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Subharmonic almost periodic functions

A.V. Rakhnin and S.Yu. Favorov

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine

E-mail:rakhnin@univer.kharkov.ua
favorov@assa.kharkov.ua

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We prove that almost periodicity in the sense of distributions coincides with almost periodicity with respect to Stepanov's metric for the class of subharmonic functions in a strip $\{z \in \mathbb{C} : a < \mathrm{Im} z < b\}$. We also prove that Fourier coefficients of these functions are continuous functions in $\mathrm{Im} z$. Further, if the logarithm of a subharmonic almost periodic function is a subharmonic function, then it is almost periodic.

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Subharmonic almost periodic functions were introduced in [2] in connection with investigation of zero distribution of holomorphic almost periodic functions in a strip. In this paper almost periodicity was defined in the sense of distributions, namely as almost periodicity of the convolution with a test function. However, subharmonic functions $\log |f(z)|$, where $f(z)$ is a holomorphic almost periodic function, were considered much earlier in papers [5] and [6], where the important point was to prove almost periodicity of such functions in the sense of distributions. In [4] this was extended to a subharmonic uniformly almost periodic function whose logarithm is a subharmonic function.

In this paper we prove that subharmonic almost periodic in the sense of distributions functions are almost periodic in the classical sense, if we consider Stepanov integral metric instead of the uniform metric. Therefore the classes of subharmonic almost periodic in the sense of distributions functions and subharmonic Stepanov almost periodic functions are the same.

Now the Fourier–Bohr coefficients of such functions can be defined in the usual way. For a horizontal strip these coefficients are functions depending on $\mathrm{Im} z$. In this paper we prove that these coefficients depend continuously on $\mathrm{Im} z$, which allows us to approximate any subharmonic almost periodic function by exponential sums with continuous coefficients in Stepanov metric. Thus we prove that subharmonic almost periodic functions are Stepanov almost periodic in the sense of the definition in [8].

In [2] it was proved that $\exp(u)$, where u is a subharmonic almost periodic in the sense of distributions function, is also almost periodic in the sense of distributions. Moreover, for an almost periodic function $\log|f(z)|$, where $f(z)$ is a holomorphic function, $|f(z)|$ is uniformly almost periodic. Conversely, we prove that the logarithm of a subharmonic almost periodic function is an almost periodic function, provided it is a subharmonic function. Thus we obtain a stronger than the one in [4], as well as the converse to the result in [2].

We start with the following definitions and notations (see [1, p. 51]).

Definition 1. *A continuous function $f(z)$ ($z = x + iy$), defined on $\mathbb{R} + iK$, where K is a compact subset of $\mathbb{R}$ (it is allows that $K = \{0\}$), is called uniformly almost periodic (Bohr almost periodic), if from any sequence $\{t_n\} \subset \mathbb{R}$ one can choose a subsequence $\{t_{n'}\}$ such that the functions $f(z + t_{n'})$ converge uniformly on $\mathbb{R} + iK$.*

Equivalent definition is the following:

For any $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that each interval of length $L(\varepsilon)$ contains a real number τ with the property

$$\sup_{z \in \mathbb{R}+iK} |f(z+\tau) - f(z)| < \varepsilon.$$

Definition 2. *A distribution $f(z) \in D'(S)$ of order 0 (S is an open horizontal strip) is called almost periodic, if for any test function $\varphi \in D(S)$ the convolution*

$$\int u(z)\varphi(z-t)dxdy$$

is uniformly almost periodic on the real axis.

Note that according to [6], for an almost periodic distribution $f(z)$ from any sequence $\{h_n\} \subset \mathbb{R}$ one can choose a subsequence $\{h_{n'}\}$, such that $\int \varphi(z)f(z + h_{n'})dxdy$ converge uniformly on every set $\Gamma_K = \{\varphi(z+t) : t \in \mathbb{R}, \varphi \in K\}$, where K is a compact subset of $D(S)$.

Any subharmonic function is locally integrable, so we can consider it as a distribution.

A class of subharmonic almost periodic functions in an open strip S will be denoted by $WAP(S)$.

Furthermore, for $-\infty < \alpha < \beta < +\infty$ we define

$$S_{[\alpha,\beta]} = \{z \in \mathbb{C} : \alpha \le \mathrm{Im} z \le \beta\},$$

$$\mathrm{Im}S = \{\mathrm{Im}z : z \in S\},$$

and for functions u, v, which are integrable on horizontal intervals in $S_{[\alpha,\beta]}$, we denote

$$d_{[\alpha,\beta]}(u,v) := \sup_{z\in S_{[\alpha;\beta]}} \int_0^1 |u(z+t) - v(z+t)|dt.$$

Definition 3. *A function $f(z)$ integrable on horizontal intervals in an open horizontal strip S is called Stepanov almost periodic, if from any sequence $\{h_n\} \subset \mathbb{R}$ one can choose a subsequence $\{h_{n'}\}$ and a function $g(z)$ such that the functions $f(z+h_{n'})$ converge to $g(z)$ in the topology defined by seminorms $d_{[\alpha,\beta]}$, $\alpha, \beta \in \mathrm{Im}S$.*

A class of a subharmonic Stepanov almost periodic functions in an open strip S will be denoted by $StAP(S)$. Since such functions are Stepanov almost periodic on every line $y = const$, for $u \in StAP(S)$ there exists the mean value

$$M(u,y) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(x+iy)dx.$$

To each such u we can associate Fourier–Bohr series

$$u(z) \sim \sum_{\lambda\in\mathbb{R}} a_\lambda(u,y)e^{i\lambda x},$$

where

$$a_\lambda(u,y) := M(ue^{-i\lambda x}, y).$$

are Fourier–Bohr coefficients.

