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Partially Observed Discrete-valued Time Series

(Recommended by Prof. E. Dshalalow)

The analysis of time series of counts is a rapidly developing area. It has very broad application in view of the host of integer-valued time series which cannot be satisfactorily handled within the classical framework of Gaussian- like series. In this paper we derive recursive filters for partially observed discrete-valued time series. These processes are regulated by thinning binomial and multinomial operators (to be defined below).

Анализ временных последовательностей отсчетов — интенсивно развивающееся направление. Такой анализ широко используется для базовых целочисленных временных последовательностей, с которыми нельзя удовлетворительно работать в рамках классических последовательностей гауссова типа. Получены рекурсивные фильтры для частично наблюдаемых дискретизированных временных последовательностей. Показано, что эти процессы регулируются прореживающими биномиальными и полиномиальными операторами.

K e y w o r d s: filtering, time series, change of measre, binomial thinning.

1. Introduction. The analysis of time series of counts is a rapidly developing area [1–6] and the book by MacDonald [7]. It has very broad application in view of the host of integer-valued time series which cannot be satisfactorily handled within the classical framework of Gaussianlike series. Many of the statistical which occur in practice are by their very nature discrete-valued (see [7] for more details). These models are also adequate for the study of branching processes with immigration [8].

In this paper we derive recursive filters for partially observed discretevalued time series. The dynamics of these processes are regulated by thinning binomial and multinomial operators.

The Binomial thining operator «$\circ$» [2, 5] is defined as follows. For any nonnegative integer-valued random variable X and $\alpha \in \{0, 1\}$,

$$a \circ X = \sum_{j=1}^{X} Y_j , \tag{1}$$

where $Y_1, Y_2, \ldots$ is a sequence of of i.i.d. random variables independent of X, such that $P(Y_j = 1) = 1 - P(Y_j = 0) = \alpha$.

2. Scalar dynamics. Consider a system whose state at time k is $x_k \in Z_+$. The time index k of the state evolution will be discrete and identified with $\mathbb{N} = \{0, 1, 2, \ldots, \}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space upon which $\{v_k\}$, $\{w_k\}$, $k \in \mathbb{N}$ are independent and identically distributed (i.i.d.) sequences of random variables such that, for all k, $v_k \in \mathbb{Z}_+$ has probability function φ and w_k is Gaussian random variables, having zero means and variances 1 ($N(0, 1)$). Let $\{\mathcal{F}_k\}$, $k \in \mathbb{N}$ be the complete filtration (that is $\mathcal{F}_0$ contains all the P-null events) generated by $\{x_0, x_1, \ldots, x_k\}$. The state of the system satisfies the dynamics

$$x_{k+1} = \alpha(X_k) \circ x_k + v_{k+1}. \tag{2}$$

Here $\{X_k\}_{k\in\mathbb{N}}$ is a stochastic process with finite state space S_X of size N which we identify, without loss of generality, with the canonical basis $\{e_1, \ldots, e_N\}$ of $\mathbb{R}^N$. Since X_n takes only a finite number of values we may write

$$\alpha(X_k) = (\alpha(e_1), \ldots, \alpha(e_N)) = (\alpha_1, \ldots, \alpha_N)) \triangleq \boldsymbol{\alpha}.$$

Therefore $\alpha(X_k) = \langle \boldsymbol{\alpha}, X_k \rangle$. Here $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^N$. Let's assume the process X is a Markov chain with semimartingale representation [9, 10].

$$X_k = AX_{k-1} + M_k \tag{3}$$

where $\{M_k\}_{k\in\mathbb{N}}$ is a sequence of martingale increments with respect to the complete filtration generated by X and A denotes the probability transition matrix of the Markov chain X.

A useful and simple model for a noisy observation of x_k is to suppose it is given as a linear function of x_k plus a random «noise» term. That is, we suppose that for some real numbers c_k and positive real numbers d_k our observations have the form

$$y_k = c_k x_k + d_k w_k. \tag{4}$$

We shall also write $\{\mathcal{Y}_k\}$, $k \in \mathbb{N}$ for the complete filtration generated by $\{y_0, y_1, \ldots, y_k\}$.

