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Totally geodesic submanifolds in the tangent bundle of a Riemannian 2-manifold

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We give a full description of totally geodesic submanifolds in the tangent bundle of a Riemannian 2-manifold of constant curvature and present a new class of a cylinder-type totally geodesic submanifolds in the general case.

Introduction

Let (M^n, g) be a Riemannian manifold with metric g and TM^n its tangent bundle. S. Sasaki [7] introduced on TM^n a natural Riemannian metric Tg. With respect to this metric, all the fibers are totally geodesic and intrinsically flat submanifolds. Probably M.-S. Liu [5] was the first who noticed that the base manifold embedded into TM^n by the zero section is totally geodesic, as well. Soon afterwards, K. Sato [9] described geodesics (the totally geodesic submanifolds of dimension 1) in the tangent bundle over space forms. The next step was made by P. Walczack [10] who tried to find a nonzero section $\xi : M^n \to TM^n$ such that the image $\xi(M^n)$ is a totally geodesic submanifold. He proved that if ξ is of constant length and $\xi(M^n)$ is totally geodesic, then ξ is a parallel vector field. As a consequence, the base manifold should be reducible. The irreducible case stays out of considerations up to now. A general conjecture stated by A. Borisenko claims that, in irreducible case, the zero vector field is the unique one which generates a totally geodesic submanifold $\xi(M^n)$ or, equivalently, the base manifold is the unique totally geodesic submanifold of dimension n in TM^n transversal to fibers. A dimensional restriction is essential. M.T.K. Abbassi and the author [1] treated the case of fiber transversal submanifolds in TM^n of dimension $l < n$ and have found some examples of totally geodesic submanifolds of this type. Earlier this problem had been considered in [11].

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It is also worthwhile to mention that in the case of *tangent sphere bundle* the situation is different. S. Sasaki [8] described geodesics in the tangent sphere bundle over space forms and P. Nagy [6] described geodesics in the tangent sphere bundle over symmetric spaces. The author has given a full description of totally geodesic vector fields on 2-dimensional manifolds of constant curvature [12] and an example of a totally geodesic unit vector field on positively/negatively curved manifolds of nonconstant curvature [13]. A full description of 2-manifolds which admit a totally geodesic unit vector field was given in [14].

In this paper we consider a more general problem concerning the description of all possible totally geodesic submanifolds in the *tangent bundle* of Riemannian 2-manifold with a sign-preserving curvature. For the spaces of constant curvature this problem was posed by A. Borisenko in [2].

In Section 2 we prove the following theorems.

Theorem 1. *Let M^2 be Riemannian manifold of constant curvature $K \neq 0$. Suppose that $\tilde{F}^2 \subset TM^2$ is a totally geodesic submanifold. Then locally $\tilde{F}^2$ is one of the following submanifolds:*

(a) a single fiber T_qM^2;

(b) a cylinder-type surface based on a geodesic γ in M^2 with elements generated by a parallel unit vector field along γ;

(c) the base manifold embedded into TM^2 by zero vector field.

Remark that the item (b) of Theorem 1 is a consequence of more general result.

Theorem 2. *Let M^2 be a Riemannian manifold of sign-preserving curvature. Suppose that $\tilde{F}^2 \subset TM^2$ is a totally geodesic submanifold having nontransversal intersection with the fibers. Then locally $\tilde{F}^2$ is a cylinder-type surface based on a geodesic γ in M^2 with elements generated by a parallel unit vector field along γ.*

Moreover, a general Riemannian manifold M^n admits this class of totally geodesic surfaces in TM^n (see Prop. 2.4).

In Section 3 we prove the following general result.

Theorem 3. *Let M^2 be a Riemannian manifold with sign-preserving curvature. Then TM^2 does not admit a totally geodesic 3-manifold even locally.*

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1. Necessary facts about the Sasaki metric

Let (M^n, g) be an n-dimensional Riemannian manifold with metric g. Denote by $\langle \cdot\,, \cdot \rangle$ the scalar product with respect to g. The *Sasaki metric* on TM^n is defined by the following scalar product: if $\tilde X, \tilde Y$ are tangent vector fields on TM^n, then

$$\langle\langle \tilde X, \tilde Y \rangle\rangle := \langle \pi_* \tilde X, \pi_* \tilde Y \rangle + \langle K\tilde X, K\tilde Y \rangle, \tag{1}$$

where $\pi_* : TTM^n \to TM^n$ is the differential of the projection $\pi : TM^n \to M^n$ and $K : TTM^n \to TM^n$ is the *connection map* [3]. The local representations for π_* and K are the following ones.

Let $(x^1, \dots, x^n)$ be a local coordinate system on M^n. Denote by $\partial_i := \partial/\partial x^i$ the natural tangent coordinate frame. Then, at each point $q \in M^n$, any tangent vector ξ can be decomposed as $\xi = \xi^i \, \partial_i|_q$. The set of parameters $\{x^1, \dots, x^n; \xi^1, \dots, \xi^n\}$ forms the natural induced coordinate system in TM^n, i.e., for a point $Q = (q, \xi) \in TM^n$, with $q \in M^n$, $\xi \in T_qM^n$, we have $q = (x^1, \dots, x^n)$, $\xi = \xi^i \, \partial_i|_q$. The natural frame in T_QTM^n is formed by

$$\tilde\partial_i := \frac{\partial}{\partial x^i}|_Q, \quad \tilde\partial_{n+i} := \frac{\partial}{\partial \xi^i}|_Q,$$

and for any $\tilde X \in T_QTM^n$ we have the decomposition

$$\tilde X = \tilde X^i \tilde\partial_i + \tilde X^{n+i} \tilde\partial_{n+i} \ .$$

Now locally, the *horizontal* and *vertical* projections of $\tilde X$ are given by

$$\begin{aligned} &\pi_* \tilde X|_Q = \tilde X^i \, \partial_i|_q, \\ &K\tilde X|_Q = (\tilde X^{n+i} + \Gamma^i_{jk}(q) \, \xi^j \tilde X^k) \, \partial_i|_q, \end{aligned} \tag{2}$$

where Γ^i_{jk} are the Christoffel symbols of the metric g.

The inverse operations are called *lifts* . If $X = X^i \, \partial_i$ is a vector field on M^n then the vector fields on TM given by

$$\begin{aligned} &X^h = X^i \, \tilde\partial_i - \Gamma^i_{jk} \, \xi^j X^k \, \tilde\partial_{n+i}, \\ &X^v = X^i \, \tilde\partial_{n+i} \end{aligned}$$

are called the *horizontal* and *vertical* lifts of X respectively. Remark that for any vector field X on M^n it holds

$$\begin{aligned} &\pi_* X^h = X, \quad KX^h = 0, \\ &\pi_* X^v = 0, \quad KX^v = X. \end{aligned} \tag{3}$$

There is a natural decomposition

$$T_Q(TM^n) = \mathcal{H}_Q(TM^n) \oplus \mathcal{V}_Q(TM^n),$$

where $\mathcal{H}_Q(TM^n) = \ker K$ is called the *horizontal distribution* and $\mathcal{V}_Q(TM^n) = \ker \pi_*$ is called the *vertical distribution* on TM^n. With respect to the Sasaki metric, these distributions are mutually orthogonal. The vertical distribution is *integrable* and the fibers are precisely its integral submanifolds. The horizontal distribution is *never integrable* except the case of a flat base manifold.

For any vector fields X, Y on M^n, the covariant derivatives of various combinations of lifts to the point $Q = (q, \xi) \in TM^n$ can be found by the formulas [4]

$$\begin{aligned} &\tilde\nabla_{X^h} Y^h|_Q = (\nabla_X Y|_q)^h - \tfrac{1}{2}(R_q(X,Y)\xi)^v, \quad && \tilde\nabla_{X^v} Y^h|_Q = \tfrac{1}{2}(R_q(\xi,X)Y)^h, \\ &\tilde\nabla_{X^h} Y^v|_Q = (\nabla_X Y|_q)^v + \tfrac{1}{2}(R_q(\xi,Y)X)^h, \quad && \tilde\nabla_{X^v} Y^v|_Q = 0, \end{aligned} \tag{4}$$

where ∇ and R are the Levi–Civita connection and the curvature tensor of M^n respectively.

