{2n-2} K_{i^4} + \sum_{p\le q}^{2n-2} K_{p^2q^2}]] - \sum_{i=1}^{2n-2}(K_{i^2i'^2} + K_{i^2(N-i)'^2})$$

$$= 2n\sum_{i=1}^{2n-2} B^0_{i^2} - 2\sum_{i=1}^{2n-2} B^0_{i'^2} + 2\sum_{i=1}^{2n-2}\sum_{j=1}^{n-1}(B^j_{i^2} + B^j_{i'^2}) + C_w + C_{w'}. \qquad (31)$$

Multiplying equation (28) by $\frac{n-1}{n+1}$ and adding to (31), we obtain the required relation (22). Lemma 3 is proved.

As one can see, to find an expression for $\Delta^2 K_0$ (19) via the polynomial L^4, relations (21) and (22) are insufficient. There should be found a relation between the sum of the differences $C_w + C_{w'}$ and C, for which we will define a new linear combination L_{2w} of the coefficients of polynomial L^4 constructed by the linear combination L_w. Namely, if L_w contains the coefficients $K_{\alpha\beta i}$ and $K_{i\gamma\delta}$, then L_{2w} contains the coefficient $K_{\alpha\beta\gamma\delta}$ with the sign: $\mathrm{sign}(K_{\alpha\beta\gamma\delta}) = \mathrm{sign}(K_{\alpha\beta i})\cdot\mathrm{sign}(K_{i\gamma\delta})$. Moreover, by definition, L_{2w} contains only one term with given indices $\alpha\beta\gamma\delta$ and no one of the form $K_{\alpha^2\gamma^2}$ or $K_{\alpha\beta(N-\alpha)(N-\beta)}$. The sign of $K_{\alpha\beta\gamma\delta}$ does not depend on permutation of indices $\alpha, \beta, \gamma, \delta$, what can be shown by means of arguments used above (mainly in the proof of Lemma 1). A linear combination $L_{2w'}$ is defined in a similar way.

Lemma 4. *For the linear combinations L_{2w} and $L_{2w'}$ of the coefficients of polynomial $L^4(y, y')$ there is true the equality*

$$L_{2w} + L_{2w'} = (2n-5)(C_w + C_{w'}) + C, \qquad (32)$$

where $C_w, C_{w'}$ and C have the same values as in Lemma 3.

P r o o f. To prove this lemma, it is not necessary first to find the linear combination L_w and then to define the above combination L_{2w}. It is sufficient to use the system of equations (8) having the derivatives $\frac{\partial w}{\partial y_i}$, $i = 1, 2, \ldots, 2n-2$. In fact, any monomial $y_p y_q$ contained in the derivative $\frac{\partial w}{\partial y_i}$ has $\mathrm{sign}(K^w_{ipq})$ which coincides with the sign of the corresponding monomial in the polynomial w (7)

(if w does not contain the monomial $y_i y_p y_q$, we suppose $\mathrm{sign}(K^w_{ipq}) = 0$). Hence, the form of L_{2w} expressed via C_w and C can be determined by the coefficients of the fourth degree polynomial $\sum_{i=1}^{2n-2} C^i_{\{2\}} \frac{\partial w}{\partial y_i}$, where $C^i_{\{2\}}$ is a quadratic part of the polynomial C^i of (9).

Let $i = |2j|'$, $j < n$. Then, in the i-equation of system (8) the derivative $\frac{\partial w}{\partial y_i}$ contains $2n-4$ summands with coefficients equal to ± 1 and the semi-difference $\frac{1}{2}(y_j^2 - y_{N-j}^2)$. Moreover, there is a monomial $y_{N-p}y_{N-q}$ along with $y_p y_q$, $p < q < n$ or $n \le p < q$, and by definition there is neither summand $K_{pq(N-p)(N-q)}$ nor $K_{j^2(N-j)^2}$ in L_{2w}. Thus, if L_{2w} contains the summand K_{pqrs}, then it contains C^i_{pq} with the sign $\mathrm{sign}(K^w_{irs}) \cdot \mathrm{sign}(K_{pqrs}) = \mathrm{sign}(K^w_{ipq})$, as by definition $\mathrm{sign}(K_{pqrs}) = \mathrm{sign}(K^w_{ipq}) \cdot \mathrm{sign}(K^w_{irs})$, and the coefficient at $\mathrm{sign}(K^w_{ipq})C^i_{pq}$ is $(2n-4) - 2 + 2 \cdot \frac{1}{2} = 2n-5$. Besides, L_{2w} contains $\mathrm{sign}(K^w_{ij^2})C^i_{j^2} = C^{|2j|'}_{j^2}$ and $\mathrm{sign}(K^w_{i(N-j)^2})C^i_{(N-j)^2} = -C^{|2j|'}_{(N-j)^2}$ (see the first two summands of the second sum in (7)) with the coefficient $2n-4 = (2n-5)+1$. Thus, when $i = |2j|' < n$, L_{2w} contains