Using measure change techniques we shall derive a recursive expression for the conditional distribution of x_k given $\mathcal{Y}_k$.

Recursive estimation. Initially we suppose all processes are defined on an «ideal» probability space $(\Omega, \mathcal{F}, \overline{P})$; then under a new probability measure P, to be defined, the model dynamics (2) and (4) will hold.

Suppose that under $\overline{P}$:

1) $\{x_k\}$, $k \in \mathbb{N}$ is an i.i.d. sequence with density function $\phi(x)$ with support in $\mathbb{Z}_+$;

2) $\{y_k\}$, $k \in \mathbb{N}$ is an i.i.d. $N(0, 1)$ sequence with density function

$$\psi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

For $l = 0$, $\bar{\lambda}_0 = \dfrac{\psi(d_0^{-1}(y_0 - c_0 x_0))}{d_0 \psi(y_0)}$ and for $l = 1, 2, \ldots$ define

$$\bar{\lambda}_l = \frac{\phi(x_l - \langle \boldsymbol{\alpha},\ X_{l-1}\rangle \circ x_{l-1})\,\psi(d_l^{-1}(y_l - c_l x_l))}{d_l \phi(x_l)\,\psi(y_l)}, \tag{5}$$

$$\bar{\Lambda}_k = \prod_{l=0}^{k} \bar{\lambda}_l. \tag{6}$$

Let $\mathcal{G}_k$ be the complete σ-field generated by $\{x_0,\ x_1, \ldots, x_k, \langle \boldsymbol{\alpha},\ X_0\rangle \circ x_0, \ldots$ $\ldots, \langle \boldsymbol{\alpha},\ X_k\rangle \circ x_k, y_0,\ y_1, \ldots,\ y_k\}$ for $k \in \mathbb{N}$.

Lemma 1. The process $\{\bar{\Lambda}_k\}$, $k \in \mathbb{N}$ is a $\bar{P}$-martingale with respect to the filtration $\{\mathcal{G}_k\}$, $k \in \mathbb{N}$.

P r o o f . Since $\bar{\Lambda}_k$ is $\mathcal{G}_k$-measurable $\bar{E}[\bar{\Lambda}_{k+1} | \mathcal{G}_k] = \bar{\Lambda}_k \bar{E}[\bar{\Lambda}_{k+1} | \mathcal{G}_k]$. Therefore we must show that $\bar{E}[\bar{\Lambda}_{k+1} | \mathcal{G}_k] = 1$:

$$\bar{E}[\bar{\lambda}_{k+1} | \mathcal{G}_k] = \bar{E}\left[\frac{\phi(x_{k+1} - \langle \boldsymbol{\alpha},\ X_k\rangle \circ x_k)\,\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d_{k+1}\phi(x_{k+1})\,\psi(y_{k+1})} | \mathcal{G}_k\right] =$$

$$= \bar{E}\left[\frac{\phi(x_{k+1} - \langle \boldsymbol{\alpha},\ X_k\rangle \circ x_k)}{\phi(x_{k+1})}\,\bar{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d_{k+1}\psi(y_{k+1})} | \mathcal{G}_k, x_{k+1}\right] | \mathcal{G}_k\right].$$

Now,

$$\bar{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d_{k+1}\psi(y_{k+1})} | \mathcal{G}_k, x_{k+1}\right] =$$

$$= \int_{\mathbb{R}} \frac{\psi(d_{k+1}^{-1}(y - c_{k+1} x_{k+1}))}{d_{k+1}\psi(y)}\,\psi(y)\,dy = 1$$

and

$$\bar{E}\left[\frac{\phi(x_{k+1} - \langle \boldsymbol{\alpha},\ X_k\rangle \circ x_k)}{\phi(x_{k+1})} | \mathcal{G}_k\right] =$$

$$= \bar{E}\left[\sum_{x \in \mathbb{Z}_+} \frac{\phi(x - \langle \boldsymbol{\alpha},\ X_k\rangle \circ x_k)}{\phi(x)}\,\phi(x) | \mathcal{G}_k\right] = \sum_{u \in \mathbb{Z}_+} \phi(u) = 1.$$

