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Necessary and Sufficient Conditions in Inverse Scattering Problem on the Axis for the Triangular 2×2 Matrix Potential

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The characteristic properties of the scattering data for the Schrödinger operator on the axis with a triangular 2×2 matrix potential are obtained. A difference between the necessary and sufficient conditions for solvability of ISP under consideration, contained in the previous works of the authors, is eliminated.

Key words: scattering on the axis, inverse problem, triangular matrix potential.

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To the blessed memory of Alexander Ya. Povzner (27.06.1915–22.04.2008)

The monograph by V.A. Marchenko [1, Chapter 3] contains a complete solution of the inverse scattering problem (ISP) on the axis for the Schrödinger equation with a real scalar potential having the first moment. A solution of ISP on the axis for the potential having the second moment is considered in [2], [3, Ch. VI]. In the works of the authors [4, 5] for ISP on the axis for the Schrödinger system of equations with an upper triangular 2×2 matrix potential $V(x)$, having the second moment and not having a virtual level (that is, when there was no bounded solution on the axis for $k = 0$), there were obtained some necessary and close, but somewhat different, sufficient conditions on the scattering data (SD) for the problem

$$-Y'' + V(x)Y = k^2 Y, \quad -\infty < x < \infty. \tag{1}$$

In the present paper the necessary and sufficient conditions on SD to the problem above are given in four versions (see Theorems 1 and 2 below, each in two versions), where the definitions and notations from [4, 5] are used. In particular, $R^{\pm}(k)$, $k \in \mathbb{R}$; $k_j^2 < 0$, and $Z_j^{\pm}(t)$, $j = \overline{1,p}$, stand respectively for right and left upper triangular matrical reflection coefficients, negative eigenvalues of the problem, and corresponding right and left upper triangular normalizing polynomials determined by (15) in [4]. Normalizing polynomials for a scalar nonselfadjoint scattering problem on the semiaxis were introduced by V.E. Lyantse [6]. A more exhaustive list of references is available in [1]–[7]. In what follows the matrices are denoted by upper-case letters and their elements by the corresponding lower-case ones with two subscripts. The subscript '0' indicates that the corresponding values were built with a discrete spectrum not being taken into account.

Theorem 1. *Consider the problem (1) with an upper triangular 2×2 matrix potential having the second moment. The potential is assumed to have a real diagonal and to be such that the problem (1) has no virtual level. The items 1)–7) listed below provide necessary and sufficient conditions for a set of the values*

$$\left\{ R^+(k),\ k \in \mathbb{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = 1, \ldots, p < \infty \right\} \tag{2}$$

to be right SD for (1). Here $R^+(k)$ and $Z_j^+(t)$ ($j = \overline{1,p}$) are upper triangular 2×2 matrix functions. This theorem is valid in two versions: either under condition 4 or 4a.

1) *$R^+(k)$ is continuous in $k \in \mathbb{R}$; $\overline{r_{ll}^+(k)} = r_{ll}^+(-k)$, $\left|r_{ll}^+(k)\right| \le 1 - \frac{C_l k^2}{1+k^2}$, with $C_l > 0$, $l = 1, 2$, $R^+(0) = -I$; $I - R^+(-k)R^+(k) = O(k^2)$ as $k \to 0$ and $R^+(k) = O(k^{-1})$ as $k \to \pm\infty$ (here the last two conditions can be enhanced as the necessary ones up to the requirements of continuity for the function $\{I - R^+(-k)R^+(k)\}k^{-2}$ for $k \in \mathbb{R}$ and the estimate $R^+(k) = o(k^{-1})$ for $k \to \pm\infty$).*

2) *The function*

$$F_R^+(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} R^+(k) e^{ikx} dk$$

is absolutely continuous, and for every $a > -\infty$ one has

$$\int\limits_a^{+\infty} (1 + x^2) \left| \frac{d}{dx} F_R^+(x) \right| dx < \infty.$$

3) *The functions* $zc^0_{ll}(z) \equiv za^0_{ll}(z)$, $l = 1, 2$, *given by*

$$zc^0_{ll}(z) \equiv za^0_{ll}(z) := ze^{-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\ln(1-|r^+_{ll}(k)|^2)}{k-z}dk}, \quad Im\ z > 0, \qquad (3)$$

are continuously differentiable in the closed upper half-plane. Hence they are well defined on the real axis by continuity, and one has $\lim\limits_{z\to 0}(za^0_{ll}(z)) \neq 0$ *due to the absence of a virtual level.*

4) *The function*

$$F^-_R(x) \equiv -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} C(k)^{-1}R^+(-k)C(-k)e^{-ikx}dk \qquad (4)$$

is absolutely continuous, and for every $a < +\infty$ *one has*

$$\int\limits_{-\infty}^{a} \left(1+x^2\right)\left|\tfrac{d}{dx}F^-_R(x)\right| dx < \infty. \qquad (5)$$

Here the matrix $C(k)$ *is defined as follows. For* $l = 1, 2$ *its elements* $c_{ll}(k)$ *are given by*

$$zc_{ll}(z) \equiv za_{ll}(z) := zc^0_{ll}(z)\prod_{j=1}^{p}\left(\tfrac{z-k_j}{z+k_j}\right)^{s^l_j}, \quad Im\ z > 0, \qquad (6)$$

where $Im\ k_j > 0$, $s^l_j = sign\ z^{[j]+}_{ll} \geq 0$. *Furthermore,* $c_{21}(k) \equiv 0$, $c_{12}(k) \equiv c_{12}(k + i0)$ *with*

$$zc_{12}(z) = \left\{\frac{-\psi^+(0)\prod\limits_{j=1}^{p}k_j^{\kappa_j} + a_1 z + \ldots + a_\kappa z^\kappa}{\prod\limits_{j=1}^{p}(z+k_j)^{\kappa_j}} + \psi^+(z)\right\} a^0_{11}(z), \quad Im\ z > 0. \qquad (7)$$

