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Predictability in Spatially Extended Systems with Model Uncertainty *. II

Macroscopic models for spatially extended systems under random influences are often described by stochastic partial differential equations. Some techniques for understanding solutions of such equations, such as estimating correlations, Liapunov exponents and impact of noises, are discussed. They are relevant for understanding predictability in spatially extended systems with model uncertainty, for example, in physics, geophysics and biological sciences. The presentation is for a wide audience.

Рассмотрены некоторые методы представления решений стохастических дифференциальных уравнений в частных производных, в частности в задачах корреляции оценки, экспоненты Ляпунова и воздействие шумов. Методы пригодны для понимания предсказуемости в пространственно распределенных системах с неопределенностью модели, например, в физике, геофизике и биологических науках.

K e y w o r d s: Stochastic partial differential equations, correlation, Liapunov exponents, predictability, uncertainty, invariant manifolds, impact of noise

4. Correlation. In this section, we discuss correlation of solutions, at different time instants, of some linear SPDEs. We first recall some information about Fourier series in Hilbert space.

Hilbert-Schmidt theory and Fourier series in Hilbert space. A separable Hilbert space H has a countable orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Namely, $\langle e_m, e_n \rangle = \delta_{mn}$, where δ_{mn} is the Kronecker delta function. Moreover, for any $h \in H$, we have Fourier series expansion

$$h = \sum_{n=1}^{\infty} \langle h, e_n \rangle \, e_n .$$

* J. Duan. Predictability in Spatially Extended Systems with Model Uncertainty. I. — Электронное моделирование, № 2, 2009. This work was partly supported by the NSF Grants 0542450 and 0620539.

In the context of solving stochastic PDEs, we may choose to work on a Hilbert space with an appropriate orthonormal basis. This is naturally possible with the help of the Hilbert-Schmidt theory [1, p. 232].

The Hilbert-Schmidt theorem [1, p. 232] says that a linear compact symmetric operator A on a separable Hilbert space H has a set of eigenvectors that form a complete orthonormal basis for H. Moreover, all the eigenvalues of A are real, each non-zero eigenvalue has finite multiplicity, and two eigenvectors that correspond to different eigenvalues are orthogonal.

This theorem applies to a strong (self-adjoined) elliptic differential operator B

$$Bu = \sum_{0 \le |\alpha|,\ |\beta| \le m} (-1)^{|\alpha|} D^{\alpha} (a_{\alpha\beta}(x) D^{\beta} u),\ \ x \in D \subset \mathbb{R}^n,$$

where the domain of definition of B is an appropriate dense subspace of $H = L^2(D)$, depending on the boundary condition specified for $u(x)$.

In this case, $A := B^{-1}$ is a linear symmetric compact operator in a Hilbert space, e. g., $H = L^2(D)$. We may consider $A := (B + aI)^{-1}$ for some real number a. This may be necessary in order for the operator to be invertible, i.e., no zero eigenvalue, such as in the case of Laplace operator with zero Neumann boundary condition.

By the Hilbert-Schmidt theorem, eigenvectors (also called eigenfunctions in this context) of $A = B^{-1}$ form an orthonormal basis for $H = L^2(D)$. Note that A and B share the same set of eigenfunctions. So we can claim that the strong elliptic operator B's eigenfunctions form an orthonormal basis for $H = L^2(D)$.

In the case of one spatial variable, the elliptic differential operator is the so called Sturm-Liouville operator [1, p. 245]. For example $Bu = -(pu')' + qu$, $x \in (0, l)$ where $p(x)$, $p'(x)$ and $q(x)$ are continuous on $(0, l)$. This operator arises in the method of separating variables for solving linear (deterministic) partial differential equations in the next section. By the Hilbert-Schmidt theorem, eigenfunctions of the Sturm-Liouville operator form an orthonormal basis for $H = L^2(0, l)$.

The wave equation with additive noise. Consider the stochastic wave equation with additive noise

$$u_{tt} = c^2 u_{xx} + \epsilon W_t,\ \ 0 < x < l,\ \ t > 0,$$

$$u(0, t) = u(l, t) = 0,$$

$$u(x, 0) = f(x),\ \ u_t(x, 0) = g(x),$$

where ϵ is a real parameter modeling the noise intensity, $c > 0$ is a constant (wave speed), and W_t is a Brownian motion taking values in Hilbert space $H = L^2(0, l)$.