Define P on $\{\Omega, \mathcal{F}\}$ by setting the restriction of the Radon—Nykodim derivative $\frac{dP}{d\overline{P}}$ to $\mathcal{G}_k$ equal to $\overline{\lambda}_k$. Then:

Lemma 2. $\{v_k\}$, $k \in \mathbb{N}$ is an i.i.d. sequence with density function $\phi(x)$ with support in $\mathbb{Z}_+$ and $\{w_k\}$, $k \in \mathbb{N}$ are i.i.d. $N(0, 1)$ sequences of random variables, where

$$v_{k+1} \overset{\Delta}{=} (x_{k+1} - \langle \boldsymbol{\alpha}, X_k \rangle \circ x_k),$$

$$w_k \overset{\Delta}{=} (d_k^{-1}(y_k - c_k x_k).$$

P r o o f. Suppose f, g: $\mathbb{R} \to \mathbb{R}$ are «test» functions (i.e. measurable functions with compact support). Then with E (resp. $\overline{E}$) denoting expectation under P (resp. $\overline{P}$) and using Bayes' Theorem [9, 10]

$$E[f(v_{k+1})\, g(w_{k+1}) | \mathcal{G}_k] = \frac{\overline{\Lambda}_k \overline{E}\,[\overline{\lambda}_{k+1} f(v_{k+1})\, g(w_{k+1}) | \mathcal{G}_k]}{\overline{\Lambda}_k \overline{E}\,[\overline{\lambda}_{k+1} | \mathcal{G}_k]} =$$

$$= \overline{E}\,[\overline{\lambda}_{k+1} f(v_{k+1})\, g(w_{k+1}) | \mathcal{G}_k],$$

where the last equality follows from Lemma 1. Consequently

$$E[f(v_{k+1})\, g(w_{k+1}) | \mathcal{G}_k] = \overline{E}\,[\overline{\lambda}_{k+1} f(v_{k+1})\, g(w_{k+1}) | \mathcal{G}_k] =$$

$$= \overline{E}\left[\frac{\phi(x_{k+1} - \langle \boldsymbol{\alpha}, X_k \rangle \circ x_k)\, \psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d\phi(x_{k+1})\, \psi(y_{k+1})} \right] \times$$

$$\times f(x_{k+1} - \langle \boldsymbol{\alpha}, X_k \rangle \circ x_k)\, g(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1})) | \mathcal{G}_k] =$$

$$= \overline{E}\left[\frac{\phi(x_{k+1} - \langle \boldsymbol{\alpha}, X_k \rangle \circ x_k)}{\phi(x_{k+1})} f(x_{k+1} - \langle \boldsymbol{\alpha}, X_k \rangle \circ x_k) \times \right.$$

$$\left. \times \overline{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d_{k+1} \psi(y_{k+1})} g(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1})) | \mathcal{G}_k, x_{k+1} \right] | \mathcal{G}_k \right].$$

Now

$$\overline{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1}))}{d_{k+1} \psi(y_{k+1})} g(d_{k+1}^{-1}(y_{k+1} - c_{k+1} x_{k+1})) | \mathcal{G}_k, x_{k+1} \right] =$$

$$= \int_{\mathbb{R}} \frac{\psi(d_{k+1}^{-1}(y - c x_{k+1}))}{d_{k+1} \psi(y)} \psi(y)\, g(d_{k+1}^{-1}(y - c_{k+1} x_{k+1}))\, dy = \int_{\mathbb{R}} \psi(u)\, g(u)\, du$$

and

$$\overline{E}\left[\frac{\phi(x_{k+1}-\langle \boldsymbol{\alpha},\ X_k\rangle\circ x_k)}{\phi(x_{k+1})} f(x_{k+1}-\langle \boldsymbol{\alpha},\ X_k\rangle\circ x_k) \mid \mathcal{G}_k\right]=$$

$$=\overline{E}\left[\sum_{x\in\mathbb{Z}_+}\frac{\phi(x-\langle \boldsymbol{\alpha},\ X_k\rangle\circ x_k)}{\phi(x)}\phi(x)f(x-\langle \boldsymbol{\alpha},\ X_k\rangle\circ x_k)|\mathcal{G}_k\right]=\sum_{x\in\mathbb{Z}_+}\phi(z)f(z).$$

Therefore $E[f(v_{k+1})\,g(w_{k+1})|\mathcal{G}_k]=\sum_{x\in\mathbb{Z}_+}\phi(z)f(z)\int_{\mathbb{R}}\psi(u)g(u)du$ and the lemma is proved.