Definition 4. *A function $u(z) \geq 0$ is called logarithmic subharmonic in a domain $G \subset \mathbb{C}$, if the function $\log u(z)$ is subharmonic in this domain.*

It is easy to see that a logarithmic subharmonic function is subharmonic.

We prove the following theorems:

Theorem 1. *$u(z) \in WAP(S)$ if and only if $u(z) \in StAP(S)$.*

Theorem 2. *Let $u(z)$ be a logarithmic subharmonic function in a strip S. Then $\log u(z) \in WAP(S)$ if and only if $u(z) \in WAP(S)$.*

Theorem 3. *Let $u(z)$ be a subharmonic almost periodic function in a strip S. Then its Fourier–Bohr coefficients are continuous in $\mathrm{Im}S$.*

From Theorem 3 and Bessel inequality for Fourier–Bohr coefficients it follows that spectrum of an almost periodic subharmonic function $u(z)$ (i.e., the set

$\{\lambda \in \mathbb{R} : a_\lambda(u, y) \not\equiv 0\}$) it is most countable, which also follows from Theorem 1.12 in [6].

Theorem 4. *Subharmonic function $u(z)$ in an open horizontal strip S is almost periodic if and only if there exists a sequence of finite exponential sums*

$$P_m(z) = \sum_{n=1}^{N_m} a_n^{(m)}(y) e^{i\lambda_n^{(m)} x}, \tag{1}$$

where $\lambda_n \in \mathbb{R}$, $a_n^{(m)}(y) \in C(\mathrm{Im} S)$, which converges to the function $u(z)$ in the topology defined by seminorms $d_{[\alpha,\beta]}$, $\alpha, \beta \in \mathrm{Im} S$.

Moreover, $P_m(z)$, $m = 1, 2, \ldots$ are subharmonic functions in S.

To prove the theorems above we use the following propositions:

Proposition 1. *Convergence of subharmonic functions in $D'(G)$ is equivalent to the convergence in $L^1_{loc}(G)$* (see [7]).

Proposition 2. *Weak limit of subharmonic functions is subharmonic function* (see [7]).

We denote by G^μ the Green potential of a measure μ for the disk $B(R, 0)$, i.e.

$$G^\mu(z) := \int\limits_{B(R,z_0)} \log \frac{|R^2 - z\overline{\zeta}|}{R|z - \zeta|} d\mu(\zeta).$$

Lemma 1. *Let measures μ_n converge weakly to a measure μ in a neighborhood of the disk $\overline{B(R, 0)}$, and $\mu(\partial B(R, 0)) = 0$. Then for any $t_1 > 0$, $t_2 > 0$ such that $t_1^2 + t_2^2 < R^2$,*

$$\lim_{n\to\infty} \sup_{y\in[-t_2;t_2]} \int\limits_{-t_1}^{t_1} |G^{\mu_n}(z) - G^\mu(z)|\, dx = 0, \tag{2}$$

where $z = x + iy$.

P r o o f. Denote $\nu_n = \mu_n - \mu$. We have

$$\sup_{y\in[-t_2;t_2]} \int\limits_{-t_1}^{t_1} |G^{\mu_n}(z) - G^\mu(z)|\, dx \le \sup_{y\in[-t_2;t_2]} \int\limits_{-t_1}^{t_1} \left| \int\limits_{B(R,0)} \log \frac{|R^2 - z\overline{\zeta}|}{R} d\nu_n(\zeta) \right| dx$$

$$+ \sup_{y\in[-t_2;t_2]} \int\limits_{-t_1}^{t_1} \left| \int\limits_{B(R,0)} \log |z - \zeta| d\nu_n(\zeta) \right| dx. \tag{3}$$

The condition $\mu(\partial B(R,0)) = 0$ implies that the restrictions of the measures μ_n to the disk $\overline{B(R,0)}$ converge weakly to the restriction of the measure μ on the disk, and the function $\log(|R^2 - z\overline{\zeta}|R^{-1})$ is continuous for $|x| \le t_1$, $|y| \le t_2$, $\zeta \in \overline{B(R,0)}$. Thus the first term on the right-hand side of (3) is small. Without loss of generality, we can assume that $R < 1/2$, so that for $z, \zeta \in B(R,0)$ we have $\log|z-\zeta| < 0$.

