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A Property of Azarin's Limit Set of Subharmonic Functions

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Let $v(z)$ be a subharmonic function of order $\rho > 0$, and $\mathrm{Fr}(v)$ be the limit set in the sense of Azarin. Let z be fixed and $I(z) = \{u(z) : u \in \mathrm{Fr}(v)\}$. We prove that $I(z)$ is either a closed interval or a semiclosed interval which does not contain its infimum.

Key words: subharmonic function, limit set of Azarin, indicator of growth.

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The following definitions are needed to state our main result. The definition and properties of proximate order $\rho(r)$ in the sense of Valiron can be found in [1]. We denote $V(r) = r^{\rho(r)}$.

A subharmonic function v is of proximate order $\rho(r)$ if

$$\varlimsup_{z\to\infty} \frac{v(z)}{V(|z|)} < \infty.$$

Let $v(z)$ be a subharmonic function of proximate order $\rho(r)$, $\rho = \lim \rho(r) \in (0, \infty)$ $(r \to \infty)$, and let

$$v_t(z) = \frac{v(tz)}{V(t)}$$

be a trajectory of Azarin of subharmonic function v. The limit set of Azarin $\mathrm{Fr}(v)$ is defined as a set of functions given by

$$u(z) = \lim_{n\to\infty} v_{t_n}(z)$$

for some sequence (t_n), $t_n \to +\infty$.

The limit is taken in the sense of distributions. This means that

$$\lim_{n\to\infty} \iint v_{t_n}(z)\varphi(z)dm_2(z) = \iint u(z)\varphi(z)dm_2(z)$$

for any test function φ, where m_2 is a two-dimensional Lebesgue measure.

Let

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{v(re^{i\theta})}{V(r)},\ \underline{h}(\theta) = \underline{\lim_{r\to\infty}}^{*} \frac{v(re^{i\theta})}{V(r)} := \sup_{E} \liminf_{\substack{r\to\infty \\ r\overline{\in}E}} \frac{v(re^{i\theta})}{V(r)},$$

where $E \subset (0,\infty)$ runs over all sets of zero linear density which is defined by

$$\mathrm{dens}E = \lim_{r\to\infty} \frac{\mathrm{mes}(E\cap[0,r])}{r}.$$

The function h is called an indicator of function v and the function $\underline{h}$ is called a lower indicator of function v. In 1979 V.S. Azarin [2] proved that

$$H(z) := \sup\{u(z):\ u\in \mathrm{Fr}(v)\} = h(\theta)r^{\rho},\ z = re^{i\theta},$$

$$\underline{H}(z) := \inf\{u(z):\ u\in \mathrm{Fr}(v)\} = \underline{h}(\theta)r^{\rho}.$$

See [3] for other properties of the limit set.

Denote $I(z) = \{u(z) : u \in \mathrm{Fr}(v)\}$. We prove the following refined version of Azarin's theorem.

Theorem 1. *For each z, $(\underline{h}(\theta)r^{\rho}, h(\theta)r^{\rho}]$ is a subset of $I(z)$, and $I(z)$ is a subset of $[\underline{h}(\theta)r^{\rho}, h(\theta)r^{\rho}]$.*

We give an example of a subharmonic function v such that $\underline{h}(\theta)r^{\rho}\overline{\in}I(z)$ for some z. The case $\underline{h}(\theta)r^{\rho} \in I(z)$ is possible as well.