Using Bayes' Theorem [10]

$$E[I(x_k=x)X_k|\mathcal{Y}_k]=\frac{\overline{E}[\overline{\Lambda}_k I(x_k=x)\,X_k|\mathcal{Y}_k]}{\overline{E}[\overline{\Lambda}_k|\mathcal{Y}_k]}, \tag{7}$$

where $\overline{E}$ (resp. E) denotes expectations with respect to $\overline{P}$ (resp. P). Consider the unnormalized, conditional expectation which is the numerator of (7) and write

$$\overline{E}[\overline{\Lambda}_k I(x_k=x)X_k|\mathcal{Y}_k]=q_k(x)=(q_k^1(x),\ ...,q_k^N(x))'. \tag{8}$$

If $p_k(.)$ denotes the normalized conditional density, such that $E[I(x_k=$ $=x)\,X_k|\mathcal{Y}_k]=p_k(x)$, then from (7) we see that

$$p_k(x)=q_k(x)\left[\sum_z q_k(z)\right]^{-1}\ \text{for}\ x\in\mathbb{Z}_+,\ k\in\mathbb{N}.$$

Then we have the following result.

Theorem 1. The measure-valued process q satisfies the recursion

$$q_{k+1}(x)=A\sum_{z\in\mathbb{Z}_+}\mathbf{B}(z,x)q_k(z),$$

where $\mathbf{B}(z,x)$ is a diagonal matrix with entries

$$\frac{\psi(d^{-1}(y_{k+1}-cx))}{d\psi(y_{k+1})}\sum_{r=0}^{z}\phi(x-r)\binom{z}{r}\alpha_i^r(1-\alpha_i)^{z-r}.$$

P r o o f. In view of (3), (5) and (6)

$$\overline{E}[\overline{\Lambda}_k\overline{\lambda}_{k+1}I(x_{k+1}=x)\,X_{k+1}|\mathcal{Y}_{k+1}]=$$

$$=\overline{E}\left[\overline{\Lambda}_k\frac{\phi(x-\langle \boldsymbol{\alpha},\ X_k\rangle\circ x_k)\psi(d_{k+1}^{-1}(y_{k+1}-c_{k+1}x))}{d_{k+1}\phi(x)\psi(y_{k+1})}\phi(x)(\mathrm{A}X_k+M_{k+1}|\mathcal{Y}_{k+1}]\right]=$$

$$=\frac{\psi(d_{k+1}^{-1}(y_{k+1}-c_{k+1}x))}{d_{k+1}\psi(y_{k+1})}\sum_{i=1}^{N}\overline{E}\,\overline{\Lambda}_k\phi(x-\alpha_i\circ x_k)\langle X_k,e_i\rangle|\mathcal{Y}_{k+1}]Ae_i=$$

$$=\frac{\psi(d_{k+1}^{-1}(y_{k+1}-c_{k+1}x))}{d_{k+1}\psi(y_{k+1})}\times$$

$$\times\sum_{i=1}^{N}\overline{E}\left[\overline{\Lambda}_k\sum_{r=0}^{x_k}\phi(x-r)\binom{x_k}{r}\alpha_i^r(1-\alpha_i)^{x_k-r}\langle X_k,e_i\rangle|\mathcal{Y}_{k+1}]\right]Ae_i=$$

$$=\frac{\psi(d_{k+1}^{-1}(y_{k+1}-c_{k+1}x))}{d_{k+1}\psi(y_{k+1})}\times$$

$$\times\sum_{i=1}^{N}\overline{E}\left[\overline{\Lambda}_k\sum_{z\in\mathbb{Z}_+}\sum_{r=0}^{z}\phi(x-r)\binom{z}{r}\alpha_i^r(1-\alpha_i)^{z-r}I(x_k=z)\langle X_k,e_i\rangle|\mathcal{Y}_k]\right]Ae_i.$$