The constants $a_1, \ldots, a_\kappa$ *are uniquely derivable*[⋆] *by the given polynomials* $Z_j(t)$ *and by* $a_{ll}(z)$ *(6),* $\kappa = \sum\limits_{j=1}^{p}\kappa_j = \sum\limits_{j=1}^{p}(sign\ z^{[j]+}_{11} + sign\ z^{[j]+}_{22})$. *Besides,* $zc_{12}(z)$ *is bounded and continuous in the closed upper half-plane*

$$\psi^\pm(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{h(k)}{k-z}\prod_{j=1}^{p}\left(\frac{k-k_j}{k+k_j}\right)^{s^1_j} dk, \quad \pm Im\ z > 0, \qquad (8)$$

[⋆]From the equation system (20) in [4] in which one has to substitute $c_{12}(k_j)$ (7) (or (49) in [4]), $a_{12}(k_j)$ ((49) in [4]), and $a_{ll}(k_j)$, $\dot{a}_{ll}(k_j)$ (6) (cf. Remark 2 in [4] corrected in [5]).

$$h(k) = ka_{11}(-k)a_{22}(k)\{r^+_{11}(-k)r^+_{12}(k) + r^+_{12}(-k)r^+_{22}(k)\}, \quad h(0) = 0, \tag{9}$$

$$\psi^+(0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} k^{-1}h(k)\prod_{j=1}^{p}\left(\frac{k-k_j}{k+k_j}\right)^{s^1_j} dk. \tag{10}$$

4a) *(5) holds when in (4) one substitutes $C(k)$ by the matrix $C^0(k)$, where $c^0_{ll}(z) \equiv a^0_{ll}(z)$, $l = 1, 2$, are given by (3), and $c^0_{12}(z)$ is determined as follows:*

$$zc^0_{12}(z) = [\psi^+_0(z) - \psi^+_0(0)]a^0_{11}(z), \quad Im\ z > 0, \tag{11}$$

$$\psi^+_0(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h^0(k)}{k-z}dk, \tag{12}$$

$$h^0(k) = ka^0_{11}(-k)a^0_{22}(k)\{r^+_{11}(-k)r^+_{12}(k) + r^+_{12}(-k)r^+_{22}(k)\}, \quad h^0(0) = 0, \tag{13}$$

$$\psi^+_0(0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} k^{-1}h^0(k)dk. \tag{14}$$

5) $\deg Z^+_j(t) \le \sum_{l=1}^{2} sign\ z^{[j]+}_{ll} - 1$, $j = \overline{1,p}$, *with $z^{[j]+}_{ll}$ being nonnegative and constant.*

6) *$rg\ Z^+_j(t) = rg\ diag\ Z^+_j(t) = rg\ diag\ Z^+_j(0)$, $j = \overline{1,p}$.*

7) *The function $h^0(k)$ given by (13) satisfies the Hölder condition on the real axis, that is, there exist constants α and μ, $0 < \mu \le 1$, such that $|h^0(k_1) - h^0(k_2)| \le \alpha|k_1 - k_2|^\mu$ for all $-\infty < k_1 < k_2 < \infty$. (Notice that within the 'only if' part this condition can be enhanced up to $\mu = 1$.) Moreover, $h^0(k) = o(\frac{1}{k})$ as $k \to \pm\infty$ (the last estimate is a consequence of the conditions 1 and 3 of this theorem). Obviously, one gets an equivalent condition replacing in the present condition 7 $h^0(k)$ (13) with $h(k)$ (9).*

Remark 1. *The conditions of the theorem related only to the diagonal matrix elements are direct consequences of [1, Ch. 3], [2], [3, Ch. VI].*

Remark 2. *Condition 7 of Theorem 1 corresponds to the condition H of the main theorem in [4, 5] taken in somewhat reduced form containing no Hölder condition for $h(k)$ (13) in the neighborhood of infinity, that is, without the condition $|h(k_1) - h(k_2)| \le \alpha|\frac{1}{k_1} - \frac{1}{k_2}|^\mu$ for $|k_1|, |k_2| \ge 1$, $\alpha = const > 0$, $0 < \mu \le 1$. However, right at the point ∞ the Hölder condition $|h(k) - h(\infty)| \le \alpha\frac{1}{|k|^\mu}$ holds with*

$h(\infty) = 0$ *and* $\mu = 1$ *due to our condition 7.* (Notice that the notations $h^0(k)$ and $h(k)$ for the expressions (13) and (9) are not found in [4, 5]. Furthermore, to avoid a confusion, we call Theorem 1 from [4, 5] the 'main theorem' owing to the fact that the term 'main theorem' is in the title of [4], while 'Theorem 1' refers to the present work.)

P r o o f of Theorem 1. Prove that condition 7 is necessary. Recall that the condition H in [4], [5] was among sufficient conditions of the main theorem and there was no claim that this condition was necessary.

Now notice that by condition 3 of Theorem 1, the functions $ka^0_{ll}(k)$, $l = 1, 2$, defined by (3), and hence also $ka_{ll}(k)$ (6), have continuous derivatives on the real axis. Demonstrate that

$$a^0_{ll}(k) = 1 + O(1/k), \qquad da^0_{ll}(k)/dk = o(1/k), \qquad k \to \pm\infty; \tag{15}$$

$$a_{ll}(k) = 1 + O(1/k), \qquad da_{ll}(k)/dk = o(1/k), \qquad k \to \pm\infty. \tag{16}$$