Method of eigenfunction expansion:

$$u=\sum_{n=1}^{\infty} u_n(t)\, e_n(x),\ \ W_t=\sum_{n=1}^{\infty} \sqrt{q_n} W_n(t)\, e_n(x),$$

where

$$e_n(x)=\sqrt{2l}\sin\frac{n\pi x}{l},\ \ \lambda_n=(n\pi)^2,\ \ n=1, 2, \dots .$$

The final solution

$$u(x,t)=\sum_{n=1}^{\infty}\left\{\left[A_n-\epsilon\,\frac{l}{cn\pi}\sqrt{q_n}\int_0^t \sin\frac{cn\pi}{l}\, s dW_n(s)\right]\cos\frac{cn\pi}{l}\,t+\right.$$

$$\left.+\left[B_n+\epsilon\,\frac{l}{cn\pi}\sqrt{q_n}\int_0^t \cos\frac{cn\pi}{l}\, s dW_n(s)\right]\sin\frac{cn\pi}{l}\,t\right\} e_n(x),$$

where

$$A_n=\langle f, e_n\rangle,\ \ B_n=\frac{l}{cn\pi}\langle g, e_n\rangle.$$

When the noise is at one mode, say at the first mode $e_1(x)$ (i.e., $q_1>0$ but $q_n=$ $=0$, $n=2, 3, \dots$), we see that the solution contains randomness only at that mode. So for the linear stochastic diffusion system, there is no interactions between modes. In other words, if we randomly force a few fast modes, then there is no impact on slow modes.

Mean value for the solution:

$$\mathbb{E}u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{cn\pi t}{l}\right)+B_n\sin\left(\frac{cn\pi t}{l}\right)\right]e_n(x).$$

Covariance for the solution: now we calculate the covariance of solution u at different time instants t and s, i. e. $\mathbb{E}< u(x,t)-\mathbb{E}u(x,t),\ u(x,s)-\mathbb{E}u(x,s)>$. Using the Ito's isometry, we get

$$\mathbb{E}< u(x,t)-\mathbb{E}u(x,t),\ u(x,s)-\mathbb{E}u(x,s)>=$$

$$=\sum_{n=1}^{\infty}\frac{\epsilon^2 l^2 q_n}{c^2 n^2 \pi^2}\left[\int_0^{t\wedge s}\sin^2\frac{cn\pi r}{l}\,dr\cos\frac{cn\pi t}{l}\cos\frac{cn\pi s}{l}+\right.$$

$$+\int_0^{t\wedge s}\cos^2\frac{cn\pi r}{l}\,dr\sin\frac{cn\pi t}{l}\sin\frac{cn\pi s}{l}-$$

$$-\int_0^{t\wedge s} \sin\frac{cn\pi r}{l}\cos\frac{cn\pi r}{l}\,dr\left(\cos\frac{cn\pi t}{l}\sin\frac{cn\pi s}{l}+\cos\frac{cn\pi s}{l}\sin\frac{cn\pi t}{l}\right)\Bigg].$$

After integrations, we get the covariance as

$$Cov\,(u(x,t),u(x,s)) = \mathbb{E}< u(x,t)-\mathbb{E}u(x,t), u(x,s)-\mathbb{E}u(x,s)> =$$

$$=\sum_{n=1}^{\infty}\frac{\epsilon^2 l^2 q_n}{2c^2n^2\pi^2}\left[(t\wedge s)\cos\frac{cn\pi(t-s)}{l}-\frac{l}{2cn\pi}\sin\frac{2cn\pi\,(t\wedge s)}{l}\cos\frac{cn\pi\,(t+s)}{l}+\right.$$

$$\left.+\frac{l}{2cn\pi}\cos\frac{2cn\pi\,(t\wedge s)}{l}\sin\frac{cn\pi\,(t+s)}{l}-\frac{l}{2cn\pi}\sin\frac{cn\pi\,(t+s)}{l}\right]=$$

$$=\sum_{n=1}^{\infty}\frac{\epsilon^2 l^2 q_n}{2c^2n^2\pi^2}\left[(t\wedge s)\cos\frac{cn\pi\,(t-s)}{l}+\frac{l}{2cn\pi}\sin\frac{cn\pi(t+s-2\,(t\wedge s))}{l}-\right.$$

$$\left.-\frac{l}{2cn\pi}\sin\frac{cn\pi\,(t+s)}{l}\right].$$

In particular, for $t = s$ we get the variance.