The last equality follows from the fact that x_{k+1} has distribution ϕ and is independent of everything else under $\overline{P}$. Also, note that given y_{k+1} we condition only on $\mathcal{Y}_k$ to get an expression similar to notation (8), that is,

$$\overline{E}[\overline{\Lambda}_{k+1}I(x_{k+1}=x)X_{k+1}|\mathcal{Y}_{k+1}]=$$

$$=\frac{\psi(d^{-1}(y_{k+1}-cx))}{d\psi(y_{k+1})}\sum_{i=1}^{N}\left\langle\sum_{z\in\mathbb{Z}_+}\sum_{r=0}^{z}\phi(x-r)\binom{z}{r}\alpha_i^r(1-\alpha_i)^{z-r}q_k(z)\,e_i\right\rangle Ae_i=$$

$$=A\sum_{z\in\mathbb{Z}_+}\mathbf{B}(z,x)\,q_k(z),$$

where $\mathbf{B}(z,x)$ is a diagonal matrix with entries

$$\frac{\psi(d^{-1}(y_{k+1}-cx))}{d\psi(y_{k+1})}\sum_{r=0}^{z}\phi(x-r)\binom{z}{r}\alpha_i^r(1-\alpha_i)^{z-r}.$$

Which finishes the proof.

Vector dynamics. Consider a system whose state at time $k=0, 1, 2, \ldots$, is $X_k\in\mathbb{Z}_+^m$ and which can be observed only indirectly through another process $Y_k\in\mathbb{R}^d$.

Let $(\Omega,\mathcal{F},P)$ be a probability space upon which V_k and W_k are sequences of random variables such that W_k is normally distributed with means 0 and covariance identity matrices $I_{d\times d}$ and V_k has probability distribution ϕ with support in $\mathbb{Z}_+^m$. Assume that D_k, $k\ge 0$, are non singular matrices. Let $\{\mathcal{F}_k\}$, $k\in\mathbb{N}$, be the complete filtration generated by $\{X_0, X_1, \ldots, X_k\}$.

Now we wish to generalize the operator $\circ$ to vector-valued random variables with non-negative integer-valued components.

For any vector $X = (X^1, ..., X^m)'$ in $\mathbb{Z}_+^m$ and any vector $\boldsymbol{\alpha}^i = (\alpha_1^i, ..., \alpha_m^i)'$ such that $\alpha_j^i > 0$ and $\sum_i \alpha_j^i = 1$ define

$$\boldsymbol{\alpha}^i \varnothing X^i = (\alpha_1^i \varnothing X^i, ..., \alpha_m^i \varnothing X^i)' = \left(\sum_{j=1}^{Z_1^i} Y_{1j}^i, ..., \sum_{j=1}^{Z_m^i} Y_{mj}^i \right)', \tag{9}$$

where Z_ℓ^i, $i, \ell = 1, ..., m$, are non-negative, integer-valued random variables such that $\sum_{\ell=1}^m Z_\ell^i = X^i$. For each i, ℓ, $Y_{\ell 1}^i, ..., Y_{\ell m}^i$, are i.i.d. nonnegative, integer-valued random variables with probability function ρ_ℓ^i.

Let

$$A = (\boldsymbol{\alpha}^1, ..., \boldsymbol{\alpha}^m), \quad A \varnothing X = \sum_{i=1}^m \boldsymbol{\alpha}^i \varnothing X^i. \tag{10}$$

One possible interpretation of this model is that $X = (X^1, ..., X^m)'$ represents a population composed of m distinct groups of, say, cells. Some time later, each cell in the population, regardless to which group it belongs, can mutate and divide itself into a number of new cells of any of the m types. For instance, a cell of type 1 may mutate with probability α_2^1 to produce through division a new generation of cells of type 2. Let $\alpha_2^1 \varnothing X^1 = \sum_{j=1}^{Z_2^1} Y_{2j}^1$ is the (random) number of new cells of type 2 with Z_2^1 parents of type1. In other words, for $j = 1, ..., Z_2^1$, the j-th parent cell of type 1 gave birth to Y_{2j}^1 new cells of type 2. Here Y_{2j}^1 is a random variable with probability function ρ_2^1 with support in $\mathbb{Z}_+$.