P r o o f of Theorem 1. Denote

$$(F_t u)(z) = \frac{u(tz)}{t^{\rho}}.$$

V.S. Azarin [2] proved that $F_t(\mathrm{Fr}(v))(z) \subset \mathrm{Fr}(v)$. The map $F_t : \mathrm{Fr}(v) \to \mathrm{Fr}(v)$ is one-to-one. We denote $H(z) := \sup\{u(z):\ u\in \mathrm{Fr}(v)\}$, $\underline{H}(z) := \inf\{u(z):\ u\in \mathrm{Fr}(v)\}$. We have

$$H(tz) = \sup\{u(tz):\ u\in \mathrm{Fr}(v)\} = t^{\rho}\sup\left\{\frac{u(tz)}{t^{\rho}}:\ u\in\mathrm{Fr}(v)\right\} = t^{\rho}H(z).$$

Thus $H(re^{i\theta}) = r^{\rho}H(e^{i\theta})$. Analogously, $\underline{H}(re^{i\theta}) = r^{\rho}\underline{H}(e^{i\theta})$. For every $\varepsilon > 0$ there exists (see, for example, [2]) a number R such that $v(re^{i\theta}) < (h(\theta)+\varepsilon)V(r)$ is valid for $r\in[R,\infty)$ and for any θ.

Consequently,

$$v_t(z) = \frac{v(tre^{i\theta})}{V(t)} < (h(\theta)+\varepsilon)\frac{V(tr)}{V(t)}\ ,tr > R.$$

It is known [1] that

$$\frac{V(tr)}{V(t)} \rightrightarrows r^\rho, \ 0 < a \le r \le b < \infty,$$

where the double arrow means a uniform convergence on the given set. Hence there exist numbers $R_1 > 0$ and $\alpha_0 \in (0,1)$ such that

$$v_t(z) \le h(\theta) + 2\varepsilon, \ z \in C(e^{i\theta}, \alpha_0), \ t \ge R_1. \tag{1}$$

Here $C(e^{i\theta}, \alpha_0)$ is an open disk centered at $e^{i\theta}$ with the radius α_0. Let u be an arbitrary function from $\mathrm{Fr}(v)$. It follows from the definition of fine topology [4] that the set $E = \{z : \ u(z) > u(e^{i\theta}) - \varepsilon\}$ is a fine neighborhood of $e^{i\theta}$. Then there exists a compact set K such that $K \subset E \cap C(e^{i\theta}, \alpha_0)$ and $\mathrm{cap} K > 0$. Therefore there exists a positive measure ν such that $\nu(K) > 0$, $\mathrm{supp}(\nu) \subset K$, and the potential

$$b(z) = \iint \ln|z - \zeta| d\nu(\zeta)$$

is continuous ([5], corollary to Th. 3.7).

Further we need the following results. Let $v_n(z)$ be a sequence of subharmonic functions converging in the sense of distributions to a distribution w. Then w is a regular distribution and may be represented by a subharmonic function $w(z)$. We recall that the distribution T

$$T\varphi = \iint f(z)\varphi(z) dm_2(z),$$

where f is a locally integrable function, is called a regular distribution. Let μ_n and μ be the Riesz masses of v_n and w, respectively. We have $\mu_n = \frac{1}{2\pi}\Delta v_n, \mu = \frac{1}{2\pi}\Delta w$, where Δ is the Laplace operator. Differentiation is continuous in the space of distributions. It follows that $\mu_n \to \mu$ in the sense of distributions. Theorem 0.4 [5] states that μ_n converges weakly to μ as a sequence of Radon measures. This means that $(\mu_n, \varphi) \to (\mu, \varphi)$ for any continuous compactly supported function φ. In addition, if a compact set K is Jordan measurable with respect to the measure μ (this means that $\mu(\partial K) = 0$), then μ_n converges weakly to μ as a sequence of elements of the Banach space $C^*(K)$. This means that $(\mu_n, \varphi) \to (\mu, \varphi)$ for any function φ which is continuous on K. If $K = B(z_0, R)$ and $\mu(\partial B(z_0, R)) = 0$, then

$$\lim_{n\to\infty} \iint\limits_{B(z_0,R)} \ln|z - \zeta| d\mu_n(\zeta) = \iint\limits_{B(z_0,R)} \ln|z - \zeta| d\mu(\zeta)$$