Let $\varepsilon > 0$ be an arbitrary fixed number. We denote $\log_\varepsilon |z-\zeta| = \max\{\log|z-\zeta|, \log\varepsilon\}$. This function is continuous for $|x| \le t_1$, $|y| \le t_2$, $\zeta \in \overline{B(R,0)}$ and for any $\varepsilon > 0$. We have

$$\sup_{y\in[-t_2;t_2]} \int_{-t_1}^{t_1} \left| \int_{B(R,0)} \log|z-\zeta| d\nu_n(\zeta) \right| dx \le \sup_{y\in[-t_2;t_2]} \int_{-t_1}^{t_1} \left| \int_{B(R,0)} \log_\varepsilon |z-\zeta| d\nu_n(\zeta) \right| dx$$

$$+ \sup_{y\in[-t_2;t_2]} \int_{-t_1}^{t_1} \int_{B(R,0)} |\log|z-\zeta| - \log_\varepsilon |z-\zeta|| \, d|\nu_n|(\zeta) dx.$$

The first term on the right-hand side of this inequality is small when n is sufficiently large. Then

$$\sup_{y\in[-t_2;t_2]} \int_{-t_1}^{t_1} \int_{B(R,0)} |\log|z-\zeta| - \log_\varepsilon |z-\zeta|| \, d|\nu_n|(\zeta) dx$$

$$= \sup_{y\in[-t_2;t_2]} \int_{B(R,0)} \int_{[-t_1;t_1]\cap\{x:|x+iy-\zeta|\le\varepsilon\}} (\log\varepsilon - \log|z-\zeta|) dx d|\nu_n|(\zeta)$$

$$\le \int_{B(R,0)} \int_{-\varepsilon}^{\varepsilon} (\log\varepsilon - \log|x|) dx d|\nu_n|(\zeta) \le 2\varepsilon |\nu_n|(B(R,0)).$$

Note that since $\varepsilon > 0$ is arbitrary, and measures ν_n weakly converge to zero, one can choose a constant $C \in \mathbb{R}$ with $|\nu_n|(B(R,0)) < C$. The lemma is proved.

Lemma 2. *Let $u_n(z)$ be a sequence of subharmonic functions in a domain $G \subset \mathbb{C}$ converging to a function $u_0(z) \not\equiv -\infty$ in $D'(G)$, and let*

$$\sup_{z\in G'} u_n(z) \le W(G') < \infty$$

for any subdomain $G' \subset G$. Then for any rectangle $[a;b] \times [\alpha;\beta] \subset G$,

$$\lim_{n\to\infty} \sup_{y\in[\alpha;\beta]} \int_a^b |u_n(z) - u_0(z)| \, dx = 0. \tag{4}$$

P r o o f. For every disk $B(z_0,R)\subset\subset G$ we have the following representation

$$u_n(z)=-G_R^{\mu_n}(z;z_0)+H_R(z;z_0;u_n)\ \ n=0,1,2\dots,$$

where μ_n are the Riesz measures of the functions $u_n(z)$, $G_R^{\mu_n}(z;z_0)$ is the Green potential of the measure μ_n in the disk $B(z_0,R)$, and $H_R(z;z_0;u_n)$ are the best harmonic majorants of the functions $u_n(z)$ in this disk. Conditions of the lemma imply that μ_n converge weakly to the measure μ_0. Without loss of generality, we can assume that $\mu(\partial B(z_0,R))=0$, and using Lemma 1 we conclude that for any t_1,t_2, $t_1^2+t_2^2<R^2$

$$\sup_{y_0-t_2\le y\le y_0+t_2}\int\limits_{-t_1+x_0}^{t_1+x_0}|G_R^{\mu_n}(z;z_0)-G_R^{\mu_0}(z;z_0)|dx\longrightarrow 0,$$

when $n\to\infty$. From this it follows that the functions $H_R(z;z_0;u_n)$ converge to the function $H_R(z;z_0;u_0)$ in $D'(B(z_0,R))$. Now using the mean value property, Harnak inequality, and obvious inequality

$$H_R(z;z_0;u_n)\le W(B(z_0,R))<\infty,\ \ n=0,1,2\dots,$$

we obtain uniform convergence of $H_R(z;z_0;u_n)$ to the function $H_R(z;z_0;u_0)$ in the rectangle $[-t_1+x_0,t_1+x_0]\times[-t_2+y_0,t_2+y_0]$. Covering the rectangle $[\alpha,\beta]\times[a,b]$ by a finite number of such rectangles, we prove the lemma.