The state and observations of the system are given by the dynamics

$$X_{k+1} = A_k \varnothing X_k + V_{k+1} \in \mathbb{Z}_+^m, \tag{11}$$

$$Y_k = C_k X_k + D_k W_k \in \mathbb{R}^d. \tag{12}$$

Here C_k is a matrix of appropriate dimensions and $A_k \varnothing X_k$ is defined in (10).

We write again $\{\mathcal{Y}_k\}$, $k \in \mathbb{N}$, for the complete filtration generated by the observed data $\{Y_0, Y_1, ..., Y_k\}$ up to time k. Using measure change techniques we shall derive a recursive expression for the conditional distribution of X_k given $\mathcal{Y}_k$.

Recursive estimation. Initially we suppose all processes are defined on an «ideal» probability space $(\Omega, \mathcal{F}, \overline{P})$; then under a new probability measure P, to be defined, the model dynamics (11) and (12) will hold.

Suppose that under $\overline{P}$:

1) $\{X_k\}$, $k \in \mathbb{N}$, is an i.i.d. sequence with probability function $\phi(x)$ defined on $\mathbb{Z}_+^m$;

2) $\{Y_k\}$, $k \in \mathbb{N}$, is an i.i.d. $N(0, I_{d\times d})$ sequence with density function $\psi(y) = \frac{1}{(2\pi)^{d/2}} e^{-y'y/2}$.

For any square matrix B write $|B|$ for the absolute value of its determinant.

For $l = 0$, $\overline{\lambda}_0 = \frac{\psi(D_0^{-1}(Y_0 - C_0 X_0)}{|D_0|\psi(Y_0)}$ and for $l = 1, 2, \ldots$ define

$$\overline{\lambda}_l = \frac{\phi(X_l - A_{l-1} \varnothing X_{l-1})\psi(D_l^{-1}(Y_l - C_l X_l)}{|D_l|\phi(X_l)\psi(Y_l)},$$

$$\overline{\Lambda}_k = \prod_{l=0}^{k} \overline{\lambda}_l.$$

Let $\{\mathcal{G}_k\}$ be the complete σ-field generated by $\{X_0, X_1, \ldots, X_k, Y_0, Y_1, \ldots, Y_k\}$ for $k \in \mathbb{N}$.

The process $\{\overline{\Lambda}_k\}, k \in \mathbb{N}$, is an $\overline{P}$-martingale with respect to the filtration $\{\mathcal{G}_k\}$.

Define P on $\{\Omega, \mathcal{F}\}$ by setting the restriction of the Radon—Nykodim derivative $\frac{dP}{d\overline{P}}$ to $\mathcal{G}_k$ equal to $\overline{\Lambda}_k$. It can be shown that on $\{\Omega, \mathcal{F}\}$ and under P, W_k is normally distributed with means 0 and covariance identity matrix $I_{d\times d}$, and V_k has probability function ϕ defined on $\mathbb{Z}_+^m$ where

$$V_{k+1} \triangleq X_{k+1} - A_k \varnothing X_k,\ W_k \triangleq D_k^{-1}(Y_k - C_k X_k),$$

write

$$\overline{E}[\overline{\Lambda}_k I(x_k = x) X_k | \mathcal{Y}_k] = q_n(x).$$

Then we have the following result.

Theorem 2. For $k \geq 0$

$$q_{k+1}(x) = \frac{\psi(D_{k+1}^{-1}(Y_{k+1} - C_{k+1}x)}{|D_{k+1}|\psi(Y_{k+1})} \times$$

$$\times \sum_{u \in \mathbb{Z}_+^m} \sum_{i=1}^{m} \sum_{z_1^i + \ldots + z_m^i = u^i} \prod_{i=1}^{m} \binom{x_k^i}{z_1^i \ldots z_m^i} (\alpha_1^i)^{z_1^i} \ldots (\alpha_m^i)^{z_m^i} \times$$

$$\times \phi\left(x - \sum_{i=1}^{m}\left(\sum_{j=1}^{z_1^i} y_{1j}^i, \ldots, \sum_{j=1}^{z_m^i} y_{mj}^i\right)'\right) \prod_{i,\ell=1}^{m}\prod_{j=1}^{z_\ell^i} \rho_\ell^i(y_{\ell j}^i)\, q_k(u).$$

P r o o f. The proof is similar to the scalar case and is skipped.