R e m a r k. The formulas (4) are applicable to the *lifts* of vector fields only. A formal application to a general field on tangent bundle may lead to wrong result. For example,

$$\begin{aligned} \tilde\nabla_{X^v}(\xi^i(\partial_i)^h) = {} & X^v(\xi^i)\,\partial_i^h + \xi^i \tilde\nabla_{X^v}\partial_i^h \\ = {} & X^i\partial_i^h + \xi^i \tfrac{1}{2}\big(R(\xi,X)\partial_i\big)^h = X^h + \tfrac{1}{2}\big(R(\xi,X)\xi\big)^h \end{aligned}$$

and we have an additional term in the formulas. We will use this rule in our calculations without special comments.

2. Local description of 2-dimensional totally geodesic submanifolds in TM^2

In this section we prove Theorem 1. The proof is given in a series of subsections. Namely, in Subsection 2.1 we prove the item (a), in Subsection 2.2 we prove the item (c) and finally, in Subsection 2.3 we prove Theorem 2 and therefore, the item (b) of Theorem 1.

2.1. Preliminary considerations

Let $\tilde F^2$ be a submanifold in TM^2. Let $(x^1, x^2; \xi^1, \xi^2)$ be a local chart on TM^2. Then locally $\tilde F^2$ can be given by mapping f of the form

$$f : \begin{cases} x^1 = x^1(u^1, u^2), & \xi^1 = \xi^1(u^1, u^2), \\ x^2 = x^2(u^1, u^2), & \xi^2 = \xi^1(u^1, u^2), \end{cases}$$

where u^1, u^2 are the local parameters on $\tilde{F}^2$. The Jacobian matrix f_* of the mapping f is of the form

$$f_* = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} \\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{pmatrix}.$$

Since *rank* $f_* = 2$, we have three *geometrically different* possibilities to achieve the rank, namely

$$(a) \quad \det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} \end{pmatrix} \neq 0; \qquad (b) \quad \det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \end{pmatrix} \neq 0;$$

$$(c) \quad \det \begin{pmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{pmatrix} \neq 0.$$

Without loss of generality we can consider these possibilities in a way that (b) excludes (a), and (c) excludes (a) and (b) restricting the considerations to a smaller neighbourhood or even to an open and dense subset.

Case (a). In this case one can locally parameterize the submanifold under consideration as

$$f : \begin{cases} x^1 = u^1, & \xi^1 = \xi^1(u^1, u^2), \\ x^2 = u^1, & \xi^2 = \xi^2(u^1, u^2), \end{cases}$$

and we can consider the submanifold $\tilde{F}^2$ as an image of the vector field $\xi(u^1, u^2)$ on the base manifold. Denote $\tilde{F}^2$ in this case by $\xi(M^2)$. We analyze this case in subsection 2.2.

Case (b). In this case one can parameterize the submanifold F^2 as

$$f : \begin{cases} x^1 = u^1, & \xi^1 = u^2, \\ x^2 = x^2(u^1, u^2), & \xi^2 = \xi^2(u^1, u^2). \end{cases}$$

Taking into account that we exclude the case (a) in considerations of the case (b), we should set

$$\det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} \end{pmatrix} = \frac{\partial x^2}{\partial u^2} = 0.$$

Therefore, $x^2(u^1,u^2)$ does not depend on u^2 and the local representation takes the form

$$f : \begin{cases} x^1 = u^1, & \xi^1 = u^2, \\ x^2 = x^2(u^1), & \xi^2 = \xi^2(u^1,u^2). \end{cases}$$

Remark that $\pi(\tilde F^2) = (u^1, x^2(u^1))$ is a regular curve on M^2. If we denote this projection by $\gamma(s)$ parameterized by the arc-length parameter and set $u^2 := t$, the local parametrization of $\tilde F^2$ takes the form

$$\gamma(s) : \begin{cases} x^1 = x^1(s), \\ x^2 = x^2(s), \end{cases} \qquad \xi(t,s) : \begin{cases} \xi^1 = t, \\ \xi^2 = \xi^2(t,s). \end{cases} \tag{5}$$

We can interpret this kind of submanifolds in TM^2 as a one-parametric family of smooth vector fields over a regular curve on the base manifold. We will refer to this kind of submanifolds as *ruled submanifolds* in TM^2 and analyze their totally geodesic property in subsection 2.3.

Case (c). It this case a local parametrization of $\tilde F^2$ can be given as

$$f : \begin{cases} x^1 = x^1(u^1,u^2), & \xi^1 = u^1, \\ x^2 = x^2(u^1,u^2), & \xi^2 = u^2. \end{cases}$$

Taking into account that we exclude the case (b) considering the case (c), we should suppose

$$\det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ 1 & 0 \end{pmatrix} = -\frac{\partial x^1}{\partial u^2} = 0;$$

$$\det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} \\ 0 & 1 \end{pmatrix} = \frac{\partial x^1}{\partial u^1} = 0.$$

Thus, we conclude $x^1 = const$. In the same way, we get $x^2 = const$. Therefore, *a submanifold of this kind is nothing else but the fiber, which is evidently totally geodesic and there is nothing to prove.*

2.2. Totally geodesic vector fields

In [1] the author has found the conditions on a vector field to generate a totally geodesic submanifold in the tangent bundle. Namely, let ξ be a vector field on M^n. The submanifold $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector fields X, Y on M^n the following equation holds:

$$r(X,Y)\xi + r(Y,X)\xi - \nabla_{h_\xi(X,Y)}\xi = 0, \tag{6}$$

where $r(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi$ is "half" the Riemannian curvature tensor and $h_\xi(X,Y) = R(\xi,\nabla_X\xi)Y + R(\xi,\nabla_Y\xi)X$.

It is natural to rewrite this equations in terms of ρ and e_ξ where e_ξ is a *unit* vector field and ρ is the length function of ξ.

Lemma 2.1. *Let $\xi = \rho\, e_\xi$ be a vector field on a Riemannian manifold M^n. Then $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector field X the following equations hold*

$$\begin{cases} Hess_\rho(X,X) - \rho^2\Big(R(e_\xi,\nabla_X e_\xi)X\Big)(\rho) - \rho\,|\nabla_X e_\xi|^2 = 0, \\ \rho^3\nabla_{R(e_\xi,\nabla_X e_\xi)X}e_\xi - 2X(\rho)\nabla_X e_\xi - \rho(r(X,X)e_\xi + |\nabla_X e_\xi|^2\, e_\xi) = 0, \end{cases} \tag{7}$$

where $Hess_\rho(X,X)$ is the Hessian of the function ρ.

P r o o f. Indeed, the equation (6) is equivalent to

$$r(X,X)\xi = \nabla_{R(\xi,\nabla_X\xi)X}\xi, \tag{8}$$

where X is an arbitrary vector field. Setting $\xi = \rho\, e_\xi$, where e_ξ is a unit vector field, we have

$$\begin{aligned} r(X,X)\xi &= \nabla_X\nabla_X(\rho\, e_\xi) - \nabla_{\nabla_X X}(\rho\, e_\xi) \\ &= \nabla_X(X(\rho)e_\xi + \rho\,\nabla_X e_\xi) - (\nabla_X X)(\rho)e_\xi - \rho\,\nabla_{\nabla_X X}e_\xi \\ &= \Big(X(X(\rho)) - (\nabla_X X)(\rho)\Big)e_\xi + 2X(\rho)\nabla_X e_\xi + \rho\, r(X,X)e_\xi \end{aligned}$$

and

$$\nabla_{R(\xi,\nabla_X\xi)X}\xi = \rho^2\Big(R(e_\xi,\nabla_X e_\xi)X\Big)(\rho)\, e_\xi + \rho^3\nabla_{R(e_\xi,\nabla_X e_\xi)X}e_\xi.$$

If we remark that $X(X(\rho)) - (\nabla_X X)(\rho) \overset{def}{=} Hess_\rho(X,X)$ and for a unit vector field e_ξ

$$\big\langle r(X,X)e_\xi, e_\xi\big\rangle = -|\nabla_X e_\xi|^2,$$

then we can easily decompose the equation (8) into components, parallel to and orthogonal to e_ξ, which gives the equations (7). ∎

Corollary 2.1. *Suppose that M^n admits a totally geodesic vector field $\xi = \rho\, e_\xi$. Then*

(a) the function ρ has no strong maximums;

(b) there is a bivector field $e_0 \wedge \nabla_{e_0}e_0$ such that e_ξ is parallel along it.