These facts are easily deducible as *conditions 1 – 5 of Theorem 1, applied to the diagonal elements of the matrix values in (2), are necessary and sufficient for the above elements to be SD for problem (1) corresponding to the diagonal part of the potential* $V(x)$. (By Theorem 3.5.1 from [1] under assumption that there is a finite second moment for the potential.) The above allows one to establish (15), (16) using the properties of a direct scattering problem, in particular, by Lemma 6 from [5]*. In the lemma the matrix functions $A(k)$, $B(k)$, defined by formulas (10) from [4] as matrix Wronskians, divided by $2ik$, for several Jost solutions of the problem (1), are represented in the form

$$A(k) = I - \tfrac{1}{2ik}\left\{\int_{-\infty}^{\infty} V(x)dx + \int_{-\infty}^{0} A_1(t)e^{-ikt}dt\right\} = I + O(\tfrac{1}{k}),$$
$$B(k) = \tfrac{1}{2ik}\int_{-\infty}^{\infty} B_1(t)e^{-ikt}dt = o(\tfrac{1}{k}), \quad k \to \pm\infty,$$

so that $a_{ll}(k)$ (6) appear to be the diagonal elements of $A(k)$. These expressions provide differentiability for $kA(k)$ and $kB(k)$ as well as for their asymptotics and the asymptotics of their derivatives, since $A_1(t)$, $B_1(t)$ are summable matrix functions on the left semiaxis (correspondingly, on the axis) having the first moment, which allows one to differentiate in k under the integral sign in the right-hand sides of the above relations. For this, one has to use also formula (53) from [5]: $\lim_{k\to 0} kA(k) = C_1$, where $\det C_1 \neq 0$ due to the absence of a virtual level.

*Notice that Lemma 6 from [5] is an analog of Lemma 3.5.1 from [1] with regard to the considered case of matrix potential, which has the second moment on the axis.

Thus we have proved not only asymptotics (15), (16), but also, using the definition $R^+(k) = -A(k)^{-1}B(-k)$ (see (24) from [4]), established the following lemma.

Lemma 1. *Suppose a matrix potential $V(x)$ has the second moment on the axis. The matrix reflection coefficient $R^+(k)$ for the problem (1), as in Theorem 1, is a bounded function of k on the axis with a continuous derivative and such that $\frac{d}{dk}R^+(k) = o(\frac{1}{k})$ as $k \to \pm\infty$.*

Use conditions 1 and 3 of Theorem 1, the estimates (15), (16), and Lemma 1 to deduce from (13) that $h^0(k) = o(\frac{1}{k})$ as $k \to \pm\infty$, $h^0(k) = O(k)$ as $k \to 0$ and that there exists a bounded on the axis derivative of $h^0(k)$: $|dh^0(k)/dk| < const$, $-\infty < k < \infty$ providing the Hölder condition for $h^0(k)$ (even with $\mu = 1$). Thus it is proved that condition 7 of Theorem 1 is necessary for $h^0(k)$, hence also for $h(k)$.

As for the claim that conditions 1–6 of Theorem 1 are necessary, it was established in [4], [5]. However, when proving the necessity of condition 4, we used Remark 2 from [4] corrected in [5], providing the expression (7)–(10) for $zc_{12}(z)$ as a bounded solution (together with $-za_{12}(-z)$) of the Riemann–Hilbert problem with the factorized coefficient $\frac{a_{11}(k)}{a_{22}(-k)}$ for the half-plane

$$\frac{kc_{12}(k)}{a_{11}(k)} = \frac{-ka_{12}(-k)}{a_{22}(-k)} + h(k), \ -\infty < k < \infty. \tag{17}$$

Here $h(k)$ satisfies not only condition 7, but also the Hölder condition in the neighborhood of infinity, as required in [8, Chapter II], [9, Chapter 2]. Now we abandon the last restriction adhering $h(k)$ only to condition 7. It is easy to see that in this case Remark 2 from [4] corrected in [5] is also valid. In particular, (7)–(10) of this paper hold for $zc_{12}(z)$ as one can see from the Sokhotski–Plemelj formulas. Furthermore, a bounded in both half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ solution of the problem (17), given by formulas (7)–(10) and Remark 2 of [4] corrected in [5], is unique, because a bounded solution of the homogeneous equation corresponding to (17) is identically zero. In fact, it turns both sides of this homogeneous equation into a single constant by the Liouville theorem. On the other hand, with $k \to 0$ both sides of this homogeneous equation vanish due to condition 3. So, a bounded solution of the homogeneous equation corresponding to (17) is trivial.

It is established that conditions 1–7 of Theorem 1, except condition 4a, are necessary.

Now notice that condition 7 of Theorem 1, as well as the condition H in the main theorem of [4, 5], is used only in constructing $c_{12}(z)$ in the formulation of condition 4 and in constructing $c^0_{12}(z)$ in the formulation of condition 4a. In both theorems these constructions use the same argument based on solving either the Riemann–Hilbert problem (17) or a similar problem with all functions from (17) but with subscript '0'. Thereby conditions 1–7 of Theorem 1 (with condition 4a) appear to be sufficient in parallel with the sufficiency of conditions 1–6 and H of the main theorem (with condition 4 used in the form of 4a) proved in [4, 5]. Thus we have

Claim 1. *The 'only if' part of Theorem 1 with condition 4 and its 'if' part with condition 4a are proved.*

Lemma 2. *Claim 1 implies that condition 4a is necessary.*[⋆]

First, we sketch a **proof** of Lemma 2 and then we will be able to claim that Theorem 1 is proved completely in both versions, that is, either with condition 4 or with condition 4a. If one proves that conditions 1–7 (with condition 4 used) imply conditions 1–7 (with condition 4a used, the sufficiency of which has been shown earlier), one also proves that conditions 1–7 (with condition 4) are sufficient, while their necessity has already been established.

So, let (2) be an SD for the problem considered, hence it satisfies condition 4 of Theorem 1. By omitting an eigenvalue (either simple or multiple), prove that the values

$$\left\{ R^+(k),\ k \in \mathbb{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = 1, \ldots, p-1 < \infty \right\} \tag{18}$$

also satisfy condition 4 of Theorem 1, along with conditions 1–3 and 5–7, which are obviously valid. This procedure is to be repeated p times to prove that condition 4a is necessary for SD (2).