in the sense of distributions. We have the Riesz representation

$$v_n(z) = \iint\limits_{B(z_0,R)} \ln|z - \zeta| d\mu_n(\zeta) + u_n(z),$$

where u_n is a harmonic function in disk $C(z_0, R) = \{z : |z - z_0| < R\}$. It is clear that (u_n) is a convergent sequence in the sense of distributions. Then the sequence (u_n) is uniformly convergent on every compact set $K \subset C(z_0, R)$ by Th. 4.4.2 ([6]). Thus, modulo the uniformly convergent sequence (u_n) of harmonic functions, the (v_n) is a sequence of potentials, and so many classical results from potential theory may be extended to (v_n). In particular,

$$\lim_{n\to\infty} \iint v_{t_n}(z) d\nu(z) = \iint w(z) d\nu(z). \tag{2}$$

The proof of an analogous proposition for potentials appeared in [5, Th. 3.8]. Now we have

$$\left(u(e^{i\theta}) - \varepsilon\right) \nu(K) \le \iint u(z) d\nu(z) \le \left(h(\theta) + 2\varepsilon\right) \nu(K).$$

The left-hand side follows from the inequality $u(z) > u(e^{i\theta}) - \varepsilon$ for $z \in K$, and the right-hand side follows from (1). This gives $u(e^{i\theta}) \le h(\theta)$ for any $u \in \mathrm{Fr}(v)$, $H(e^{i\theta}) \le h(\theta)$.

Further we prove that there exists a function $u_0 \in \mathrm{Fr}(v)$ such that $u_0(e^{i\theta}) = h(\theta)$. Since

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{v(re^{i\theta})}{V(r)},$$

then there exists a sequence (t_n), $t_n \to \infty$ as $n \to \infty$, such that the sequence of real numbers $v_{t_n}(e^{i\theta})$ converges to $h(\theta)$ as $n \to \infty$.

The set $\{v_t(z) : t \in \overline{0, \infty}\}$ is compact in the sense of distributions, see [7, Th. 2.7.1.1]. Hence we can find a subsequence t_{n_k} such that $v_{t_{n_k}}(z) \to u_0(z)$ in the sense of distributions.

According to the principle of ascent (for potentials it is Th. 1.3 [5]),

$$h(\theta) = \overline{\lim_{k\to\infty}} v_{t_{n_k}}(e^{i\theta}) \le u_0(e^{i\theta}) \le H(e^{i\theta}).$$

This yields

$$h(\theta) = H(e^{i\theta}),\ u_0(e^{i\theta}) = h(\theta).$$

Our next step is to prove the inequality $u(e^{i\theta}) \ge \underline{h}(\theta)$ for $u \in \mathrm{Fr}(v)$. Let $u \in \mathrm{Fr}(v)$ and $\varepsilon > 0$. It is evident that we may assume $\underline{h}(\theta) > -\infty$. Since u is upper semicontinuous, then there exists $\alpha \in (0, 1)$ such that $u(z) < u(e^{i\theta}) + \varepsilon$ for $z \in B(e^{i\theta}, \alpha)$. Let $t_n \to \infty$ as $n \to \infty$ be such a sequence that $v_{t_n} \to u$ as a sequence of distributions. Then, for $\alpha \in (0, 1)$

$$\lim_{n\to\infty} \int_{1-\alpha}^{1+\alpha} \left| u(te^{i\theta}) - v_{t_n}(te^{i\theta}) \right| dt \to 0. \tag{3}$$

Using the similar arguments as those used to prove (2), one can reduce the proposition to the following. Let a sequence μ_n of Borel measures in disk $B(z_0,R)$ converge weakly to a measure μ. Then the real sequence

$$a_n = \int_{1-\alpha}^{1+\alpha} \left| \iint_{B(z_0,R)} \ln|te^{i\theta} - \zeta| d(\mu_n - \mu)(\zeta) \right| dt$$

converges to zero.