P r o o f o f T h e o r e m 1. Inclusion $StAP(S)\subset WAP(S)$ is obvious. We prove the opposite inclusion. We consider arbitrary substrip $S_{[\alpha,\beta]}$, $\alpha,\beta\in\mathrm{Im}S$ and a sequence $\{h_j\}\subset\mathbb{R}$. Since $u(z)$ is a subharmonic almost periodic distribution, there exists a subsequence $\{h_{j_k}\}$ such that for some subharmonic (clearly also almost periodic) function $v(z)$ and for any $\varphi\in D(S_{[\alpha,\beta]})$, uniformly in $t\in\mathbb{R}$,

$$\lim_{k\to\infty}\int\limits_S(u(z+h_{j_k}+t)-v(z+t))\varphi(z)dxdy=0. \tag{5}$$

Now we will show that the functions $u(z+h_k)$ converge to $v(z)$ in the topology defined by seminorms $d_{[\alpha,\beta]}$, $\alpha,\beta\in\mathrm{Im}S$. Assuming the contrary, there exist $\varepsilon_0>0$, $\alpha,\beta\in\mathrm{Im}S$ such that for an infinite sequence k'

$$d_{[\alpha,\beta]}(u(z+h_{j_{k'}}),v(z))>\varepsilon_0,$$

and therefore there exists a subsequence $\{t_{k'}\}\in\mathbb{R}$ such that

$$\sup_{y\in[\alpha;\beta]}\int\limits_0^1|u(z+h_{j_{k'}}+t_{k'})-v(z+t_{k'})|dx>\varepsilon_0. \tag{6}$$

Passing to a subsequence (if necessary), we can assume that

$$u(z+h_{j_{k'}}+t_{k'}) \to w(z),\ v(z+t_{k'}) \to w_1(z) \text{ in } D'(S_{[\alpha,\beta]}).$$

Lemma 2 implies

$$\sup_{y\in[\alpha,\beta]} \int_0^1 |u(z+h_{j_{k'}}+t_{k'}) - w(z)|dx \to 0,\ \ k' \to \infty,$$

and

$$\sup_{y\in[\alpha,\beta]} \int_0^1 |v(z+t_{k'}) - w_1(z)|dx \to 0,\ \ k' \to \infty,$$

and thus inequality (6) implies

$$\sup_{y\in[\alpha;\beta]} \int_0^1 |w(z) - w_1(z)|dx \geq \varepsilon_0. \tag{7}$$

On the other hand, using (5), for any test function $\varphi(z)$,

$$\int\limits_{S_{[\alpha,\beta]}} (w(z) - w_1(z))\varphi(z)dxdy = \lim_{k'\to\infty} \int\limits_{S_{[\alpha,\beta]}} (u(z+h_{j_{k'}}+t'_k) - v(z+t_{k'}))\varphi(z)dxdy$$

$$= \lim_{k'\to\infty} \int\limits_{S_{[\alpha,\beta]}} (u(z+h_{j_{k'}}) - v(z))\varphi(z-t_{k'})dxdy = 0,$$

and thus $w(z) = w_1(z)$ almost everywhere. Since $w(z)$ and $w_1(z)$ are subharmonic functions, then $w(z) \equiv w_1(z)$, which contradicts (7). The theorem is proved.

To prove Theorem 2 we need the following lemmas.

Lemma 3. *Let $\varphi(t)$ be a function continuous in $[-c,c]$. Then for any $\varepsilon > 0$ there exists δ, depending on φ and ε, such that for any two integrable on compact set K functions $f, g:\ K \to [-c,c]$ the inequality*

$$\int\limits_K |f(x) - g(x)|dm < \delta$$

implies the inequality

$$\int\limits_K |\varphi(f(x)) - \varphi(g(x))|dm < \varepsilon. \tag{8}$$

P r o o f. Choose $\tau > 0$ such that $|t_1 - t_2| < \tau$ implies $|\varphi(t_1) - \varphi(t_2)| < \frac{\varepsilon}{2m(K)}$, and denote

$$A_1 = \{x \in K : |f(x) - g(x)| < \tau\},$$

$$A_2 = \{x \in K : |f(x) - g(x)| \geq \tau\}.$$

Notice that $m(A_2) \leq \frac{1}{\tau}\int_{A_2} |f(x) - g(x)|dm$, and therefore

$$\int\limits_K |\varphi(f(x)) - \varphi(g(x))|dm \leq \int\limits_{A_1} |\varphi(f(x)) - \varphi(g(x))|dm + \int\limits_{A_2} |\varphi(f(x)) - \varphi(g(x))|dm$$

$$\leq \frac{m(A_1)\varepsilon}{2m(K)} + \frac{2\sup|\varphi(t)|}{\tau}\int\limits_K |f(x) - g(x)|dm.$$

Choosing suitable δ, (8) follows. The lemma is proved.

Lemma 4. *Let $u_n(z)$ be a sequence of uniformly bounded from above logarithmic subharmonic functions in a domain $G \subset \mathbb{C}$, converging to a function $u_0(z) \not\equiv 0$ in the sense of distributions. Then the functions $\log u_n(z)$ converge to the function $\log u_0(z)$ in the sense of distributions.*

P r o o f. The functions $u_n(z)$ are logarithmic subharmonic, and in particular subharmonic. Using Proposition 1, $u_n(z)$ converge to $u_0(z)$ in $L^1_{loc}(G)$.