$$(2n-5)[\sum_{i=1}^{n-1} \sum_{p<q} \mathrm{sign}(K^w_{ipq})C^i_{pq} + \sum_{j=1}^{n-1}(C^{|2j|'}_{j^2} - C^{|2j|'}_{(N-j)^2})] + \sum_{j=1}^{n-1}(C^{|2j|'}_{j^2} - C^{|2j|'}_{(N-j)^2})$$

as summands. When $i = N - |2j|' \ge n$, the derivative $\frac{\partial w}{\partial y_i}$ contains $2n-3$ summands with the coefficients ± 1. What is more, it contains the summand $y_p y_q$, $p < n \le q$, when $p + q \ne N$, as well as the summand $y_{N-p}y_{N-q}$ and the term $y_j y_{N-j}$. Therefore, in L_{2w} there is a summand $\mathrm{sign}(K^w_{ipq})C^i_{pq}$ with the coefficient $(2n-4)-2+1 = 2n-5$ and a summand $\mathrm{sign}(K^w_{ij(N-j)})C^i_{j(N-j)} = (-1)^{|2j|'}C^{|2j|'}_{j(N-j)}$ (see the last summand of the second sum in (7)) with the coefficient $2n-4 = (2n-5)+1$ if $\mathrm{sign}(K^w_{ij(N-j)}) \ne 0$. Thus, when $i \ge n$, L_{2w} contains the following summands:

$$(2n-5)[\sum_{i=n}^{2n-2} \sum_{p<q} \mathrm{sign}(K^w_{ipq})C^i_{pq} + \sum_{j=1}^{n-1}(-1)^{|2j|'}C^{|2j|'}_{j(N-j)}] + \sum_{j=1}^{n-1}(-1)^{|2j|'}C^{|2j|'}_{j(N-j)}.$$

Summing up the obtained relations termwise and taking into account that $L_{2w'}$ satisfies similar equalities, we arrive to equality (32). Lemma 4 is proved.

As we can see, by Lemmas 3 and 4, the expressions $B, C_w\ C_{w'}$, contained in the difference $\Delta^2 K_0$ (19), are combinations of the coefficients $L^4(y, y')$ at monomials of fourth degree. Thus, by (20), the constant term K_0 of the polynomial L^4 is a linear combination of its other coefficients equal to null by (9), and K_0 must turn into null too, what contradicts to (9). Hence, a degree of the polynomial $L(y, y')$ of (9) should be greater than four.

Our method of constructing of half-circulant Hadamard matrices of order $4n$ (see Basic Lemma in [4]) is universal, i.e., it can be applied to any integer n, and, consequently, universal is the initial equation system (6) for the proposed inductive approach (notice that the well-known method used by J. Williamson [10] works only for odd n). Hence, the obtained above results give an opportunity to use the inductive approach to the proof of the existence of antipodal n-gons for any integer $n \geq 3$ as well as of half-circulant Hadamard matrices of order $4n$. Thus, there is valid the following theorem.

Therem 3. *If for any natural $m \geq 3$ the assumption that a constant term of the polynomial $L^m(y, y')$ defined by (9) is a linear combination of its other coefficients involves the same assertion for the polynomial L^{m+1}, then there exists a half-circulant Hadamard matrix of any order $4n$, $n \geq 3$.*

Notice that (by analogy with the considered above passage from degree $m = 3$ to degree $m + 1 = 4$ of polynomial L) to prove the specified in Theorem 3 properties of the polynomial L of any finite degree $m + 1$, it is sufficient to show that the difference $\Delta^{m-1}K_0$ obtained by replacing the degree m by $m + 1$ of L, can be expressed linearly via the coefficients of L^{m+1} at terms of degree $(m + 1)$.

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