We have

$$a_n = \iint_{B(z_0,R)} \left(\int_{1-\alpha}^{1+\alpha} h_n(t) \ln|te^{i\theta} - \zeta| dt \right) d(\mu_n - \mu)(\zeta),$$

where

$$h_n(t) = \operatorname{sign} \iint_{B(z_0,R)} \ln|te^{i\theta} - \zeta| d(\mu_n - \mu)(\zeta).$$

The function $h_n(t)$ is measurable and $|h_n(t)| \leq 1$. Consider a family of functions

$$H_n(\zeta) = \int_{1-\alpha}^{1+\alpha} h_n(t) \ln|te^{i\theta} - \zeta| dt, \ n = 1, 2, \dots .$$

The inequality

$$|H_n(\zeta)| \leq \int_{1-\alpha}^{1+\alpha} \Big| \ln|te^{i\theta} - \zeta| \Big| dt$$

shows that the family $H_n(\zeta)$ is uniformly bounded in $B(z_0,R)$.

Further,

$$|H_n(\zeta_2) - H_n(\zeta_1)| \leq \int_{1-\alpha}^{1+\alpha} \left| \ln|\frac{te^{i\theta} - \zeta_2}{te^{i\theta} - \zeta_1}| \right| dt$$

$$= \int_{1-\alpha}^{1+\alpha} \max \left(\ln|\frac{te^{i\theta} - \zeta_2}{te^{i\theta} - \zeta_1}|, \ln|\frac{te^{i\theta} - \zeta_1}{te^{i\theta} - \zeta_2}| \right) dt$$

$$\leq \int_{1-\alpha}^{1+\alpha} \ln \left(1 + \frac{|\zeta_2 - \zeta_1|}{|te^{i\theta} - \zeta_1|} \right) dt + \int_{1-\alpha}^{1+\alpha} \ln \left(1 + \frac{|\zeta_2 - \zeta_1|}{|te^{i\theta} - \zeta_2|} \right) dt = J_1 + J_2.$$

In the integral J_1 we introduce a new variable r by the formula $r = |te^{i\theta} - \zeta_1|$. We obtain

$$J_1 = \int\limits_0^{\infty} \ln\left(1 + \frac{|\zeta_2 - \zeta_1|}{r}\right) d\nu(r),$$

where $\nu(r) = \text{mes}\left([(1-\alpha)e^{i\theta}, (1+\alpha)e^{i\theta}] \cap B(\zeta_1, r)\right)$.

The function $\nu(r)$ is constant in $[R, \infty)$, where $R = \max\left(|(1-\alpha)e^{i\theta} - \zeta_1|\,, |(1+\alpha)e^{i\theta} - \zeta_1|\right)$. The inequality $\nu(r) \le 2r$ is obvious. From the properties given above it follows that

$$J_1 = \int\limits_0^{R} \ln\left(1 + \frac{|\zeta_2 - \zeta_1|}{r}\right) d\nu(r) \le 2\int\limits_0^{R} \ln\left(1 + \frac{|\zeta_2 - \zeta_1|}{r}\right) dr.$$

It can be shown by integrating by parts. The integral J_2 is estimated in a similar way. Be specific about what estimates show equicontinuity $H_n(\zeta)$. Arzela–Ascoli's theorem gives a compactness of the family $H_n(\zeta)$. Consequently, the sequence

$$a_n = \iint\limits_{B(z_0,R)} H_n(\zeta) d(\mu_n - \mu)(\zeta)$$

converges to zero. Formula (3) is proved.

If $A_n = \left\{t \in [1-\alpha, 1+\alpha] : \ \left|u(te^{i\theta}) - v_{t_n}(te^{i\theta})\right| \ge \varepsilon\right\}$, then $\text{mes}(A_n) \to 0$ as $n \to \infty$. If $B(\varepsilon) = \left\{r \in (0,\infty) : \ v(re^{i\theta}) < \underline{h}(\theta) - \varepsilon\right\}$, then the formula for $\underline{h}(\theta)$ gives that the linear density of $B(\varepsilon)$ is zero.