A sampling observation model. The state of the system is again given by the dynamics in (11). Write $N_k = \sum_{i=1}^{m} X_k^i$ and $\Pi(N_k)$ for the set of all partitions of N_k into m summands; that is, $x \in \Pi(N_k)$ if $x = (x^1, x^2, \ldots, x^m)$ where each x^i is a non-negative integer and $x^1 + x^2 + \ldots + x^m = N_k$. In this section we assume that the total number of individual N_k is approximately known but it is practically very difficult to measure directly their distribution between the *m* types. Therefore the population is sampled by withdrawing, (with replacement), at each time *k*, *n* individuals and observing to which type they belong. That is, at each time *k* a sample

$$Y_k = (Y_k^1, Y_k^2, \ldots, Y_k^m) = \Pi(n)$$

is obtained, where $\Pi(n)$ is the set of partitions of *n*.

We assume that

$$P(Y_k = y \mid X_k = x) = \binom{n}{y^1\ y^2 \ldots y^m}\left(\frac{x^1}{N_k}\right)^{y^1}\left(\frac{x^2}{N_k}\right)^{y^2}\ldots\left(\frac{x^m}{N_k}\right)^{y^m}. \quad (13)$$

Clearly this sequence of samples, $Y(0)$, $Y(1)$, $Y(2)$, ... enables us to revise our estimates of the state X_k.

Recursive estimates. Initially we suppose all processes are defined on an «ideal» probability space $(\Omega, \mathcal{F}, \overline{P})$; then under a new probability measure *P*, to be defined, the model dynamics (11) and (13) will hold.

Suppose that under $\overline{P}$:

1) $\{X_k\}$, $k \in \mathbb{N}$, is an i.i.d. sequence with probability function $\boldsymbol{\xi}(x)$ defined on $\mathbb{Z}_+^m$;

2) $\{Y_k\}$, $k \in \mathbb{N}$, is an i.i.d. sequence such that for $y \in \Pi(n)$,

$$\overline{P}(Y_k = y \mid \mathcal{G}_k) = \binom{n}{y^1\ y^2 \ldots y^m}\left(\frac{1}{m}\right)^n.$$

For $l = 0$, $\overline{\lambda}_0 = 1$ and for $l = 1, 2, \ldots$ define

$$\overline{\lambda}_l = \frac{\boldsymbol{\xi}(X_l - A_{l-1} \varnothing X_{l-1})}{\boldsymbol{\xi}(X_l)} m^n \left(\frac{X_k^1}{N_k}\right)^{Y_k^1}\left(\frac{X_k^2}{N_k}\right)^{Y_k^2}\ldots\left(\frac{X_k^m}{N_k}\right)^{Y_k^m}, \quad \overline{\Lambda}_k = \prod_{l=0}^{k}\overline{\lambda}_l.$$

Let $\{\mathcal{G}_k\}$ be the complete σ-field generated by $\{X_0, X_1, ..., X_k, Y_0, Y_1, ..., Y_k\}$ for $k \in \mathbb{N}$. The process $\{\overline{\Lambda}_k\}$, $k \in \mathbb{N}$, is an $\overline{P}$-martingale with respect to the filtration $\{\mathcal{G}_k\}$.

Define P on $\{\Omega, \mathcal{F}\}$ by setting the restriction of the Radon—Nykodim derivative $\frac{dP}{d\overline{P}}$ to $\mathcal{G}_k$ equal to $\overline{\Lambda}_k$. It can be shown that on $\{\Omega, \mathcal{F}\}$ and under P, V_k has probability function $\boldsymbol{\xi}(x)$ defined on $\mathbb{Z}_+^m$ where $V_{k+1} \overset{\Delta}{=} X_{k+1} - A_k \varnothing X_k$ and (13) is true. For $r \in \Pi(N_{k+1})$ write $q_{k+1}(r) = \overline{E}\,[\overline{\Lambda}_{k+1} I(X_{k+1} = r) | \mathcal{Y}_{k+1}]$.