Particulary, if $n = 2$ then either M^2 is flat or e_0 is a geodesic vector field and ρ is linear with respect to the natural parameter along each e_0 geodesic line. Moreover, the field ξ makes a constant angle with each e_0 geodesic line.

P r o o f. Indeed, for any unit vector field η consider the linear mapping $\nabla_Z\eta|_q : T_qM^n \to \eta_q^\perp$, where $\eta_q^\perp$ is an orthogonal complement to η in T_qM^n. For dimensional reasons it follows that the kernel of this mapping is not empty. In other words, there exists a (unit) vector field e_0 such that $\nabla_{e_0}\eta = 0$.

Let e_0 be a unit vector field such that $\nabla_{e_0}e_\xi = 0$. Then from $(7)_1$ we conclude

$$Hess_\rho(e_0, e_0) = 0$$

at each point of M^n. Therefore, the Hessian of ρ can not be positively definite.

Moreover, from $(7)_2$ we see that $r(e_0, e_0)e_\xi = 0$, which gives $\nabla_{e_0}\nabla_{e_0}e_\xi - \nabla_{\nabla_{e_0}e_0}e_\xi = -\nabla_{\nabla_{e_0}e_0}e_\xi = 0$. Setting $Z = e_0 \wedge \nabla_{e_0}e_0$, we get $\nabla_Z e_\xi = 0$.

Suppose now that $n = 2$. If $Z \neq 0$ then e_ξ is a parallel vector field on M^2 which means that M^2 is flat. If $Z = 0$ then evidently e_0 is a geodesic vector field. Since in this case $Hess_\rho(e_0, e_0) = e_0(e_0(\rho)) = 0$, we conclude that ρ is linear with respect to the natural parameter along each e_0 geodesic line.

As concerns the angle function $\langle e_0, e_\xi\rangle$, we have

$$e_0\langle e_0, e_\xi\rangle = \langle\nabla_{e_0}e_0, e_\xi\rangle + \langle e_0, \nabla_{e_0}e_\xi\rangle = 0.$$

■

Taking into account the Corollary 2.1, introduce on M^2 a semi-geodesic coordinate system (u, v) such that e_ξ is parallel along u-geodesics. Let

$$ds^2 = du^2 + b^2(u, v)\,dv^2 \tag{9}$$

be the first fundamental form of M^2 with respect to this coordinate system. Denote by ∂_1 and ∂_2 the corresponding coordinate vector fields. Then the following equations should be satisfied:

$$\nabla_{\partial_1}e_\xi = 0, \qquad \partial_1^2(\rho) = 0.$$

Introduce the unit vector fields

$$e_1 = \partial_1, \qquad e_2 = \frac{1}{b}\partial_2.$$

Then the following rules of covariant derivation are valid:

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, && \nabla_{e_1}e_2 = 0, \\ &\nabla_{e_2}e_1 = -k\,e_2 && \nabla_{e_2}e_2 = k\,e_1, \end{aligned} \tag{10}$$

where k is a (signed) geodesic curvature of v-curves. Remark that

$$k = -\frac{\partial_1 b}{b}.$$

With respect to chosen coordinate system, the field ξ can be expressed as

$$\xi = \rho\,(\cos\omega\, e_1 + \sin\omega\, e_2), \tag{11}$$

where $\omega = \omega(u, v)$ is an angle function, i.e.,

$$e_\xi = \cos\omega\, e_1 + \sin\omega\, e_2.$$

Introduce a unit vector field ν_ξ by

$$\nu_\xi = -\sin\omega\, e_1 + \cos\omega\, e_2.$$

Then we can easily find

$$\begin{aligned} \nabla_{e_1} e_\xi &= \partial_1\omega\, \nu_\xi, \\ \nabla_{e_2} e_\xi &= (e_2(\omega) - k)\, \nu_\xi. \end{aligned}$$

Since e_ξ is parallel along u-curves, we conclude that $\partial_1\omega = 0$, so that $\omega = \omega(v)$.

Now the problem can be formulated as

On a Riemannian 2-manifold with the metric (9), find a vector field of the form (11) with

$$\partial_1^2\rho = 0 \ \text{and}\ \omega = \omega(v) \tag{12}$$

satisfying the equation (8).

Lemma 2.2. *Let M^2 be a Riemannian 2-manifold with the metric (9) and ξ be a local vector field on M^2 satisfying (12). Then ξ is totally geodesic if and only if*

$$\begin{aligned} &\nabla_{e_2}\nabla_{e_2}\xi - (k + c\,K)\nabla_{e_1}\xi = 0, \\ &\nabla_{e_1}\nabla_{e_2}\xi + \nabla_{e_2}\nabla_{e_1}\xi + (k + c\,K)\nabla_{e_2}\xi = 0, \end{aligned} \tag{13}$$

or in a scalar form

$$\begin{cases} e_2(e_2(\rho)) - (k + c\,K)\, e_1(\rho) = \rho\lambda^2, \\ e_2(c) = 0, \\ 2e_1(e_2(\rho)) + c\,K\, e_2(\rho) = 0, \\ e_1(c) + c\,(k + c\,K) = 0, \end{cases} \tag{14}$$

where $\lambda := \langle \nabla_{e_2} e_\xi, \nu_\xi \rangle = e_2(\omega) - k$, $c := \rho^2\lambda = \pm|\,\xi \wedge \nabla_{e_2}\xi|$ and K is the Gaussian curvature of M^2.

P r o o f. Indeed,

$$\nabla_{e_1}\xi = e_1(\rho)\, e_\xi,$$
$$\nabla_{e_2}\xi = e_2(\rho)\, e_\xi + \rho\lambda\nu_\xi.$$

So, taking into account (10) and (12), we have

$$\begin{aligned}
r(e_1,e_1)\xi &= \nabla_{e_1}\nabla_{e_1}\xi - \nabla_{\nabla_{e_1}e_1}\xi = e_1(e_1(\rho))\, e_\xi = \partial_1^2\rho\, e_\xi = 0,\\
r(e_1,e_2)\xi &= \nabla_{e_1}\nabla_{e_2}\xi - \nabla_{\nabla_{e_1}e_2}\xi = \nabla_{e_1}\nabla_{e_2}\xi,\\
r(e_2,e_1)\xi &= \nabla_{e_2}\nabla_{e_1}\xi - \nabla_{\nabla_{e_2}e_1}\xi = \nabla_{e_2}\nabla_{e_1}\xi + k\nabla_{e_2}\xi,\\
r(e_2,e_2)\xi &= \nabla_{e_2}\nabla_{e_2}\xi - \nabla_{\nabla_{e_2}e_2}\xi = \nabla_{e_2}\nabla_{e_2}\xi - k\nabla_{e_1}\xi.
\end{aligned}$$