Begin with considering the diagonal elements of the values (18). To simplify our notation, we omit the indices ll of the diagonal elements. Prove that the functions of the form $f_R^{[p-1]-}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} r_p^-(k) \left(\frac{k-k_p}{k+k_p}\right)^2 e^{-ikx} dk = f_R^{[p]-}(x) - 4\mu_p^2 \int\limits_{-\infty}^{x} f_R^{[p]-}(t)(x-t)e^{-\mu_p(x-t)} dt + 4\mu_p \int\limits_{-\infty}^{x} f_R^{[p]-}(t) e^{-\mu_p(x-t)} dt$, with $\mu_j \equiv -ik_j > 0$, $r_j^-(k) \equiv -\frac{r^+(-k)a_j(-k)}{a_j(k)}$, $f_R^{[j]-}(x) \equiv \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} r_j^-(k) e^{-ikx} dk$, $j = 0, \ldots, p$, derived from the diagonal elements (2), satisfy condition 4 of Theorem 1. Obviously, $f_R^{[p-1]-}(x)$ is absolutely continuous.

Prove that $\frac{d}{dx} f_R^{[p-1]-}(x)$ satisfies (5). After simple computations, which include integration by parts and changing integration order in some multiple inte-

[⋆]This fact was mentioned without proof in the formulation of the main theorem in [4] and [5], with a footnote on a possible method of proving it in [5].

grals, we get

$$\int_{-\infty}^{a}(1+x^2)\left|\tfrac{d}{dx}f_R^{[p-1]-}(x)\right|dx$$
$$\leq 9\int_{-\infty}^{a}(1+x^2)\left|\tfrac{d}{dx}f_R^{[p]-}(x)\right|dx+\tfrac{24}{\mu_p}\int_{-\infty}^{a}t\left|\tfrac{d}{dt}f_R^{[p]-}(t)\right|dt$$
$$+\tfrac{32}{\mu_p^2}\int_{-\infty}^{a}\left|\tfrac{d}{dt}f_R^{[p]-}(t)\right|dt-4\left(\mu_p+\mu_p a^2+2a+\tfrac{2}{\mu_p}\right)e^{-\mu_p a}\int_{-\infty}^{a}t\left|\tfrac{d}{dt}f_R^{[p]-}(t)\right|e^{\mu_p t}dt$$
$$-\tfrac{4}{\mu_p^2}(\mu_p^3 a(1+a^2)+4(a\mu_p+1)^2+2\mu_p^2+4)e^{-\mu_p a}\int_{-\infty}^{a}\left|\tfrac{d}{dt}f_R^{[p]-}(t)\right|e^{\mu_p t}dt.$$

On the other hand, since condition 4 of Theorem 1 holds for $f_R^{[p]-}(x)$, the inequality of the form (5) holds for $\frac{d}{dx}f_R^{[p-1]-}(x)$, hence $f_R^{[p-1]-}(x)$ satisfies condition 4 of Theorem 1.

Now consider a nondiagonal element of the values (18) and prove that this element satisfies condition 4 of Theorem 1. Here three cases are possible, depending on the specific form of the matrix polynomial to be omitted.

Case I. Let $s_p^1=\text{sign } z_{11}^{[p]+}=1$, $s_p^2=\text{sign } z_{22}^{[p]+}=0$, that is $Z_p^+(t)=\begin{pmatrix} z_{11}^{[p]+} & z_{12}^{[p]+}(t)\\ 0 & 0\end{pmatrix}$. Then $c_{11}^p(k)=c_{11}^{p-1}(k)\frac{k-k_p}{k+k_p}$, $c_{22}^p(k)=c_{22}^{p-1}(k)$, $c_{12}^p(k)=c_{12}^{p-1}(k)+\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}c_{11}^{p-1}(k)$, where $\kappa=\sum_{j=1}^{p}\kappa_j=\sum_{j=1}^{p}\left(\text{sign } z_{11}^{[j]+}+\text{sign } z_{22}^{[j]+}\right)$, $P_{\kappa-1}(k)$ $=\psi^+(0)\prod_{j=1}^{p-1}k_j^{\kappa_j}+a_1^p-a_1k_p+k(a_2^p-a_1-a_2k_p)+\cdots+k^{\kappa-2}(a_{\kappa-1}^p-a_{\kappa-2}-a_{\kappa-1}k_p)+$ $k^{\kappa-1}(a_\kappa^p-a_{\kappa-1})$, $\deg P_{\kappa-1}(k)\leq\kappa-1$; $Q_\kappa(k)=(k+k_p)\prod_{j=1}^{p-1}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1}$, $\deg Q_\kappa(k)=\kappa$. (Here $\kappa=\kappa(p)$.)

Thus we have

$$r_{12}^{[p-1]-}(k)=\frac{k-k_p}{k+k_p}r_{12}^{[p]-}(k)-r_{11}^{[p]-}(k)\frac{P_{\kappa-1}(-k)}{Q_\kappa(-k)}\left(\frac{k-k_p}{k+k_p}\right)^2+r_{22}^{[p]-}(k)\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}.$$