For $\alpha \in (0,1)$ we denote

$$B_n = \left\{t \in [1-\alpha, 1+\alpha] : \ (\underline{h}(\theta) - \varepsilon)\,\frac{V(t_n t)}{V(t_n)} \ge v_{t_n}(te^{i\theta})\right\}.$$

If $t \in B_n$, then

$$v_{t_n}(te^{i\theta}) = \frac{v(t_n te^{i\theta})}{V(t_n)} \le (\underline{h}(\theta) - \varepsilon)\,\frac{V(t_n t)}{V(t_n)},$$

$$v(t_n te^{i\theta}) \le (\underline{h}(\theta) - \varepsilon)\, V(t_n t).$$

It follows that $t_n t \in B(\varepsilon)$, $t_n B_n \subset B(\varepsilon)$, $\text{mes}(t_n B_n) \le \text{mes}\left(B(\varepsilon) \cap [(1-\alpha)t_n, (1+\alpha)t_n]\right)$, and

$$\text{mes}(B_n) \le \frac{\text{mes}\left(B(\varepsilon) \cap [0, (1+\alpha)t_n]\right)}{t_n}.$$

Now the property that density of $B(\varepsilon)$ is zero implies $\text{mes}(B_n) \to 0$ as $n \to \infty$. We have

$$\left(\underline{h}(\theta) - \varepsilon\right) \frac{V(t_n t)}{V(t_n)} < v_{t_n}(te^{i\theta}) \le u(te^{i\theta}) + \left|u(te^{i\theta}) - v_{t_n}(te^{i\theta})\right| < u(e^{i\theta}) + 2\varepsilon$$

for $t \in [1-\alpha, 1+\alpha] \setminus (A_n \cup B_n)$. The convergence

$$\frac{V(t_n t)}{V(t_n)} \rightrightarrows t^\rho,\ t \in [1-\alpha, 1+\alpha]$$

leads to the inequality

$$(\underline{h}(\theta) - \varepsilon)\,\frac{V(t_n t)}{V(t_n)} > \underline{h}(\theta) - 2\varepsilon$$

for sufficiently large n and small α. Thus we obtain the claimed inequality $u(e^{i\theta}) \geq \underline{h}(\theta)$.

Now is the final step of the proof. With θ fixed, we denote

$$A(r,\alpha) = \frac{1}{rV(r)} \int_r^{(1+\alpha)r} v(te^{i\theta})dt.$$

The function $A(r,\alpha)$ is continuous and bounded in the variable $r \in [1,\infty)$. Then the limit set, i.e., the set of all subsequential limits, of $A(r,\alpha)$ as $r \to +\infty$ is a closed interval $J(\alpha) = [A(\alpha), B(\alpha)]$. We claim that

$$J(\alpha) = \left\{ \int_1^{1+\alpha} u(te^{i\theta})dt :\ u \in \mathrm{Fr}(v) \right\}. \qquad (4)$$

In fact, let

$$a(\alpha) = \lim_{n\to\infty} A(r_n, \alpha) = \lim_{n\to\infty} \int_1^{1+\alpha} v_{r_n}(te^{i\theta})dt.$$

In addition, we may assume that $v_{r_n}(z) \to u(z)$ in the sense of distributions. Then the equality

$$\lim_{n\to\infty} \int_1^{1+\alpha} v_{r_n}(te^{i\theta})dt = \int_1^{1+\alpha} u(te^{i\theta})dt,$$

which is a special case of (2), gives $a(\alpha) = \int_1^{1+\alpha} u(te^{i\theta})dt$. Clearly, for any $u \in \mathrm{Fr}(v)$ the value of integral $\int_1^{1+\alpha} u(te^{i\theta})dt$ belongs to the interval $J(\alpha)$. Relation (4) is proved. Note that it also follows from the results obtained by V.S. Azarin [2].