Next, these functions are uniformly bounded from above by some constant $V > 0$, bounded from below by 0, and the function $l_\varepsilon(t) = \log\max\{\varepsilon, t\}$ is continuous in the interval $[0, V]$. Lemma 3 implies that for fixed ε the functions $l_\varepsilon(u_n)(z)$ converge to the function $l_\varepsilon(u_0)(z)$ in $L^1_{loc}(G)$, and thus in the sense of distributions. From Proposition 2 it follows that the functions $l_\varepsilon(u_0)(z)$ are subharmonic for all ε, and their monotone limit when $\varepsilon \to 0$, i.e. the function $\log u_0$, is also subharmonic.

Now we consider a disk $B(z_0, r) \subset\subset G$. From the convergence in $L^1(B(z_0, r))$ of the sequence $u_n(z)$ it follows that the subsequence $\{u_{n'}(z)\}$ converges uniformly on every fixed compact set $K_1 \subset B(z_0, r)$ with positive Lebesgue measure. Since the function $\log u_0(z)$ is subharmonic and not identically $-\infty$ on K_1,

$$\sup_{z \in K_1} (\log u_0(z)) \geq C_0,$$

or

$$\sup_{z \in K_1} (u_0(z)) \geq e^{C_0}.$$

Thus for all $n' > n_0$

$$\sup_{z \in K_1} (u_{n'}(z)) \geq e^{C_0 - 1},$$

and

$$\sup_{z\in K_1}(\log u_{n'}(z)) \geq C_0 - 1,\ \forall n = 0, 1 \ldots .$$

Since the functions $u_n(z)$ are uniformly bounded from above on compact subsets of G, it follows that the family $\{\log u_{n'}(z)\}$ is compact in $D'(G)$. Therefore there exists a subsequence $\log u_{n''}(z)$ which converges in $D'(G)$ (and also in $L^1_{loc}(G)$) to some subharmonic function $v(z)$ in G.

Note that for any compact set $K \subset G$ and for any $\varepsilon > 0$ we have the following inequality:

$$\int_K |max\{\log u_{n'}(z), \log\varepsilon\} - max\{v(z), \log\varepsilon\}| dxdy \leq \int_K |\log u_{n'}(z) - v(z)| dxdy.$$

Hence, the functions $max\{\log u_{n'}(z), \log\varepsilon\}$ converge to the function $max\{v(z), \log\varepsilon\}$ in $L^1_{loc}(G)$ for any $\varepsilon > 0$.

On the other hand, as was shown above, $l_\varepsilon(u_n)(z)$ converge to $l_\varepsilon(u_0)(z)$ in $L^1_{loc}(G)$. Thus almost everywhere (and, since the functions are subharmonic, everywhere)

$$\max\{v(z), \log\varepsilon\} = \max\{\log u_0(z), \log\varepsilon\}. \tag{9}$$

Since a set on which a subharmonic function equals to $-\infty$ has Lebesgue measure zero, then $\varepsilon \to 0$ implies that $\mathrm{mes}(\{z \in G : v(z) < \log\varepsilon\}) \to 0$, $\mathrm{mes}(\{z \in G : \log u_0(z) < \log\varepsilon\}) \to 0$, and $v(z) = \log u_0(z)$ almost everywhere, and hence everywhere. Thus the sequence of the functions $\log u_{n'}(z)$ converges to the function $\log u_0(z)$ in $D'(G)$ and in $L^1_{loc}(G)$.

If for some subsequence of the functions $\log u_{n_j}(z)$, $\varepsilon_0 > 0$ and compact set $K_0 \in G$

$$\int_{K_0} |\log u_{n_j}(z) - \log u_0(z)| dxdy \geq \varepsilon_0, \tag{10}$$

then, using the above construction of the sequence $u_{n_j}(z)$, we have that some subsequence of the sequence $\{\log u_{n_j}\}$ converges to $\log u_0(z)$ in $L^1_{loc}(G)$, which contradicts (10). The lemma is proved.

P r o o f o f T h e o r e m 2. From Proposition 3 in [2] it follows that the inclusion $\log u \in WAP(S)$ implies that inclusion $u \in WAP(S)$. We are going to show the opposite inclusion. Let $u(z) \in WAP(S)$ and $\{h_n\} \subset \mathbb{R}$ be an arbitrary sequence. Passing to a subsequence if necessary, we can assume that for some subharmonic function u_0, uniformly in $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \int_S (u(z + h_n) - u_0(z))\varphi(z - t) dxdy = 0. \tag{11}$$

To prove the theorem it is sufficient to verify that uniformly in $t \in \mathbb{R}$

$$\lim_{n\to\infty} \int_S \log u(z + h_n + t)\varphi(z)dxdy = \int_S \log u_0(z + t)\varphi(z)dxdy. \tag{12}$$

Assuming that this fails, for some $\varepsilon > 0$ and some sequence $t_n \to \infty$,

$$\left| \int_S \log u(z + h_n + t_n)\varphi(z)dxdy - \int_S \log u_0(z + t_n)\varphi(z)dxdy \right| \geq \varepsilon. \tag{13}$$

Here $u_0(z)$ is a logarithmic subharmonic function with $u_0(z) \in WAP(S)$. Passing to a subsequence and using almost periodicity of the function $u_0(z)$, we can assume that

$$\lim_{n\to\infty} \int_S u_0(z + t_n)\varphi(z)dxdy = \int_S v(z)\varphi(z)dxdy \tag{14}$$

for some subharmonic in the strip S function $v(z)$. Since the limit in (11) is uniform in $t \in \mathbb{R}$, (14) implies

$$\lim_{n\to\infty} \int_S u(z + h_n + t_n)\varphi(z)dxdy = \int_S v(z)\varphi(z)dxdy.$$

Now Lemma 4 implies that both integrals in (13) have the same limit $\int \log v(z)\varphi(z)dxdy$, when $n \to \infty$, which is impossible. Thus (12) holds and Theorem 2 is proved.