As concerns the right-hand side of (8), we have

$$\begin{aligned}
&R(\xi,\nabla_{e_1}\xi)e_1 = 0, \quad R(\xi,\nabla_{e_1}\xi)e_2 = 0,\\
&R(\xi,\nabla_{e_2}\xi)e_1 = \rho^2\lambda\, R(e_\xi,\nu_\xi)e_1 = -\rho^2\lambda\, K\, e_2,\\
&R(\xi,\nabla_{e_2}\xi)e_2 = \rho^2\lambda\, R(e_\xi,\nu_\xi)e_2 = \rho^2\lambda\, K\, e_1.
\end{aligned}$$

Therefore, setting $X = e_1$ in (8), we obtain an identity. Setting $X = e_2$, we have

$$\nabla_{e_2}\nabla_{e_2}\xi - k\nabla_{e_1}\xi = \rho^2\lambda\, K\nabla_{e_1}\xi.$$

Setting $X = e_1 + e_2$, we obtain

$$r(e_1,e_2)\xi + r(e_2,e_1)\xi = -\rho^2\lambda\, K\,\nabla_{e_2}\xi,$$

which can be reduced to

$$\nabla_{e_1}\nabla_{e_2}\xi + \nabla_{e_2}\nabla_{e_1}\xi + k\nabla_{e_2}\xi = -\rho^2\lambda\, K\nabla_{e_2}\xi.$$

It remains to mention that

$$|\,\xi\wedge\nabla_{e_2}\xi|^2 = |\xi|^2\,|\nabla_{e_2}\xi|^2 - \langle\xi,\nabla_{e_2}\xi\rangle^2 = \rho^2(e_2(\rho)^2 + \rho^2\lambda^2) - (e_2(\rho)\rho)^2 = \rho^4\lambda^2.$$

So, if we set $c = \rho^2\lambda$, we evidently obtain (13).

Moreover, continuing calculations, we see that

$$\begin{aligned}
\nabla_{e_2}\nabla_{e_2}\xi = &\ \Big[e_2(e_2(\rho)) - \rho\lambda^2\Big]\, e_\xi + \Big[e_2(\rho)\lambda + e_2(\rho\lambda)\Big]\,\nu_\xi\\
= &\ \Big[e_2(e_2(\rho)) - \rho\lambda^2\Big]\, e_\xi + \tfrac{1}{\rho}e_2(c)\,\nu_\xi,\\
\nabla_{e_1}\nabla_{e_2}\xi + \nabla_{e_2}\nabla_{e_1}\xi = &\ \Big[e_2(e_1(\rho)) + e_1(e_2(\rho))\Big]e_\xi + \Big[e_1(\rho)\lambda + e_1(\rho\lambda)\Big]\nu_\xi\\
= &\ \Big[e_2(e_1(\rho)) + e_1(e_2(\rho))\Big]e_\xi + \tfrac{1}{\rho}e_1(c)\nu_\xi.
\end{aligned}$$

Taking into account that $e_1(e_2(\rho)) - e_2(e_1(\rho)) = k\, e_2(\rho)$, the equations (13) can be written as

$$\begin{aligned}
&\Big[e_2(e_2(\rho)) - \rho\lambda^2\Big]\, e_\xi + \tfrac{1}{\rho} e_2(c)\,\nu_\xi - (k + cK) e_1(\rho)\, e_\xi = 0,\\
&\Big[2\, e_1(e_2(\rho)) - k\, e_2(\rho)\Big]\, e_\xi + \tfrac{1}{\rho} e_1(c)\,\nu_\xi + (k + cK)\Big[e_2(\rho)\, e_\xi + \rho\lambda\,\nu_\xi\Big] = 0,
\end{aligned}$$

and after evident simplifications we obtain the equations (14). ■

Proposition 2.1. *Let M^2 be a Riemannian manifold of constant curvature. Suppose ξ is a nonzero local vector field on M^2 such that $\xi(M^2)$ is totally geodesic in TM^2. Then M^2 is flat.*

P r o o f. Let M^2 a Riemannian manifold of constant curvature $K \neq 0$. Then the function b in (9) should satisfy the equation

$$-\frac{\partial_{11} b}{b} = K.$$

The general solution of this equation can be expressed in 3 forms:

(a) $b(u, v) = A(v)\cos(u/r + \theta(v))$ or $b(u, v) = A(v)\sin(u/r + \theta(v))$ for $K = 1/r^2 > 0$;

(b) $b(u, v) = A(v)\cosh(u/r + \theta(v))$ or $b(u, v) = A(v)\sinh(u/r + \theta(v))$ for $K = -1/r^2 < 0$;

(c) $b(u, v) = A(v)e^{u/r}$ for $K = -1/r^2 < 0$.

Evidently, we may set $A(v) \equiv 1$ (making a v-parameter change) in each of these cases.

The equation $(14)_2$ means that c does not depend on v. Since K is constant, the equation $(14)_4$ implies

$$e_2(k) = 0.$$

If we remark that $k = -\frac{\partial_1 b}{b}$ then one can easily find $\theta(v) = const$ in cases (a) and (b).

After a u-parameter change, the function b takes one of the forms:

(a) $b(u, v) = \cos(u/r)$ or $b(u, v) = \sin(u/r)$ for $K = 1/r^2 > 0$;

(b) $b(u, v) = \cosh(u/r)$ or $b(u, v) = \sinh(u/r)$ for $K = -1/r^2 < 0$;

(c) $b(u, v) = e^{u/r}$ for $K = -1/r^2 < 0$.

From the equation $(14)_4$ we find

$$cK = -\frac{e_1(c)}{c} - k = -\frac{e_1(c)}{c} + \frac{e_1(b)}{b} = e_1(\ln b/c).$$

Suppose first that $e_2(\rho) \neq 0$. Multiplying $(14)_3$ by $e_2(\rho)$, we can easily solve this equation with respect to $e_2(\rho)$ by a chain of simple transformations:

$$\begin{aligned}
&2e_2(\rho) \cdot e_1(e_2(\rho)) + e_1(\ln b/c) \cdot [e_2(\rho)^2] = 0, \\
&e_1[e_2(\rho)^2] + e_1(\ln b/c) \cdot [e_2(\rho)^2] = 0, \\
&\tfrac{e_1[e_2(\rho)^2]}{e_2(\rho)^2} + e_1(\ln b/c) = 0, \\
&e_1[\ln e_2(\rho)^2] + e_1(\ln b/c) = 0, \\
&e_1(\ln[e_2(\rho)^2\, b/c]) = 0
\end{aligned}$$

and therefore, $e_2(\rho)^2\, b/c = h(v)^2$ or

$$\partial_2\rho = h(v)\sqrt{c\,b}.$$

Since ρ is linear with respect to the u-parameter, say $\rho = a_1(v)u + a_2(v)$, then $\partial_2\rho = a_1'u + a_2'$ and therefore $\sqrt{cb}$ is also linear with respect to u, namely $\sqrt{cb} = m_1(v)u + m_2(v) = \frac{a_1'}{h}\,u + \frac{a_2'}{h}$. But the functions c and b do not depend on v. Therefore m_1 and m_2 are constants, so $a_1 = m_1 \int h(v)\,dv, a_2 = m_2 \int h(v)\,dv$. Thus

$$\sqrt{cb} = m_1 u + m_2.$$

Now the function c takes the form

$$c(u) = \frac{(m_1 u + m_2)^2}{b}$$

and therefore

$$e_1(c) = \frac{2m_1(m_1 u + m_2)}{b} - \frac{(m_1 u + m_2)^2 \partial_1 b}{b^2}.$$

Substitution into $(14)_4$ gives

$$\frac{2m_1(m_1 u + m_2)}{b} - \frac{2(m_1 u + m_2)^2 \partial_1 b}{b^2} + \frac{(m_1 u + m_2)^4}{b^2} K = 0$$

or

$$\frac{(m_1 u + m_2)}{b^2}\Big[2m_1 b - 2(m_1 u + m_2)\partial_1 b + (m_1 u + m_2)^3 K\Big] = 0.$$

The expression in brackets is an algebraic one and can not be identically zero if $K \neq 0$. Therefore $m_1 = m_2 = 0$ and hence $\rho^2\lambda := c = 0$. But this identity implies

$\lambda = 0$ or $\rho = 0$. If $\lambda = 0$ then e_ξ is a parallel unit vector field and therefore, M^2 is flat and we come to a contradiction. Therefore $\rho = 0$.