Therefore, using the specific form of the Fourier transform of a ratio of polynomials (see Sect. 2, Ch. I, Ex. 5 [10]), one can deduce that the function

$$f_{12}^{[p-1]-}(x)\equiv\tfrac{1}{2\pi}\int_{-\infty}^{\infty}r_{12}^{[p-1]-}(k)e^{-ikx}dk$$
$$=f_{12}^{[p]-}(x)+2\mu_p e^{-\mu_p x}\int_{-\infty}^{x}f_{12}^{[p]-}(t)e^{\mu_p t}dt-\widetilde{\alpha_p}e^{-\mu_p x}\int_{-\infty}^{x}f_{11}^{[p]-}(t)(x-t)e^{\mu_p t}dt$$

$$-\sum_{j=1}^{p} e^{-\mu_j x}\int_{-\infty}^{x}\left\{s_j^1\alpha_j f_{11}^{[p]-}(t)-s_j^2\gamma_j f_{22}^{[p]-}(t)\right\}e^{\mu_j t}dt$$
$$-\sum_{j=1}^{p-1} e^{\mu_j x}\int_{x}^{+\infty}\left\{s_j^2\beta_j f_{11}^{[p]-}(t)-s_j^1\delta_j f_{22}^{[p]-}(t)\right\}e^{-\mu_j t}dt$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants with $\widetilde{\alpha_p}\neq 0$, $\gamma_p\neq 0$. Assume $\beta_p=\delta_p=0$ to deduce that for all $a<\infty$

$$\int_{-\infty}^{a}(1+x^2)\left|\tfrac{d}{dx}f_{12}^{[p-1]-}(x)\right|dx$$
$$\leq 3\int_{-\infty}^{a}(1+x^2)\left|\tfrac{d}{dx}f_{12}^{[p]-}(x)\right|dx+\tfrac{4}{\mu_p}\int_{-\infty}^{a}t\left|\tfrac{d}{dt}f_{12}^{[p]-}(t)\right|dt$$
$$+\tfrac{4}{\mu_p^2}\int_{-\infty}^{a}\left|\tfrac{d}{dt}f_{12}^{[p]-}(t)\right|dt+\int_{-\infty}^{a}(1+t^2)\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j}+\frac{|\widetilde{\alpha_p}|}{\mu_p^2}\right)\left|\tfrac{d}{dt}f_{11}^{[p]-}(t)\right|\right.$$
$$\left.+\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j}\left|\tfrac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+2\int_{-\infty}^{a}t\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|-s_j^2|\beta_j|}{\mu_j^2}+\frac{2|\widetilde{\alpha_p}|}{\mu_p^3}\right)\left|\tfrac{d}{dt}f_{11}^{[p]-}(t)\right|\right.$$
$$\left.+\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|-s_j^1|\delta_j|}{\mu_j^2}\left|\tfrac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+2\int_{-\infty}^{a}\left\{\left(\sum_{j=1}^{p}\frac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j^3}+\frac{3|\widetilde{\alpha_p}|}{\mu_p^4}\right)\left|\tfrac{d}{dt}f_{11}^{[p]-}(t)\right|\right.$$
$$\left.+\sum_{j=1}^{p}\frac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j^3}\left|\tfrac{d}{dt}f_{22}^{[p]-}(t)\right|\right\}dt+M_p(a)<\infty,$$

where $M_p(a)\equiv -2e^{-\mu_p a}[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}|\frac{d}{dt}f_{12}^{[p]-}(t)|e^{\mu_p t}dt-\sum_{j=1}^{p}\frac{e^{-\mu_j a}}{\mu_j}[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}(s_j^1|\alpha_j||\frac{d}{dt}f_{11}^{[p]-}(t)|+s_j^2|\gamma_j||\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_j t}dt+\sum_{j=1}^{p-1}\frac{e^{\mu_j a}}{\mu_j}[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{a}^{+\infty}(s_j^2|\beta_j|\cdot|\frac{d}{dt}f_{11}^{[p]-}(t)|+s_j^1|\delta_j||\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{-\mu_j t}dt-|\widetilde{\alpha_p}|\frac{e^{-\mu_p a}}{\mu_p}[(1+a^2)a-\frac{3a^2+1}{\mu_p}+\frac{6a}{\mu_p^2}+\frac{6}{\mu_p^3}]\cdot$
$\int_{-\infty}^{a}|\frac{d}{dt}f_{11}^{[p]-}(t)|e^{\mu_p t}dt+|\widetilde{\alpha_p}|\frac{e^{-\mu_p a}}{\mu_p}[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}t|\frac{d}{dt}f_{11}^{[p]-}(t)|e^{\mu_p t}dt<\infty.$

Case II. Now let $s_p^1=0$, $s_p^2=1$, that is $Z_p^+(t)=\begin{pmatrix}0 & z_{12}^{[p]+}(t)\\ 0 & z_{22}^{[p]+}\end{pmatrix}$. Then $c_{11}^p(k)=c_{11}^{p-1}(k)$, $c_{22}^p(k)=c_{22}^{p-1}(k)\frac{k-k_p}{k+k_p}$, $c_{12}^p(k)=c_{12}^{p-1}(k)\frac{k-k_p}{k+k_p}+\frac{P_{\kappa-1}(k)}{Q_\kappa(k)}c_{11}^{p-1}(k)$, where $P_{\kappa-1}(k)=\psi^+(0)\prod_{j=1}^{p-1}k_j^{\kappa_j}+a_1^p+a_1k_p+k(a_2^p-a_1+a_2k_p)+\dots+k^{\kappa-1}(a_\kappa^p-a_{\kappa-1})+\frac{2k_p\psi^-(-k_p)}{k}[\prod_{j=1}^{p-1}(k+k_j)^{\kappa_j}-\prod_{j=1}^{p-1}k_j^{\kappa_j}]$, $\deg P_{\kappa-1}(k)\leq\kappa-1$; $Q_\kappa(k)=\prod_{j=1}^{p}(k+k_j)^{s_j^2}(k-k_j)^{s_j^1}$, $\deg Q_\kappa(k)=\kappa$.