P r o o f o f T h e o r e m 3. Without loss of generality, we can assume that S is a strip with finite width. Let S_0 be an arbitrary substrip, $S_0 \subset\subset S$. Since the function $u(z)$ is almost periodic, its Riesz measure $\mu := \frac{1}{2\pi}\Delta u$ is also almost periodic in the sense of distributions.

Denote

$$K(w) = \frac{1}{2} \log |e^{-\gamma w^2} - 1|,$$

where

$$0 < \gamma < \frac{\pi}{\max\limits_{y_1,y_2 \in \mathrm{Im} S} (y_1 - y_2)^2}.$$

Note that the kernel $K(w)$ is a subharmonic function which is bounded from above in S and its restriction to S_0 satisfies the equation

$$\Delta K(w) = 2\pi\delta(w), \tag{15}$$

where $\delta(w)$ is a standard Dirac measure. Denote

$$V(z) = \int\limits_S K(w-z)\varphi(\mathrm{Im}w)d\mu(w), \tag{16}$$

where $\varphi \geq 0$ is a test function on $\mathrm{Im}S$ such that $\varphi(y) = 1$ for $y \in \mathrm{Im}S_0$.

Denote $P_n = \{(n-1/2, n+3/4) \times \mathrm{Im}S\} \subset S$. We are going to show that $V(z)$ is a subharmonic function in every P_n. Fixing $n_0 \in \mathbb{Z}$, we have

$$\int\limits_S K(w-z)\varphi(\mathrm{Im}w)d\mu(w) = \int\limits_{[n_0-1,n_0+1)\times\mathrm{Im}S} K(w-z)\varphi(\mathrm{Im}w)d\mu(w)$$

$$+ \sum_{n\in\mathbb{Z}\setminus\{n_0-1,n_0\}} \int\limits_{[n,n+1)\times\mathrm{Im}S} K(w-z)\varphi(\mathrm{Im}w)d\mu(w). \tag{17}$$

Every term in the right hand side of (17) is obviously a subharmonic function. For $\mathrm{Re}w \in [n, n+1)$, $\mathrm{Im}w \in \mathrm{supp}\,\varphi$, $z \in P_{n_0}$, $n \neq n_0$, $n \neq n_0 - 1$ we have

$$\left|e^{-\gamma(w-z)^2}\right| = e^{-\gamma(\mathrm{Re}w-\mathrm{Re}z)^2+\gamma(\mathrm{Im}w-\mathrm{Im}z)^2} \leq e^{\pi-\gamma(|n-n_0|-3/4)^2}.$$

Thus

$$\sum_{n\in\mathbb{Z}\setminus\{n_0-1,n_0\}} \left| \int\limits_{[n,n+1)\times\mathrm{Im}S} K(w-z)\varphi(\mathrm{Im}w)d\mu(w) \right|$$

$$\leq \sum_{n\in\mathbb{Z}\setminus\{n_0-1,n_0\}} \sup_{z\in P_{n_0}} \sup_{w\in[n,n+1)\times\mathrm{supp}\,\varphi} \left|\frac{1}{2}\log|1-e^{-\gamma(z-w)^2}|\right| \mu([n,n+1)\times\mathrm{supp}\,\varphi).$$

Since the measure μ is almost periodic, $\mu([n, n+1) \times \mathrm{supp}\,\varphi)$ is bounded from above uniformly in n (see [2]), and therefore the series (17) converges uniformly in $z \in P_{n_0}$ and the function $V(z)$ is subharmonic in P_{n_0}, and also in S.

Now we are going to show that the function $V(z)$ is subharmonic almost periodic in S. We consider a test function $\psi(z)$ on S and verify that the function

$$f(t) = \int\limits_S V(z)\psi(z-t)dxdy$$

is uniformly almost periodic on the real axis. We have

$$f(t) = \int\limits_S \left(\int\limits_S K(w-z)\psi(z)dxdy \right) \varphi(\mathrm{Im}w)d\mu(w+t).$$

Note that the function

$$\Psi(w) := \int_S K(w-z)\psi(z)dxdy$$

is continuous in S, because the difference $K(w) - \log|w|$ is continuous in some neighborhood of zero. Moreover, $\Psi(w) = O(e^{-\gamma|w|^2})$ when $|\mathrm{Re}w| \to \infty$.