According to Theorem 2 [8],

$$\lim_{\alpha\to+0} \frac{A(\alpha)}{\alpha} = \underline{h}(\theta),\ \lim_{\alpha\to+0} \frac{B(\alpha)}{\alpha} = h(\theta).$$

If $\underline{h}(\theta) < h < h(\theta)$, then the inequalities

$$\frac{A(\alpha)}{\alpha} < h\frac{(1+\alpha)^{\rho+1}-1}{(\rho+1)\alpha} < \frac{B(\alpha)}{\alpha}$$

are valid for all sufficiently small α. Therefore there exists a strictly positive number α and a function $u \in \mathrm{Fr}(v)$ such that

$$h\frac{(1+\alpha)^{\rho+1}-1}{\rho+1} = \int\limits_1^{1+\alpha} u(te^{i\theta})dt,$$

$$\int\limits_1^{1+\alpha} \Big(u(te^{i\theta}) - ht^\rho\Big)\, dt = 0.$$

We claim that there exists $t_0 \in [1, 1+\alpha]$ with $u(t_0 e^{i\theta}) = ht_0^\rho$. Consider the function $w(z) = u(z) - h|z|^\rho$. Further we will assume that θ is fixed and consider $w(te^{i\theta})$ as a function in variable t. We have either $w(te^{i\theta}) = 0$ almost everywhere on the interval $[1, 1+\alpha]$ or the function $w(te^{i\theta})$ has strictly positive and strictly negative values on this interval. In the first case t_0 is obtained. We consider the second case. The function $w(te^{i\theta})$ is upper semicontinuous. Hence the set $E = \big\{t > 0 : \ w(te^{i\theta}) < 0\big\}$ is open. Under the assumption E is nonempty, the set E meets the interval $[1, 1+\alpha]$. We have

$$E = \bigcup_{k=1}^{\infty} (a_k, b_k),$$

where (a_k, b_k) is a disjoint system of intervals. There exists k such that $(a_k, b_k) \cap [1, 1+\alpha] \neq \varnothing$. The point a_k or the point b_k necessarily belongs to the interval $(1, 1+\alpha)$, assume that $b_k \in (1, 1+\alpha)$. Because $b_k \overline{\in} E$, the inequality $w(b_k e^{i\theta}) \geq 0$ is valid. The function $w(z)$ is continuous in fine topology. Hence there exists a fine neighborhood G of $b_k e^{i\theta}$ such that

$$w(b_k e^{i\theta}) = \lim_{\substack{z \to b_k e^{i\theta} \\ z \in G}} w(z).$$

According to the theorem of Lebesgue and Beurling ([4, Prop. IX.6]), the point $b_k e^{i\theta}$ is a limit point for the set $G \cap (a_k e^{i\theta}, b_k e^{i\theta})$. This gives $w(b_k e^{i\theta}) \leq 0$ and then $w(b_k e^{i\theta}) = 0$. Thus b_k is the required point t_0. For the function

$$u^{(1)}(z) = \frac{u(t_0 z)}{t_0^\rho} \in \mathrm{Fr}(v)$$

we have $u^{(1)}(e^{i\theta}) = h$.

Now the assertions of the theorem follow from the above. The theorem is proved.

We produce a subharmonic function v such that $\underline{h}(0) = \underline{H}(1)\overline{\in}I(1)$ and construct the limit set of Azarin of this function in the form

$$\mathrm{Fr}(v) = \{\overline{u_t(z) : \ t \in (0,\infty)}\}. \tag{5}$$

Consider the function

$$a(z) = \sum_{n=1}^{\infty} \frac{1}{n^3} \ln \left| 1 - \frac{z}{1-e^{-n}} \right|.$$

We have

$$a(z) = \left(\sum_{n=1}^{\infty} \frac{1}{n^3} \right) \ln|z| + \sum_{n=1}^{\infty} \frac{1}{n^3} \ln \frac{1}{1-e^{-n}} + O\left(\frac{1}{|z|}\right), \ z \to \infty.$$