R e m a r k. If $K = 0$, we can not conclude that $c = 0$. In this case the expression in brackets can be identically zero for $m_1 = 0$ and $b = const$. And we have $c = m_2 = const$.

Suppose now that $e_2(\rho) = 0$. Then

$$\rho = a_1 u + a_2,$$

where a_1, a_2 are constants and we obtain the following system:

$$\begin{aligned} &-(k + cK)\partial_1\rho = \rho\lambda^2, \\ &\partial_2\, c = 0, \\ &\partial_1 c + c(k + cK) = 0. \end{aligned} \tag{15}$$

If $\partial_1\rho = 0$ then immediately $\rho = 0$ or $\lambda = 0$. The identity $\lambda = 0$ implies $K = 0$ as above. Therefore, $\rho = 0$.

Suppose $\partial_1\rho \neq 0$ or equivalently $a_1 \neq 0$. Then from $(15)_1$ we get

$$(k + cK) = -\frac{\rho\lambda^2}{a_1}. \tag{16}$$

Since $c = \rho^2\lambda$, from $(15)_2$ we see that $\partial_2\lambda = 0$ or $\partial_2\left[\frac{\partial_2\omega + \partial_1 b}{b}\right] = 0$. Since b does not depend on v, we have $\partial_{22}\omega = 0$ or equivalently $\partial_2\omega = \alpha = const$. Thus, $\lambda = \frac{\alpha + \partial_1 b}{b}$.

Now we can find $\partial_1 c$ in two ways. First, from $(15)_3$ using (16) and keeping in mind that $c = \rho^2\lambda$:

$$\partial_1 c = c\frac{\rho\lambda^2}{a_1} = \frac{\rho^3\lambda^3}{a_1}.$$

Second, directly:

$$\partial_1 c = 2\rho\partial_1\rho\lambda + \rho^2\partial_1\lambda.$$

It is easy to see that $\partial_1\lambda = k\lambda - K$, and hence we get

$$\partial_1 c = 2a_1\rho\lambda + \rho^2(k\lambda - K).$$

Equalizing, we have

$$2a_1\rho\lambda + \rho^2(k\lambda - K) - \frac{\rho^3\lambda^3}{a_1} = 0$$

or

$$\frac{\rho}{a_1}\left[2a_1^2\lambda + a_1\rho(k\lambda - K) - \rho^2\lambda^3\right] = 0.$$

The expression in brackets is an algebraic one and can not be identically zero for $K \neq 0$. Since $\rho \neq 0$, we obtain a contradiction.

R e m a r k. We do not obtain a contradiction if $K = 0$, since we have another solution $\lambda = 0$ which gives $\partial_1 b + \alpha = 0$ and hence $b = -\alpha u + m$. ∎

We have achieved the result by putting a restriction on the geometry of the base manifold. Putting a restriction on the vector field, we are able to achieve a similar result. Recall that a totally geodesic vector field necessarily makes a constant angle with some family of geodesics on the base manifold (see Cor. 2.1). It is not parallel along this family and this fact is essential for its totally geodesic property. Namely,

Proposition 2.2. *Let M^2 be a Riemannian manifold. Suppose ξ is a nonzero local vector field on M^2 which is parallel along some family of geodesics of M^2. If $\xi(M^2)$ is totally geodesic in TM^2 then M^2 is flat.*

R e m a r k. Geometrically, this assertion means that if $\xi(M^2)$ is not transversal to the horizontal distribution on TM^2 then $\xi(M^2)$ is never totally geodesic in TM^2 except when M^2 is flat.

P r o o f. Let M^2 be a *nonflat* Riemannian manifold and suppose that the hypothesis of the theorem is fulfilled. Then, choosing a coordinate system as in Lemma 2.2, we have

$$\nabla_{e_1}\xi = 0,$$

and we can reduce (13) to

$$\begin{aligned} &\nabla_{e_2}\nabla_{e_2}\xi = 0, \\ &\nabla_{e_1}\nabla_{e_2}\xi + (k + cK)\nabla_{e_2}\xi = 0. \end{aligned} \tag{17}$$

Now make a simple computation.

$$\begin{aligned} R(e_2, e_1)\nabla_{e_2}\xi &= \nabla_{e_2}\nabla_{e_1}\nabla_{e_2}\xi - \nabla_{e_1}\nabla_{e_2}\nabla_{e_2}\xi - \nabla_{[e_2,e_1]}\nabla_{e_2}\xi \\ &= \nabla_{e_2}\nabla_{e_1}\nabla_{e_2}\xi - k\nabla_{e_2}\nabla_{e_2}\xi = \nabla_{e_2}\nabla_{e_1}\nabla_{e_2}\xi. \end{aligned}$$

On the other hand, differentiating $(17)_2$, we find

$$\nabla_{e_2}\nabla_{e_1}\nabla_{e_2}\xi = -e_2(k + cK)\nabla_{e_2}\xi.$$

So we have

$$R(e_2, e_1)\nabla_{e_2}\xi = -e_2(k + cK)\nabla_{e_2}\xi.$$

Therefore, either $\nabla_{e_2}\xi = 0$ or $e_2(k + cK) = 0$. If we accept the first case we see that ξ is a parallel vector field on M^2 and we get a contradiction.

If we accept the second case, we obtain

$$R(e_2, e_1)\nabla_{e_2}\xi = 0,$$

which means that $\nabla_{e_2}\xi$ belongs to a kernel of the curvature operator of M^2. In dimension 2 this means that M^2 is flat or, equivalently, ξ is a parallel vector field and we obtain a contradiction, as well.

■

2.3. Ruled totally geodesic submanifolds in TM^2

Proposition 2.3. *Let M^2 be a Riemannian manifold of sign-preserving curvature. Consider a ruled submanifold $\tilde{F}^2$ in TM^2 given locally by*

$$\gamma(s) : \begin{cases} x^1 = x^1(s), \\ x^2 = x^2(s), \end{cases} \qquad \xi(t,s) : \begin{cases} \xi^1 = t, \\ \xi^2 = \xi^2(t,s). \end{cases}$$

Then $\tilde{F}^2$ is totally geodesic in TM^2 if $\gamma(s)$ is a geodesic in M^2,

$$\xi(t,s) = t\,\rho(s)\,e(s),$$

where $e(s)$ is a unit vector field which is parallel along γ and $\rho(s)$ is an arbitrary smooth function.

R e m a r k. Geometrically, $\tilde{F}^2$ is a cylinder-type surface based on geodesic $\gamma(s)$ with elements directed by a unit vector field $e(s)$ parallel along $\gamma(s)$.