Since the values $\mu([n, n+1] \times \mathrm{Im}S_0)$ are uniformly bounded in n, then

$$\int_S \frac{\varphi(\mathrm{Im}w + \mathrm{Im}z)d\mu(w+z)}{1+|w|^2} \le C_1 < \infty \tag{18}$$

uniformly in $z \in S_0$.

We fix $\varepsilon > 0$ and choose a test function $\nu(t)$, $0 \le \nu(t) \le 1$ on $\mathbb{R}$, and such that $\nu(\mathrm{Re}w) = 1$ on the set

$$\left\{ w: \ |\Psi(w)| > \frac{\varepsilon}{C_1(1+|w|^2)} \right\}.$$

For all $t \in \mathbb{R}$ we have

$$f(t) = \int_S \Psi(w)\nu(\mathrm{Re}w)\varphi(\mathrm{Im}w)d\mu(w+t)$$

$$+ \int_S \Psi(w)(1-\nu(\mathrm{Re}w))\varphi(\mathrm{Im}w)d\mu(w+t).$$

Property (18) implies that the second integral in the equality is not greater than ε for all $t \in \mathbb{R}$. Since μ is an almost periodic measure, the first integral is an almost periodic function, and if τ is an ε-almost period, then it is a 2ε-almost period for f. Thus, the function $V(z)$ is a subharmonic almost periodic, and in addition (15) implies that $\Delta V(z) = 2\pi\varphi(y)\mu(z)$ in the sense of distributions. Consider the function

$$H(z) := V(z) - u(z).$$

This function is harmonic and almost periodic in the sense of distributions in S_0. Let $\varphi \ge 0$ be a test function in the disk $B(\varepsilon, 0)$, which depends only on $|z|$ and such that $\int \varphi(z)dxdy = 1$. Since the convolution $\int H(z)\varphi(z+\zeta)dxdy$ is equal to $H(\zeta)$ in some strip $S_1 \subset\subset S_0$, then the remark after Definition 2 implies uniform almost periodicity of the function $H(z)$ in S_1. So its Fourier–Bohr coefficients are continuous in $\mathrm{Im}S_1$ and, since ε is arbitrary, in $\mathrm{Im}S_0$. Thus it is enough show that the Fourier–Bohr coefficients of $V(z)$,

$$a_\lambda(V, y) = M(Ve^{-i\lambda x}, y) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V(x+iy)e^{-i\lambda x}dx,$$

are continuous.

We fix $\varepsilon > 0$. We have

$$K(w) = \max\{K(w), -2\log N\} + \min\{K(w) + 2\log N, 0\} = K_1(w) + K_2(w),$$

where $N < \infty$ will be chosen later. Denote

$$V_1(z) := \int\limits_S K_1(z-w)\varphi(\mathrm{Im}w)d\mu(w),$$

$$V_2(z) := \int\limits_S K_2(z-w)\varphi(\mathrm{Im}w)d\mu(w).$$

Since for $|\gamma w^2| < 1/2$ we have

$$K(w) = 1/2\log|1 - 1 + \gamma w^2 - \frac{\gamma^2 w^4}{2!} + \ldots| = \log|w| + \theta(w),$$

where $\theta(w)$ is a continuous function, then $K_2(w) = 0$ for $|w| \geq \delta_0 > 0$ and N sufficiently large. Moreover, if $|\theta(w)| \leq \log N$, then for all $w \in \mathbb{C}$ and $y \in \mathrm{Im}S$,

$$\int\limits_{-T}^{T} K_2(z-w)dx = \int\limits_{-T}^{T} \min\{\log|x - u + i(y-v)| + \theta(z-w) + 2\log N, 0\}dx$$

$$\geq \int\limits_{-\infty}^{\infty} \min\{\log|Nx - Nu|, 0\}dx = -\frac{C}{N}, \tag{19}$$

with some constant C, $0 < C < \infty$. Now using the property that $\mu([n, n+1] \times \operatorname{supp}\varphi)$ are bounded and the fact that $K_2(z-w) = 0$ for $|z-w| \geq \delta_0$, we have that for all T

$$\left|\frac{1}{2T}\int\limits_{-T}^{T} V_2(z)e^{-ix\lambda}dx\right| \leq \int\limits_{|\mathrm{Re}w| \leq T+\delta_0} \frac{1}{2T}\int\limits_{-T}^{T} |K_2(z-w)|dxd\mu(w)$$

$$\leq \frac{C}{2TN}\int\limits_{|\mathrm{Re}w| \leq T+\delta_0} \varphi(\mathrm{Im}w)d\mu(w) \leq \frac{C_2}{N} \leq \varepsilon, \tag{20}$$

if N is sufficiently large.