On every interval

$$\left(-\infty, 1-e^{-1}\right), \ \left(1-e^{-k}, 1-e^{-k-1}\right), \ k = 1,2,\ldots, (1,\infty),$$

the function $a(x)$ is strictly concave since

$$a''(x) = -\sum_{n=1}^{\infty} \frac{1}{n^3} \frac{1}{(x-1+e^{-n})^2}.$$

Let $x_n \in \left(1-e^{-n}, 1-e^{-n-1}\right)$ be such a point that

$$a(x_n) = \max\left\{a(x) : \ x \in \left(1-e^{-n}, 1-e^{-n-1}\right)\right\}.$$

Then the function $a(x)$ increases on the interval $(1-e^{-n}, x_n)$ and decreases on the interval $\left(x_n, 1-e^{-n-1}\right)$. First we prove the relation

$$a(x_n) \to a(1) \tag{6}$$

as $n \to \infty$. Let $\xi_k = 1 - \frac{1}{2}\left(1+\frac{1}{e}\right)e^{-k}$. We have

$$a(1) - a(\xi_k) = -\sum_{n=1}^{\infty} \frac{1}{n^3} \ln \left| 1 - \frac{1}{2}\left(1+\frac{1}{e}\right) e^{n-k} \right|.$$

From the inequalities

$$\left| \sum_{1 \le n \le \frac{k}{2}} \frac{1}{n^3} \ln\left(1 - \frac{1}{2}\left(1+\frac{1}{e}\right)e^{n-k}\right) \right| \le (1+\frac{1}{e})e^{-k/2} \sum_{n=1}^{\infty} \frac{1}{n^3},$$

$$\left| \sum_{\frac{k}{2}<n\le k} \frac{1}{n^3} \ln\left(1-\frac{1}{2}\left(1+\frac{1}{e}\right)e^{n-k}\right)\right| \le \sum_{\frac{k}{2}<n\le k} \frac{1}{n^3}\ln\frac{2}{1-\frac{1}{e}},$$

$$\left|\sum_{n=k+1}^{\infty} \frac{1}{n^3}\ln\left|1-\frac{1}{2}\left(1+\frac{1}{e}\right)e^{n-k}\right|\right| \le \sum_{n=k+1}^{\infty}\frac{1}{n^2}$$

it follows that $a(\xi_k) \to a(1)$, as $k \to \infty$. Since $\xi_k \in \left(1-e^{-k}, 1-e^{-k-1}\right)$, then $a(x_k) \ge a(\xi_k)$,

$$\underline{\lim}_{k\to\infty} a(x_k) \ge a(1).$$

The upper semicontinuity of the function $a(z)$ yields

$$\overline{\lim}_{k\to\infty} a(x_k) \le \overline{\lim}_{z\to 1} a(z) \le a(1),$$

and (6) follows.

Introduce ρ and ρ_1 with $0<\rho<\rho_1<1$. It follows from (6) that

$$-\alpha = \inf \frac{a(x_n)}{x_n^{\rho_1}} > -\infty.$$

Let β_0 be the number in the interval $\left(0, 1-\frac{1}{e}\right)$ such that $a(\beta_0) = -\alpha\beta_0^{\rho_1}$.

The existence of β_0 follows from the inequality $a_1(t) = a(t)+\alpha t^{\rho_1} > 0$ in the right neighborhood of zero and the inequality $a_1(t) < 0$ in the left neighborhood of $1-\frac{1}{e}$. The function $a_1(t)$ is strictly concave on the interval $\left[0, (1-\frac{1}{e})\right)$, and $a_1(0)=0$. Any strictly concave function has at most two zeros. This proves that β_0 is unique. We choose $c > 2\alpha$ and denote

$$A_1 = (\beta_0, 1-\frac{1}{e}),\ A_k = (x_{k-1}, 1-e^{-k}),\ k=2,3,\dots .$$

Let s_k be the unique point $t \in A_k$ such that $a(t) = -ct^{\rho_1}$. We denote $a_2(t) = a(t) + ct^{\rho_1}$ and consider the case $k \ge 2$.