Note that

$$\sum_{r \in \Pi(N_k)} I(X_{k+1} = r) = 1$$

so that

$$\sum_{r \in \Pi(N_k)} q_{k+1}(r) = \overline{E}\,[\overline{\Lambda}_{k+1} | \mathcal{Y}_{k+1}].$$

We then have the following recursion.

Theorem 3. If $Y_k = (Y_k^1, Y_k^2, ..., Y_k^m) = (y^1, y^2, ..., y^m) \in \Pi(N_k)$,

$$q_k(r) = m^n \left(\frac{r^1}{N_k}\right)^{y^1} \left(\frac{r^2}{N_k}\right)^{y^2} \cdots \left(\frac{r^m}{N_k}\right)^{y^m} \times$$

$$\times \sum_{s \in \Pi(N_{k-1})} \sum_{i=1}^{m} \sum_{z_1^i + ... + z_m^i = s^i} \prod_{i=1}^{m} \binom{s^i}{z_1^i \ldots z_m^i} (\alpha_1^i)^{z_1^i} \ldots (\alpha_m^i)^{z_m^i} \times$$

$$\times \boldsymbol{\xi}\left(r - \sum_{i=1}^{m} \left(\sum_{j=1}^{z_1^i} y_{1j}^i, \ldots, \sum_{j=1}^{z_m^i} y_{mj}^i \right)' \right) \prod_{i,\ell=1}^{m} \prod_{j=1}^{z_\ell^i} \rho_\ell^i (y_{\ell j}^i)\, q_{k-1}(s).$$

(Note we take $0^0 = 1$.)

P r o o f.

$$q_k(r) = \overline{E}\,[\overline{\Lambda}_k I(X_k = r) | \mathcal{Y}_k] =$$

$$= \overline{E}\,[\overline{\Lambda}_k I(X_k = r) | \mathcal{Y}_{k-1}, Y_k = (y^1, y^2, ..., y^m)] =$$

$$= \overline{E}\,[\overline{\Lambda}_{k-1} \overline{\lambda}_k I(X_k = r) | \mathcal{Y}_{k-1}, Y_k = (y^1, y^2, ..., y^m)] =$$

$$= m^n \left(\frac{r^1}{N_k}\right)^{y^1} \left(\frac{r^2}{N_k}\right)^{y^2} \cdots \left(\frac{r^m}{N_k}\right)^{y^m} \overline{E}\,[\overline{\Lambda}_{k-1} I(X_k = r) \frac{\boldsymbol{\xi}(r - A_k \varnothing X_{k-1})}{\boldsymbol{\xi}(r)} | \mathcal{Y}_{k-1}] =$$

$$= m^n \left(\frac{r^1}{N_k}\right)^{y^1} \left(\frac{r^2}{N_k}\right)^{y^2} \dots \left(\frac{r^m}{N_k}\right)^{y^m} \overline{E}\,[\overline{\Lambda}_{k-1} \sum_{s\in\Pi(N_{k-1})} \boldsymbol{\xi}(r - A_k \varnothing s)\, I(X_{k-1} = s) | \mathcal{Y}_{k-1}],$$

using the definition of the operator $\varnothing$ in (9) and (10) yields the result.

Remark.

$$P(X_k = r) | \mathcal{Y}_k) = E\,[I(X_k = r) | \mathcal{Y}_k] = \frac{q_k(r)}{\sum_{s\in\Pi(N_k)} q_k(s)}.$$

To obtain the expected value of X_k given the observations $\mathcal{Y}_k$ we consider the vector of values $r = r^1, r^2, \ldots, r^m$) for any $r \in \Pi\,(N_k)$. Then

$$E\,[X_k | \mathcal{Y}_k] = \frac{\sum_{r\in\Pi(N_k)} q_k(r)\, r}{\sum_{s\in\Pi(N_k)} q_k(s)}.$$

Аналіз часових послідовностей відліків — напрям, що інтенсивно розвивається. Такий аналіз широко використовується для базових цілочисельних часових послідовностей, з якими не можна задовільно працювати у рамках класичних послідовностей гаусова типу. Отримано рекурсивні фільтри для частково спостерігаємих дискретизованих часових послідовностей. Показано, що ці процеси регулюються проріжуючими біноміальними та поліноміальними операторами.

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