P r o o f. Fixing $s = s_0$, we see that F^2 meets the fiber over $x^1(s_0), x^2(s_0)$ by a curve $\xi(t, s_0)$. If F^2 is supposed to be totally geodesic, then this curve is a straight line on the fiber. Therefore, the family $\xi(t,s)$ should be of the form

$$\xi(t,s) : \begin{cases} \xi^1 = t, \\ \xi^2 = \alpha(s)t + \beta(s). \end{cases}$$

Introduce two vector fields given along $\gamma(s)$ by

$$a = \partial_1 + \alpha(s)\partial_2, \quad b = \beta(s)\partial_2. \tag{18}$$

Then we can represent $\xi(t,s)$ as

$$\xi(t,s) = a(s)\,t + b(s).$$

Denote by τ and ν the vectors of the Frenet frame of the curve $\gamma(s)$. Denote also by $(')$ the covariant derivative of vector fields with respect to the arc-length parameter on $\gamma(s)$. Then

$$\begin{cases} \tau' = k\,\nu, \\ \nu' = -k\,\tau. \end{cases}$$

Denote by $\tilde\partial_1, \tilde\partial_2$ the s and t coordinate vector fields on F^2 respectively. A simple calculation yields

$$\tilde\partial_1 = \tau^h + (\xi')^v, \quad \tilde\partial_2 = a^v.$$

One of the unit normal vector fields can be found immediately, namely $\tilde N_1 = \nu^h$. Consider the conditions on F^2 to be totally geodesic with respect to the normal vector field $\tilde N_1$. Using formulas (4),

$$\tilde\nabla_{\tilde\partial_1}\tilde N_1 = \tilde\nabla_{\tau^h+(\xi')^v}\nu^h = -k\tau^h - \frac{1}{2}\Big[R(\tau,\nu)\xi\Big]^v + \frac{1}{2}\Big[R(\xi,\xi')\nu\Big]^h.$$

Therefore,

$$\langle\langle\tilde\nabla_{\tilde\partial_1}\tilde N_1, \tilde\partial_2\rangle\rangle = -\frac{1}{2}\langle R(\tau,\nu)\xi, a\rangle = -\frac{1}{2}\langle R(\tau,\nu)b, a\rangle = 0.$$

Since M^2 is supposed to be nonflat, it follows $b\wedge a = 0$. From (18) we conclude $b = 0$. Thus, $\xi(t,s) = a(s)\,t$. Moreover,

$$\begin{aligned} \langle\langle\tilde\nabla_{\tilde\partial_1}\tilde N_1, \tilde\partial_1\rangle\rangle &= -k - \tfrac{1}{2}\langle R(\tau,\nu)\xi,\xi'\rangle + \tfrac{1}{2}\langle R(\xi,\xi')\nu,\tau\rangle \\ &= -k + \langle R(\xi,\xi')\nu,\tau\rangle = -k + t^2\langle R(a,a')\nu,\tau\rangle = 0 \end{aligned}$$

identically with respect to parameter t. Therefore, $k = 0$ and $a\wedge a' = 0$. Thus, $\gamma(s)$ is a geodesic line on M^2. In addition, $(a\wedge a' = 0) \sim (a' = \lambda a)$. Set $a = \rho(s)\,e(s)$, where $\rho = |a(s)|$. Then $(a' = \lambda a) \sim (\rho' e + \rho e' = \lambda\rho e)$, which means that $e' = 0$. From this we conclude

$$\xi(t,s) = t\rho(s)\,e(s),$$

where $\rho(s)$ is arbitrary function and $e(s)$ is a unit vector field, parallel along $\gamma(s)$. Therefore,

$$\tilde\partial_1 = \tau^h + t\rho'\,e^v, \quad \tilde\partial_2 = \rho\,e^v,$$

and we can find another unit normal vector field $\tilde N_2 = (e^\perp)^v$, where $e^\perp(s)$ is a unit vector field also parallel along $\gamma(s)$ and orthogonal to $e(s)$. For this vector field we have

$$\begin{aligned} &\tilde\nabla_{\tilde\partial_1}\tilde N_2 = \tilde\nabla_{\tau^h+(\xi')^v}(e^\perp)^v = [(e^\perp)']^v + \tfrac{1}{2}\Big[R(\xi,e^\perp)\tau\Big]^h = \tfrac{1}{2}t\rho\Big[R(e,e^\perp)\tau\Big]^h, \\ &\tilde\nabla_{\tilde\partial_2}\tilde N_2 = 0. \end{aligned}$$

Evidently, $\langle\langle\tilde\nabla_{\tilde\partial_i}\tilde N_2, \tilde\partial_k\rangle\rangle = 0$ for all $i,k = 1,2$. Thus, the submanifold is totally geodesic. ∎

The converse statement is true in general.

Proposition 2.4. *Let M^n be a Riemannian manifold. Consider a cylinder type surface $\tilde F^2 \subset TM^n$ parameterized as*

$$\{\gamma(s), t\,\rho(s)\,e(s)\},$$

where $\gamma(s)$ is a geodesic in M^n, $e(s)$ is a unit vector field, parallel along γ and $\rho(s)$ is an arbitrary smooth function. Then $\tilde F^2$ is totally geodesic in TM^n and intrinsically flat.

P r o o f. Indeed, the tangent basis of $\tilde F^2$ is consisted of

$$\tilde\partial_1 = {\gamma'}^h + t\rho' e^v, \qquad \tilde\partial_2 = \rho\, e^v.$$

By formulas (4),

$$\begin{aligned}
\tilde\nabla_{\tilde\partial_1}\tilde\partial_1 &= (\nabla_{\gamma'}\gamma')^h + \frac{1}{2}\big[R(\gamma',\gamma')\xi\big]^v = 0\,,\\
\tilde\nabla_{\tilde\partial_1}\tilde\partial_2 &= (\nabla_{\gamma'}\rho\, e)^v + \frac{1}{2}\big[R(\xi,\rho\, e)\gamma'\big]^h = \rho' e^v \ \sim\ \tilde\partial_2\,,\\
\tilde\nabla_{\tilde\partial_2}\tilde\partial_1 &= \frac{1}{2}\big[R(\xi,\rho\, e)\gamma'\big]^h = \frac{1}{2}\big[R(t\,\rho\, e,\rho\, e)\gamma'\big]^h = 0,\\
\tilde\nabla_{\tilde\partial_2}\tilde\partial_2 &= \rho e^v(\rho)\, e^v = 0.
\end{aligned}$$

It is easy to find the Gaussian curvature of this submanifold, since it is equal to the sectional curvature of TM^2 along the $\tilde\partial_1 \wedge \tilde\partial_2$- plane. Using the curvature tensor expressions [4], we find

$$Gauss(\tilde F^2) = \big\langle\big\langle \tilde R(\tau^{\,h}, e^v)e^v, \tau^{\,h}\big\rangle\big\rangle = \frac{1}{4}|R(\xi,e)\tau|^2 = 0.$$

■

3. Local description of 3-dimensional totally geodesic submanifolds in TM^2

Theorem 3.1. *Let M^2 be Riemannian manifold with Gaussian curvature K. A totally geodesic submanifold $\tilde F^3 \subset TM^2$ locally is either*

a) a 3-plane in $TM^2 = E^4$ if $K = 0$, or

b) a restriction of the tangent bundle to a geodesic $\gamma \in M^2$ such that $K|_\gamma = 0$ if $K \not\equiv 0$. If M^2 does not contain such a geodesic, then TM^2 does not admit 3-dimensional totally geodesic submanifolds.

P r o o f. Let $\tilde F^3$ be a submanifold it TM^2. Let $(x^1,x^2;\xi^1,\xi^2)$ be a local chart on TM^2. Then locally $\tilde F^3$ can be given mapping f of the form

$$f:\begin{cases} x^1 = x^1(u^1,u^2,u^3),\\ x^2 = x^2(u^1,u^2,u^3),\\ \xi^1 = \xi^1(u^1,u^2,u^3),\\ \xi^2 = \xi^2(u^1,u^2,u^3), \end{cases}$$

where u^1,u^2,u^3 are the local parameters on $\tilde F^3$. The Jacobian matrix f_* of the mapping f is of the form

$$f_* = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3}\\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3}\\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^1}{\partial u^3}\\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} & \frac{\partial \xi^2}{\partial u^3} \end{pmatrix}.$$

Since $rank\ f_* = 3$, we have two geometrically different possibilities to achieve the rank, namely

$$(a)\quad \det\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3}\\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3}\\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^1}{\partial u^3} \end{pmatrix} \neq 0; \quad (b)\quad \det\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3}\\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^1}{\partial u^3}\\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} & \frac{\partial \xi^2}{\partial u^3} \end{pmatrix} \neq 0.$$

Without loss of generality we can consider this possibilities in such a way that (b) excludes (a).