Further, since $K_1(w) = O(e^{-\gamma|w|^2})$ for $|\mathrm{Re}w| \to \infty$, then one can choose a test function $0 \le \eta(t) \le 1$ on $\mathbb{R}$ such that $\eta(\mathrm{Re}w) = 1$ on the set

$$\left\{ w : \ |K_1(w)| > \frac{\varepsilon}{C_1(1+|w|^2)} \right\},$$

where C_1 is the constant from (18). We have

$$V_1(z) = \int\limits_S K_1(w)\eta(\mathrm{Re}w)\varphi(\mathrm{Im}w + \mathrm{Im}z)d\mu(w+z)$$

$$+ \int\limits_S K_1(w)(1-\eta(\mathrm{Re}w))\varphi(\mathrm{Im}w + \mathrm{Im}z)d\mu(w+z) = V_3(z) + V_4(z).$$

From the choice of the function η it follows that

$$|V_4(z)| \le \varepsilon, \ \text{ for } \ z \in S_0. \tag{21}$$

Since the kernel $K_1(w)$ is continuous and the family of shifts of a test function in $\mathrm{Im}S_0$ is a compact set, then (see the remark to Definition 2) the function $V_3(z)$ is uniformly almost periodic in S_0 and it has continuous in $\mathrm{Im}S_0$ Fourier–Bohr coefficients (see [1, p. 145]). Thus, if $y_1, y_2 \in \mathrm{Im}S_0$ and $|y_1 - y_2| \le \delta(\varepsilon)$, then (20) and (21) imply

$$|a_\lambda(V,y_1) - a_\lambda(V,y_2)| \le |a_\lambda(V_3,y_1) - a_\lambda(V_3,y_2)| + |a_\lambda(V_4,y_1)| + |a_\lambda(V_4,y_2)|$$

$$+|a_\lambda(V_2,y_1)| + |a_\lambda(V_2,y_2)| \le 5\varepsilon.$$

Thus $a_\lambda(V,y)$ are continuous. The theorem is proved.

P r o o f o f T h e o r e m 4. For $P_m(z)$ we choose Bohner–Fejer sums of the function $u(z)$

$$P_m(z) := \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} u(z+t)\Phi^{(m)}(t)dt = \sum k_\lambda^{(m)} a_\lambda(u, \mathrm{Im}z) e^{i\lambda \mathrm{Re}z}.$$

Here $\Phi^{(m)}(t)$ is a sequence of Bohner–Fejer kernels (see. [3, p. 69]), and the set $\{k_\lambda^{(m)} : k_\lambda^{(m)} \neq 0\}$ is finite for every m. Note that, according to Theorem 3, the functions $a(u,y)$ are continuous in $y \in \mathrm{Im}S$.

We are going to show that $P_m(z)$ are subharmonic.

Note that the kernels $\Phi^{(m)}(t)$ are nonnegative, bounded, and

$$\lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \Phi^{(m)}(t)dt = 1.$$

Also note that the subharmonic almost periodic function $u(z)$ is bounded from above in any subset $S' \subset\subset S$. Thus, using Fatou's lemma, for any $m = 1, 2, \dots$, $z \in S$, and sufficiently small ρ,

$$\begin{aligned} \frac{1}{2\pi\rho}\int_0^{2\pi} P_m(z+\rho e^{i\varphi})d\varphi &\geq \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2\pi\rho}\int_0^{2\pi} u(z+\rho e^{i\varphi}+t)\Phi^{(m)}(t)d\varphi dt \\ &\geq \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T} u(z+t)\Phi^{(m)}(t)dt = P_m(z). \end{aligned}$$

As it is shown in [6], for any test function $\varphi(z)$ in S and for some (depending only on the spectrum of the function $u(z)$) sequence of Bohner–Fejer sums, for $m \to \infty$, uniformly in $t \in \mathbb{R}$

$$\int_S P_m(z)\varphi(z+t)dxdy \to \int_S u(z)\varphi(z+t)dxdy. \tag{22}$$

Now we are going to verify that it implies the convergence in the topology defined by seminorms $d_{[\alpha,\beta]}$, $\alpha, \beta \in \mathrm{Im}S$. Indeed, if it is not true, then for some sequence $x_m \to \infty$ and some $\alpha, \beta \in \mathrm{Im}S$, $\varepsilon_0 > 0$,

$$\sup_{y\in[\alpha,\beta]}\int_0^1 |u(x_m+iy+t) - P_m(x_m+iy+t)|dt \geq \varepsilon_0.$$

Since $u \in StAP(S)$, then, passing to a subsequence if necessary, one can assume that functions $u(z+x_m)$ converge to some function $v \in StAP(S)$ with respect to metric $d_{[\alpha,\beta]}$, and therefore

$$\sup_{y\in[\alpha,\beta]}\int_0^1 |P_m(x_m+iy+t) - v(t+iy)|dt \geq \varepsilon_0/2. \tag{23}$$

Moreover, according to Theorem 2,

$$\int_S u(z+x_m)\varphi(z)dxdy \to \int_S v(z)\varphi(z)dxdy.$$

Therefore, by setting $t = -x_m$ in (22), we have for any test function $\varphi(z)$

$$\lim_{m\to\infty}\int_S P_m(z+x_m)\varphi(z)dxdy = \int_S v(z)\varphi(z)dxdy.$$

According to Lemma 2, this contradicts to (23). The theorem is proved.

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