This implies

$$r_{12}^{[p-1]-}(k) = \frac{k-k_p}{k+k_p} r_{12}^{[p]-}(k) - r_{11}^{[p]-}(k) \frac{P_{\kappa-1}(-k)}{Q_\kappa(-k)} \frac{k-k_p}{k+k_p} + r_{22}^{[p]-}(k) \frac{P_{\kappa-1}(k)}{Q_\kappa(k)} \frac{k-k_p}{k+k_p}.$$

Using again the quoted above Example 5 to Ch. I, Section 2 of [10], one can deduce that the function

$$f_{12}^{[p-1]-}(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} r_{12}^{[p-1]-}(k) e^{-ikx} dk = f_{12}^{[p]-}(x) + 2\mu_p e^{-\mu_p x} \int_{-\infty}^{x} f_{12}^{[p]-}(t) e^{\mu_p t} dt$$
$$- \sum_{j=1}^{p} e^{-\mu_j x} \int_{-\infty}^{x} \left\{ s_j^1 \alpha_j f_{11}^{[p]-}(t) - s_j^2 \gamma_j f_{22}^{[p]-}(t) \right\} e^{\mu_j t} dt$$
$$- \sum_{j=1}^{p-1} e^{\mu_j x} \int_{x}^{+\infty} \left\{ s_j^2 \beta_j f_{11}^{[p]-}(t) - s_j^1 \delta_j f_{22}^{[p]-}(t) \right\} e^{-\mu_j t} dt$$
$$+ \widetilde{\gamma}_p e^{-\mu_p x} \int_{-\infty}^{x} f_{22}^{[p]-}(t)(x-t) e^{\mu_p t} dt$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants with $\alpha_p \neq 0$, $\widetilde{\gamma}_p \neq 0$. Again set $\beta_p = \delta_p = 0$ to get for all $a < \infty$

$$\int_{-\infty}^{a} (1+x^2) \left| \frac{d}{dx} f_{12}^{[p-1]-}(x) \right| dx \leq 3 \int_{-\infty}^{a} (1+x^2) \left| \frac{d}{dx} f_{12}^{[p]-}(x) \right| dx + \frac{4}{\mu_p} \int_{-\infty}^{a} t \left| \frac{d}{dt} f_{12}^{[p]-}(t) \right| dt$$
$$+ \frac{4}{\mu_p^2} \int_{-\infty}^{a} \left| \frac{d}{dt} f_{12}^{[p]-}(t) \right| dt + \int_{-\infty}^{a} (1+t^2) \left\{ \sum_{j=1}^{p} \frac{s_j^1 |\alpha_j| + s_j^2 |\beta_j|}{\mu_j} \left| \frac{d}{dt} f_{11}^{[p]-}(t) \right| \right.$$
$$\left. + \left(\sum_{j=1}^{p} \frac{s_j^2 |\gamma_j| + s_j^1 |\delta_j|}{\mu_j} + \frac{|\widetilde{\gamma}_p|}{\mu_p^2} \right) \left| \frac{d}{dt} f_{22}^{[p]-}(t) \right| \right\} dt$$
$$+2 \int_{-\infty}^{a} t \left\{ \sum_{j=1}^{p} \frac{s_j^1 |\alpha_j| - s_j^2 |\beta_j|}{\mu_j^2} \left| \frac{d}{dt} f_{11}^{[p]-}(t) \right| \right.$$
$$\left. + \left(\sum_{j=1}^{p} \frac{s_j^2 |\gamma_j| - s_j^1 |\delta_j|}{\mu_j^2} - \frac{2|\widetilde{\gamma}_p|}{\mu_p^3} \right) \left| \frac{d}{dt} f_{22}^{[p]-}(t) \right| \right\} dt$$
$$+2 \int_{-\infty}^{a} \left\{ \sum_{j=1}^{p} \frac{s_j^1 |\alpha_j| + s_j^2 |\beta_j|}{\mu_j^3} \left| \frac{d}{dt} f_{11}^{[p]-}(t) \right| \right.$$
$$\left. \left(\sum_{j=1}^{p} \frac{s_j^2 |\gamma_j| + s_j^1 |\delta_j|}{\mu_j^3} + \frac{3|\widetilde{\gamma}_p|}{\mu_p^4} \right) \left| \frac{d}{dt} f_{22}^{[p]-}(t) \right| \right\} dt + M_p(a) < \infty,$$

where $M_p(a) \equiv -2e^{-\mu_p a}[1 + a^2 + \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}] \int\limits_{-\infty}^{a} |\frac{d}{dt} f_{12}^{[p]-}(t)| e^{\mu_p t} dt - \sum\limits_{j=1}^{p} \frac{e^{-\mu_j a}}{\mu_j}[1 + a^2 + \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}] \int\limits_{-\infty}^{a} (s_j^1 |\alpha_j| |\frac{d}{dt} f_{11}^{[p]-}(t)| + s_j^2 |\gamma_j| |\frac{d}{dt} f_{22}^{[p]-}(t)|) e^{\mu_j t} dt + \sum\limits_{j=1}^{p-1} \frac{e^{\mu_j a}}{\mu_j}[1 + a^2 - \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}] \int\limits_{a}^{+\infty} (s_j^2 |\beta_j| \cdot |\frac{d}{dt} f_{11}^{[p]-}(t)| + s_j^1 |\delta_j| |\frac{d}{dt} f_{22}^{[p]-}(t)|) e^{-\mu_j t} dt - |\widetilde{\gamma_p}| \frac{e^{-\mu_p a}}{\mu_p}[(1 + a^2)a - \frac{3a^2+1}{\mu_p} + \frac{6a}{\mu_p^2} + \frac{6}{\mu_p^3}] \cdot$
$\int\limits_{-\infty}^{a} |\frac{d}{dt} f_{22}^{[p]-}(t)| e^{\mu_p t} dt + |\widetilde{\gamma_p}| \frac{e^{-\mu_p a}}{\mu_p}[1 + a^2 - \frac{2a}{\mu_p} + \frac{2}{\mu_p^2}] \int\limits_{-\infty}^{a} t |\frac{d}{dt} f_{22}^{[p]-}(t)| e^{\mu_p t} dt < \infty$.