Consider the case (a). In this case we can locally parameterize the submanifold F^3 as

$$f:\begin{cases} x^1 = u^1,\\ x^2 = u^2,\\ \xi^1 = u^3,\\ \xi^2 = \xi^2(u^1,u^2,u^3). \end{cases}$$

By hypothesis, the submanifold $\tilde F^3$ is totally geodesic in TM^2. Therefore, it intersects each fiber of TM^2 by a vertical geodesic, i.e., by a straight line. Fix $u_0 = (u_0^1,\, u_0^2)$. Then the parametric equation of $\tilde F^3 \cap T_{u_0}M^2$ with respect to fiber parameters is

$$\begin{cases} \xi^1 = u^3,\\ \xi^2 = \xi^2(u_0^1,u_0^2,u^3). \end{cases}$$

On the other hand, this equation should be the equation of a straight line and hence

$$\begin{cases} \xi^1 = u^3, \\ \xi^2 = \alpha(u_0^1, u_0^2)\, u^3 + \beta(u_0^1, u_0^2), \end{cases}$$

where $\alpha(u) = \alpha(u^1, u^2)$ and $\beta(u) = \beta(u^1, u^2)$ some smooth functions on M^2. From this viewpoint, after setting $u^3 = t$ *the submanifold under consideration can be locally represented as a one-parametric family of smooth vector fields* ξ_t *on* M^2 of the form

$$\xi_t(u) = t\,\partial_1 + \big(\alpha(u)t + \beta(u)\big)\,\partial_2$$

with respect to the coordinate frame $\partial_1 = \partial/\partial u^1$, $\partial_2 = \partial/\partial u^2$.

Introduce the vector fields

$$a(u) = \partial_1 + \alpha(u)\,\partial_2, \quad b(u) = \beta(u)\,\partial_2. \tag{19}$$

Then ξ_t can be expressed as

$$\xi_t(u) = t\,a(u) + b(u).$$

It is natural to denote by $\xi_t(M^2)$ a submanifold $\tilde{F}^3 \subset TM^2$ of this kind.

Denote by $\tilde{\partial}_i$ $(i = 1, \dots, 3)$ the coordinate vector fields of $\xi_t(M^2)$. Then

$$\begin{aligned} \tilde{\partial}_1 &= \big\{1, 0, 0, t\,\partial_1\alpha + \partial_1\beta\big\}, \\ \tilde{\partial}_2 &= \big\{0, 1, 0, t\,\partial_2\alpha + \partial_2\beta\big\}, \\ \tilde{\partial}_3 &= \big\{0, 0, 1, \alpha\big\}. \end{aligned}$$

A direct calculation shows that these fields can be represented as

$$\begin{aligned} \tilde{\partial}_1 &= \partial_1^h + t(\nabla_{\partial_1} a)^v + (\nabla_{\partial_1} b)^v, \\ \tilde{\partial}_2 &= \partial_2^h + t(\nabla_{\partial_2} a)^v + (\nabla_{\partial_2} b)^v, \\ \tilde{\partial}_3 &= a^v. \end{aligned}$$

Denote by $\tilde{N}$ a normal vector field of $\xi_t(M^2)$. Then

$$\tilde{N} = (a^\perp)^v + Z_t^h,$$

where $\langle a^\perp, a\rangle = 0$ and the field $Z_t = Z_t^1\partial_1 + Z_t^2\partial_2$ can be found easily from the equations

$$\langle\langle \tilde{\partial}_i, \tilde{N}\rangle\rangle = \langle Z_t, \partial_i\rangle + t\langle \nabla_{\partial_i} a, a^\perp\rangle + \langle \nabla_{\partial_i} b, a^\perp\rangle = 0, \quad i = 1, 2.$$

Using the formulas (4), one can find

$$\begin{aligned}\tilde\nabla_{\tilde\partial_i} a^v &= \tilde\nabla_{\partial_i^h + t(\nabla_{\partial_i} a)^v + (\nabla_{\partial_i} b)^v} a^v \\ &= (\nabla_{\partial_i} a)^v + \tfrac12\Big[R(\xi_t, a)\partial_i\Big]^h = (\nabla_{\partial_i} a)^v + \tfrac12\Big[R(b,a)\partial_i\Big]^h.\end{aligned}$$

If the submanifold $\xi_t(M^2)$ is totally geodesic, then the following equations should be satisfied identically:

$$\big\langle\big\langle \tilde\nabla_{\tilde\partial_i}\tilde\partial_3, \tilde N\big\rangle\big\rangle = \big\langle \nabla_{\partial_i} a, a^\perp\big\rangle + \frac12\big\langle R(b,a)\partial_i, Z_t\big\rangle = 0$$

with respect to the parameter t. To simplify the further calculations, suppose that the coordinate system on M^2 is the orthogonal one, so that $\big\langle \partial_1, \partial_2\big\rangle = 0$ and

$$R(b,a)\partial_2 = g^{11} K\,|b\wedge a|\partial_1, \quad R(b,a)\partial_1 = -g^{22}K\,|b\wedge a|\partial_2,$$

where K is the Gaussian curvature of M^2 and g^{11}, g^{22} are the contravariant metric coefficients. Then we have

$$\begin{aligned}\big\langle R(b,a)\partial_1, Z_t\big\rangle &= -g^{22}K\,|b\wedge a|\big\langle Z_t, \partial_2\big\rangle \\ &= g^{22}K\,|b\wedge a|\Big(t\big\langle \nabla_{\partial_2} a, a^\perp\big\rangle + \big\langle \nabla_{\partial_2} b, a^\perp\big\rangle\Big), \\ \big\langle R(b,a)\partial_2, Z_t\big\rangle &= g^{11}K\,|b\wedge a|\big\langle Z_t, \partial_1\big\rangle \\ &= -g^{11}K\,|b\wedge a|\Big(t\big\langle \nabla_{\partial_1} a, a^\perp\big\rangle + \big\langle \nabla_{\partial_1} b, a^\perp\big\rangle\Big).\end{aligned}$$

Thus we get the system

$$\begin{cases} g^{22}K|b\wedge a|\big\langle\nabla_{\partial_2}a, a^\perp\big\rangle t + \big\langle\nabla_{\partial_1}a, a^\perp\big\rangle + g^{22}K|b\wedge a|\big\langle\nabla_{\partial_2}b, a^\perp\big\rangle = 0, \\ g^{11}K|b\wedge a|\big\langle\nabla_{\partial_1}a, a^\perp\big\rangle t - \big\langle\nabla_{\partial_2}a, a^\perp\big\rangle + g^{11}K|b\wedge a|\big\langle\nabla_{\partial_1}b, a^\perp\big\rangle = 0, \end{cases}$$

which should be satisfied identically with respect to t. As a consequence, we have 3 cases:

(i) $\quad K = 0, \begin{cases} \big\langle \nabla_{\partial_1} a, a^\perp\big\rangle = 0 \\ \big\langle \nabla_{\partial_2} a, a^\perp\big\rangle = 0 \end{cases};$

(ii) $\quad K \neq 0, \quad |b\wedge a| = 0, \begin{cases} \big\langle \nabla_{\partial_1} a, a^\perp\big\rangle = 0 \\ \big\langle \nabla_{\partial_2} a, a^\perp\big\rangle = 0 \end{cases};$

(iii) $\quad K \neq 0, \ |b\wedge a| \neq 0, \begin{cases} \big\langle \nabla_{\partial_1} a, a^\perp\big\rangle = 0 \\ \big\langle \nabla_{\partial_2} a, a^\perp\big\rangle = 0 \end{cases}, \begin{cases} \big\langle \nabla_{\partial_1} b, a^\perp\big\rangle = 0 \\ \big\langle \nabla_{\partial_2} b, a^\perp\big\rangle = 0 \end{cases}.$

Case (i). In this case the base manifold is flat and we can choose a Cartesian coordinate system, so that the covariant derivation becomes a usual one and we have

$$\begin{cases} \nabla_{\partial_i} a = \{0, \partial_i \alpha\}, \quad i = 1, 2, \\ a^{\perp} = \{-\alpha, 1\}. \end{cases}$$