We have $a_2(x_{k-1}) > 0$, $a_2(t) < 0$ in the left neighborhood of $1-e^{-k}$. This shows that s_k exists. Analogously, there exists $s_{1k} \in \left(1-e^{-k+1}, x_{k-1}\right)$ such that $a_2(s_{1k}) = 0$. The function $a_2(t)$ is strictly concave in each interval $\left(1-e^{-k+1}, 1-e^{-k}\right)$. This proves that s_k is unique. The existence and uniqueness of s_1 is proved in the same way as for β_0.

We have $a(z) = \mathrm{Re}\lambda(z)$,

$$\lambda(z) = \sum_{n=1}^{\infty}\frac{1}{n^3}\ln\left(1-\frac{z}{1-e^{-n}}\right),$$

$$\lambda'(z) = \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{1}{z-1+e^{-n}}.$$

A level-line of $a(z)$ is a real-analytic curve if it does not meet zeros and poles of $\lambda'(z)$. We wish to find zeros of $\lambda'(z)$. The following identity holds

$$\mathrm{Im}\lambda'(z) = -y \sum_{n=1}^{\infty} \frac{1}{n^3 |z-1+e^{-n}|^2}. \tag{7}$$

It gives that all zeros of $\lambda'(z)$ are real. Now it is easy to verify that the set of zeros of $\lambda'(z)$ is $\{x_n,\ n = 1, 2, \dots\}$.

Let σ_k be the unique point in the interval $\left(1 - e^{-k}, x_k\right)$ such that $a(\sigma_k) = a(s_k)$. The inequality $a(x) < a(s_k)$ is realized on the interval (s_k, σ_k). From the identity $\frac{\partial a}{\partial y}(z) = -\mathrm{Im}\lambda'(z)$ and (7) it follows that the function $a(x, y)$ strictly increases on $(0, \infty)$ and strictly decreases on $(-\infty, 0)$ in the variable y. Therefore there exist functions $y_1(x) > 0$ and $y_2(x) < 0$ on the interval (s_k, σ_k) such that

$$a(x, y_1(x)) = a(s_k),\ a(x, y_2(x)) = a(s_k).$$

The collection of curves $z = x + iy_1(x),\ z = x + iy_2(x),\ x \in (s_k, \sigma_k)$, and points $z = s_k,\ z = \sigma_k$ is a closed Jordan curve L_k that is a level curve of $a(z)$. It is a real analytic curve. Let G_k be a bounded domain with boundary L_k.

Let $u(z)$ be a function such that $u(z) = a(z)$ if $z \overline{\in} \bigcup_{k=1}^{\infty} G_k$, and $u(z) = a(s_k)$ if $z \in G_k$. The function $u(z)$ is subharmonic. It is important for us that the inequalities

$$u(x) \geq -cx^{\rho_1},\ u(x) > -cx^{\rho} \tag{8}$$

are realized on the semi-axis $(0, \infty)$.

Consider the Azarin trajectory of function u,

$$u_t(z) = \frac{u(tz)}{t^\rho},\ t \in (0, \infty).$$

One can prove that $u_t(z) \to 0$ in the sense of distributions when $t \to 0$ or $t \to \infty$.

Theorem 9 [3] asserts that there exists a subharmonic function v of order ρ such that

$$\mathrm{Fr}(v) = \{u_t(z) :\ t \in (0, \infty)\} \cup \{0\}.$$

It follows from (8) that

$$u_t(1) = \frac{u(t)}{t^\rho} > -c.$$

In addition,

$$u_{s_k}(1) = \frac{u(s_k)}{s_k^{\rho}} = -c s_k^{\rho_1 - \rho} \to -c$$

as $k \to \infty$.

This gives $\underline{H}(1) = -c$, $\underline{H}(1) \overline{\in} I(1)$. The function v is a required example.

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