Case III. Finally, let $s_p^1 = 1$, $s_p^2 = 1$, that is $Z_p^+(t) = \begin{pmatrix} z_{11}^{[p]+} & z_{12}^{[p]+}(t) \\ 0 & z_{22}^{[p]+} \end{pmatrix}$. Then $c_{11}^p(k) = c_{11}^{p-1}(k) \frac{k-k_p}{k+k_p}$, $c_{22}^p(k) = c_{22}^{p-1}(k) \frac{k-k_p}{k+k_p}$, $c_{12}^p(k) = c_{12}^{p-1}(k) \frac{k-k_p}{k+k_p} + \frac{P_{\kappa-1}(k)}{Q_\kappa(k)} c_{11}^{p-1}(k)$, where $\kappa = \sum\limits_{j=1}^{p} \kappa_j = \sum\limits_{j=1}^{p} \left(\text{sign } z_{11}^{[j]+} + \text{sign } z_{22}^{[j]+}\right)$, $P_{\kappa-1}(k) = a_1^p + a_1 k_p^2 + k(\psi^+(0) \prod\limits_{j=1}^{p-1} k_j^{\kappa_j} + a_2^p + a_2 k_p^2) + \cdots + k^{\kappa-1}(a_\kappa^p - a_{\kappa-2}) + 2k_p \psi^-(-k_p) \prod\limits_{j=1}^{p-1} (k + k_j)^{\kappa_j} + \frac{2k_p^2 \psi^-(-k_p)}{k} [\prod\limits_{j=1}^{p-1} (k + k_j)^{\kappa_j} - \prod\limits_{j=1}^{p-1} k_j^{\kappa_j}]$, $\deg P_{\kappa-1}(k) \le \kappa - 1$; $Q_\kappa(k) = (k + k_p)^2 \prod\limits_{j=1}^{p-1} (k + k_j)^{s_j^2} (k - k_j)^{s_j^1}$, $\deg Q_\kappa(k) = \kappa$.

Thus we obtain

$$r_{12}^{[p-1]-}(k) = r_{12}^{[p]-}(k) - \frac{4k_p}{k+k_p} r_{12}^{[p]-}(k) + \frac{4k_p^2}{(k+k_p)^2} r_{12}^{[p]-}(k)$$
$$r_{11}^{[p]-}(k) \frac{R_\kappa(k)}{(k+k_p)^3 \prod\limits_{j=1}^{p-1} (k-k_j)^{s_j^2} (k+k_j)^{s_j^1}} + r_{22}^{[p]-}(k) \frac{T_\kappa(k)}{(k+k_p)^3 \prod\limits_{j=1}^{p-1} (k+k_j)^{s_j^2} (k-k_j)^{s_j^1}},$$

where $R_\kappa(k) \equiv (-1)^\kappa P_{\kappa-1}(-k)(k - k_p)$, $\deg R_\kappa(k) \le \kappa$; $T_\kappa(k) \equiv P_{\kappa-1}(k)(k - k_p)$, $\deg T_\kappa(k) \le \kappa$.

Use again Example 5 to Chapter I, Section 2 of [10] to conclude that the function

$$f_{12}^{[p-1]-}(x) \equiv \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} r_{12}^{[p-1]-}(k) e^{-ikx} dk = f_{12}^{[p]-}(x) + 4\mu_p e^{-\mu_p x} \int\limits_{-\infty}^{x} f_{12}^{[p]-}(t) e^{\mu_p t} dt$$
$$-4\mu_p^2 e^{-\mu_p x} \int\limits_{-\infty}^{x} f_{12}^{[p]-}(t)(x - t) e^{\mu_p t} dt$$
$$-\sum\limits_{j=1}^{p} e^{-\mu_j x} \int\limits_{-\infty}^{x} \{s_j^1 \alpha_j f_{11}^{[p]-}(t) - s_j^2 \gamma_j f_{22}^{[p]-}(t)\} e^{\mu_j t} dt$$

$$-\sum_{j=1}^{p-1} e^{\mu_j x}\int_x^{+\infty}\{s_j^2\beta_j f_{11}^{[p]-}(t)-s_j^1\delta_j f_{22}^{[p]-}(t)\}e^{-\mu_j t}dt$$
$$-e^{-\mu_p x}\int_{-\infty}^{x}(\alpha_p^{[1]}f_{11}^{[p]-}(t)-\gamma_p^{[1]}f_{22}^{[p]-}(t))(x-t)e^{\mu_p t}dt$$
$$-e^{-\mu_p x}\int_{-\infty}^{x}(\alpha_p^{[2]}f_{11}^{[p]-}(t)-\gamma_p^{[2]}f_{22}^{[p]-}(t))(x-t)^2e^{\mu_p t}dt$$

is absolutely continuous. Here α_j, β_j, γ_j, δ_j are constants with $\alpha_p^{[l]}\neq 0\neq\gamma_p^{[l]}$, $l=1,2$. Set $\beta_p=\delta_p=0$ to deduce that for all $a<\infty$ one has