From $\langle \nabla_{\partial_i} a, a^{\perp} \rangle = 0$ it follows that $\alpha = const$, i.e., a is a parallel vector field. Moreover, in this case

$$\begin{aligned} \tilde{\partial}_1 &= \{1, 0, 0, \partial_1 \beta\} = \partial_1^h + (\partial_1 b)^v, \\ \tilde{\partial}_2 &= \{0, 1, 0, \partial_2 \beta\} = \partial_1^h + (\partial_1 b)^v, \\ \tilde{\partial}_3 &= \{0, 0, 1, \alpha\}, \\ \tilde{N} &= \{-\partial_1 \beta, -\partial_2 \beta, -\alpha, 1\}. \end{aligned}$$

Now we can find

$$\tilde{\nabla}_{\tilde{\partial}_i} \tilde{\partial}_k = (\nabla_{\partial_i} \partial_k b)^v = \{0, 0, 0, \partial_{ik} \beta\}$$

and the conditions

$$\langle\langle \tilde{\nabla}_{\tilde{\partial}_i} \tilde{\partial}_k, \tilde{N} \rangle\rangle = 0$$

imply $\partial_{ik}\beta = 0$. Thus, $\beta = m_1 u^1 + m_2 u^2 + m_0$, where m_1, m_2, m_0 are arbitrary constants. As a consequence, the submanifold $\xi_t(M^2)$ is described by parametric equations of the form

$$\begin{cases} x^1 = u^1, \\ x^2 = u^2, \\ \xi^1 = t, \\ \xi^2 = \alpha t + m_1 u^1 + m_2 u^2 + m_0 \end{cases}$$

and we have a hyperplane in $TM^2 = E^4$.

Case(ii). Keeping in mind (19), the condition $b \wedge a = 0$ implies $b = 0$. The conditions

$$\begin{cases} \langle \nabla_{\partial_1} a, a^{\perp} \rangle = 0, \\ \langle \nabla_{\partial_2} a, a^{\perp} \rangle = 0 \end{cases}$$

imply $\nabla_{\partial_1} a = \lambda_1(u)\, a$, $\nabla_{\partial_2} a = \lambda_2(u)\, a$. As a consequence, we have

$$\begin{aligned} \xi_t &= t\, a, \\ \tilde{\partial}_1 &= \partial_1^h + t(\nabla_{\partial_1} a)^v = \partial_1^h + t\lambda_1\, a^v, \\ \tilde{\partial}_2 &= \partial_2^h + t(\nabla_{\partial_2} a)^v = \partial_2^h + t\lambda_2\, a^v, \\ \tilde{\partial}_3 &= a^v, \\ \tilde{N} &= (a^{\perp})^v. \end{aligned}$$

Using formulas (4),

$$\begin{aligned}
\tilde\nabla_{\tilde\partial_i}\tilde\partial_k &= \tilde\nabla_{\partial_i^h+t\lambda_i\,a^v}\Big(\partial_k^h+t\lambda_k\,a^v\Big)\\
&= \tilde\nabla_{\partial_i^h}\partial_k^h+t\lambda_i\tilde\nabla_{a^v}\partial_k^h+\tilde\nabla_{\partial_i^h}(t\lambda_k a^v)+t^2\lambda_i\lambda_k\tilde\nabla_{a^v}a^v\\
&= (\nabla_{\partial_i}\partial_k)^h-\tfrac12\Big[R(\partial_i,\partial_k)\xi_t\Big]^v+t\lambda_i\tfrac12\Big[R(\xi_t,a)\partial_k\Big]^h\\
&+ t\partial_i(\lambda_k)a^v+t\lambda_k(\nabla_{\partial_i}a)^v+t\lambda_k\tfrac12\Big[R(\xi_t,a)\partial_i\Big]^h\\
&= (\nabla_{\partial_i}\partial_k)^h-t\tfrac12\Big[R(\partial_i,\partial_k)a\Big]^v+t\partial_i(\lambda_k)a^v+t\lambda_k\lambda_i\,a^v.
\end{aligned}$$

Evidently, for $i\neq k$

$$\big\langle\big\langle\tilde\nabla_{\tilde\partial_i}\tilde\partial_k,\tilde N\big\rangle\big\rangle=-t\frac12\big\langle R(\partial_i,\partial_k)a,a^\perp\big\rangle\neq0,$$

since M^2 is nonflat and $a\neq0$. Contradiction.

Case (iii). The conditions imply

$$\nabla_i a=\lambda_i(u)\,a,\quad \nabla_i b=\mu_i(u)\,a,\quad i=1,2,$$

and we have

$$\begin{aligned}
&\xi_t=t\,a+b,\\
&\tilde\partial_1=\partial_1^h+(t\lambda_1+\mu_1)\,a^v,\\
&\tilde\partial_2=\partial_1^h+(t\lambda_2+\mu_2)\,a^v,\\
&\tilde\partial_3=a^v,\\
&\tilde N=(a^\perp)^v.
\end{aligned}$$

A calculation as above leads to the identity

$$\begin{aligned}
\big\langle\big\langle\tilde\nabla_{\tilde\partial_i}\tilde\partial_k,\tilde N\big\rangle\big\rangle &= -\frac12\big\langle R(\partial_i,\partial_k)\xi_t,a^\perp\big\rangle\\
&= -t\frac12\big\langle R(\partial_i,\partial_k)a,a^\perp\big\rangle-\frac12\big\langle R(\partial_i,\partial_k)b,a^\perp\big\rangle=0,
\end{aligned}$$

which can be true if and only if

$$\begin{cases}\big\langle R(\partial_i,\partial_k)a,a^\perp\big\rangle=0,\\ \big\langle R(\partial_i,\partial_k)b,a^\perp\big\rangle=0.\end{cases}$$

The first condition contradicts $K\neq0$.

Consider the case (b). In this case the submanifold $\tilde{F}^3$ can be locally parametrized by

$$\begin{cases} x^1 = u^1, \\ x^2 = x^2(u^1, u^2, u^3), \\ \xi^1 = u^2, \\ \xi^2 = u^3. \end{cases}$$

Since we exclude the case (a), we should suppose

$$\det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^1}{\partial u^3} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ 0 & 1 & 0 \end{pmatrix} = -\frac{\partial x^2}{\partial u^3} = 0;$$

$$\det \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} & \frac{\partial \xi^2}{\partial u^3} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ 0 & 0 & 1 \end{pmatrix} = \frac{\partial x^2}{\partial u^2} = 0.$$

Therefore, in this case we have a submanifold, which can be parametrized by

$$\begin{cases} x^1 = x^1(s), \\ x^2 = x^2(s), \\ \xi^1 = u^2, \\ \xi^2 = u^3, \end{cases}$$

where s is a natural parameter of the regular curve $\gamma(s) = \{x^1(s), x^2(s)\}$ on M^2. Geometrically, a submanifold of this class is nothing else but the restriction of TM^2 to the curve $\gamma(s)$. Denote by τ and ν the Frenet frame of $\gamma(s)$. It is easy to verify that

$$\tilde{\partial}_1 = \tau^{\,h}, \quad \tilde{\partial}_2 = \partial_1^v, \quad \tilde{\partial}_3 = \partial_2^v, \quad \tilde{N} = \nu^{\,h}.$$

By formulas (4), for $i = 1, 2$

$$\langle\langle \tilde{\nabla}_{\tilde{\partial}_{1+i}} \tilde{N}, \tilde{\partial}_1 \rangle\rangle = \langle\langle \tilde{\nabla}_{\partial_i^v} \nu^h, \tau^h \rangle\rangle = \frac{1}{2}\langle R(\xi, \partial_i)\nu, \tau\rangle = \frac{1}{2}\langle R(\tau, \nu)\partial_i, \xi\rangle = 0$$

for arbitrary ξ. Evidently, M^2 must be flat along $\gamma(s)$. ■

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