$$\int_{-\infty}^{a}(1+x^2)|\tfrac{d}{dx}f_{12}^{[p-1]-}(x)|dx\le 9\int_{-\infty}^{a}(1+x^2)|\tfrac{d}{dx}f_{12}^{[p]-}(x)|dx+\tfrac{24}{\mu_p}\int_{-\infty}^{a}t|\tfrac{d}{dt}f_{12}^{[p]-}(t)|dt$$
$$+\tfrac{32}{\mu_p^2}\int_{-\infty}^{a}|\tfrac{d}{dt}f_{12}^{[p]-}(t)|dt+\int_{-\infty}^{a}(1+t^2)\{(\sum_{j=1}^{p}\tfrac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j}+\tfrac{|\alpha_p^{[1]}|}{\mu_p^2}+\tfrac{2|\alpha_p^{[2]}|}{\mu_p^3})|\tfrac{d}{dt}f_{11}^{[p]-}(t)|$$
$$+(\sum_{j=1}^{p}\tfrac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j}+\tfrac{|\gamma_p^{[1]}|}{\mu_p^2}+\tfrac{2|\gamma_p^{[2]}|}{\mu_p^3})|\tfrac{d}{dt}f_{22}^{[p]-}(t)|\}dt$$
$$+2\int_{-\infty}^{a}t\{(\sum_{j=1}^{p}\tfrac{s_j^1|\alpha_j|-s_j^2|\beta_j|}{\mu_j^2}+\tfrac{2|\alpha_p^{[1]}|}{\mu_p^3}+\tfrac{6|\alpha_p^{[2]}|}{\mu_p^4})|\tfrac{d}{dt}f_{11}^{[p]-}(t)|$$
$$+(\sum_{j=1}^{p}\tfrac{s_j^2|\gamma_j|-s_j^1|\delta_j|}{\mu_j^2}+\tfrac{2|\gamma_p^{[1]}|}{\mu_p^3}+\tfrac{6|\gamma_p^{[2]}|}{\mu_p^4})|\tfrac{d}{dt}f_{22}^{[p]-}(t)|\}dt+2\int_{-\infty}^{a}\{(\sum_{j=1}^{p}\tfrac{s_j^1|\alpha_j|+s_j^2|\beta_j|}{\mu_j^3}+$$
$$\tfrac{3|\alpha_p^{[1]}|}{\mu_p^4}+\tfrac{12|\alpha_p^{[2]}|}{\mu_p^5})|\tfrac{d}{dt}f_{11}^{[p]-}(t)|+(\sum_{j=1}^{p}\tfrac{s_j^2|\gamma_j|+s_j^1|\delta_j|}{\mu_j^3}+\tfrac{3|\gamma_p^{[1]}|}{\mu_p^4}+\tfrac{12|\gamma_p^{[2]}|}{\mu_p^5})|\tfrac{d}{dt}f_{22}^{[p]-}(t)|\}dt$$
$$+M_p(a)<\infty,$$

where $M_p(a)\equiv -4e^{-\mu_p a}[\mu_p a(1+a^2)-2a^2+\frac{8a}{\mu_p}+\frac{8}{\mu_p^2}]\int_{-\infty}^{a}|\frac{d}{dt}f_{12}^{[p]-}(t)|e^{\mu_p t}dt+$ $4\mu_p e^{-\mu_p a}[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}t|\frac{d}{dt}f_{12}^{[p]-}(t)|e^{\mu_p t}dt-\sum_{j=1}^{p}\frac{e^{-\mu_j a}}{\mu_j}[1+a^2+\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}(s_j^1|\alpha_j|$ $|\frac{d}{dt}f_{11}^{[p]-}(t)|+s_j^2|\gamma_j||\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_j t}dt+\sum_{j=1}^{p-1}\frac{e^{\mu_j a}}{\mu_j}[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{a}^{+\infty}(s_j^2|\beta_j||\frac{d}{dt}f_{11}^{[p]-}(t)|$ $+s_j^1|\delta_j|\cdot|\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{-\mu_j t}dt-\frac{e^{-\mu_p a}}{\mu_p a}[(1+a^2)a-\frac{3a^2+1}{\mu_p}+\frac{6a}{\mu_p^2}+\frac{6}{\mu_p^3}]\int_{-\infty}^{a}(|\alpha_p^{[1]}||\frac{d}{dt}f_{11}^{[p]-}(t)|$ $+|\gamma_p^{[1]}|\cdot|\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_p t}dt+\frac{e^{-\mu_p a}}{\mu_p a}[1+a^2-\frac{2a}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}t(|\alpha_p^{[1]}||\frac{d}{dt}f_{11}^{[p]-}(t)|+|\gamma_p^{[1]}|$ $|\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_p t}dt-\frac{e^{-\mu_p a}}{\mu_p a}[(1+a^2)a^2+\frac{2(a^2+a(1+a^2))}{\mu_p}-\frac{2}{\mu_p^2}+\frac{24a}{\mu_p^3}+\frac{24}{\mu_p^4}]\int_{-\infty}^{a}(|\alpha_p^{[2]}||\frac{d}{dt}f_{11}^{[p]-}(t)|$ $+|\gamma_p^{[2]}||\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_p t}dt+\frac{e^{-\mu_p a}}{\mu_p a}[2a(1+a^2)+\frac{2(3a^2+1)}{\mu_p}+\frac{4a}{\mu_p^2}+\frac{12}{\mu_p^3}]\int_{-\infty}^{a}t(|\alpha_p^{[2]}||\frac{d}{dt}f_{11}^{[p]-}(t)|$ $+|\gamma_p^{[2]}||\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_p t}dt-\frac{e^{-\mu_p a}}{\mu_p a}[1+a^2+\frac{2a)}{\mu_p}+\frac{2}{\mu_p^2}]\int_{-\infty}^{a}t^2(|\alpha_p^{[2]}||\frac{d}{dt}f_{11}^{[p]-}(t)|+|\gamma_p^{[2]}|$ $|\frac{d}{dt}f_{22}^{[p]-}(t)|)e^{\mu_p t}dt<\infty.$

This case completes the proving of the fact that condition 4a is necessary for SD (2); this also completes the proof of Lemma 2 as well as of Theorem 1 in two versions.

Theorem 2. *Conditions 1–6 of Theorem 1 (either with condition 4 or 4a), together with additional condition 1a:*

1a) $R^+(k)$ *has a continuous derivative which is bounded on the entire axis and is such that* $\frac{d}{dk}R^+(k) = o(\frac{1}{k})$, $k \to \pm\infty$,

are necessary and sufficient to characterize the scattering data for the problem (1) of the case considered in Theorem 1.

P r o o f of Theorem 2. Lemma 1 establishes the necessity of condition 1a of Theorem 2, while the necessity of conditions 1–6 (either with condition 4 or 4a) are established by Theorem 1. As shown in the proof of Theorem 1, its condition 7 follows from conditions 1–6 and Lemma 1 (i.e., Th. 2, cond. 1a). Thus the conditions of Theorem 2 imply all conditions 1–7 of Theorem 1, therefore the conditions of Theorem 2 are necessary and sufficient either under condition 4 or 4a. Theorem 2 is proved.

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