Variance for the solution:

$$Var\,(u(x,t))=\sum_{n=1}^{\infty}\frac{\epsilon^2 l^2}{c^2n^2\pi^2}\,q_n\left[\frac{1}{2}\,t-\frac{l}{4cn\pi}\sin\left(\frac{2cn\pi}{l}t\right)\right].$$

Energy evolution for the solution:

$$E\,(t)=\frac{1}{2}\int_0^l [u_t^2+c^2u_x^2]\,dx.$$

Taking time derivative,

$$\dot{E}\,(t)=\int_0^l u_t[u_{tt}-c^2u_{xx}]\,dx=\epsilon\int_0^l u_t(x,t)\,\dot{W}_t(x)\,dx.$$

Or in integral form,

$$E\,(t)=E\,(0)+\epsilon\int_0^t\int_0^l u_s(x,t)\,dW_s(x)\,dx.$$

It can be shown that

$$\mathbb{E}E(t)=E(0),\quad Var(E(t))=\varepsilon^2\mathbb{E}\left(\int_0^l\int_0^t\partial_t u(x,s)\,dW_s dx\right)^2,$$

where W_t is in the following form

$$W_t=W(t)=\sum_{n=1}^{\infty}\sqrt{q_n}W_n(t)\,e_n(x),$$

and $\partial_t u$ can be written in the following form :

$$\partial_t u=\sum\left\{-A_n\frac{cn\pi}{l}\sin\left(\frac{cn\pi t}{l}\right)+B_n\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}+\right.$$

$$+\ \varepsilon\sqrt{q_n}\left[\int_0^t\sin\frac{cn\pi s}{l}\,dW_n(s)\sin\frac{cn\pi t}{l}+\right.$$

$$\left.\left.+\int_0^t\cos\frac{cn\pi s}{l}\,dW_n(s)\cos\frac{cn\pi t}{l}\right]\right\}e_n(x).$$

Set $cn\pi/l=\mu_n$ and rewrite

$$\partial_t u=\sum\left\{F_n(t)+\varepsilon\sqrt{q_n}\left[\int_0^t(\sin\mu_n s\sin\mu_n t+\cos\mu_n s\cos\mu_n t)\,dW_n(s)\right]\right\}e_n(x)=$$

$$=\sum\left\{F_n(t)+\varepsilon\sqrt{q_n}\int_0^t\cos\mu_n(t-s)\,dW_n(s)\right\}e_n(x),$$

where $F_n(t):=-A_n\mu_n s\sin\mu_n t+B_n\mu_n s\cos\mu_n t$, $n=1, 2, \ldots$. For the simplicity of notations, set

$$G_n(t):=F_n(t)+\varepsilon\sqrt{q_n}\int_0^t\cos\mu_n(t-s)\,dW_n(s),\ n=1, 2, \ldots,$$

then we have $\partial_t u=\sum G_n(t)\,e_n(x)$. Thus

$$\mathbb{E}\left(\int_0^l\int_0^t\partial_t u(x,s)\,dW_s dx\right)^2=\mathbb{E}\left[\int_0^l\left(\sum_{n=1}^{\infty}\sqrt{q_n}e_n(x)\int_0^t u_s dW_n(s)\right)dx\right]^2=$$

$$=\mathbb{E}\left[\sum_{n=1}^{\infty}\sqrt{q_n}\int_0^l\int_0^t u_s e_n(x)\,dW_n(s)\,dx\right]^2=$$

$$=\mathbb{E}\left[\sum_{n=1}^{\infty}\sqrt{q_n}\int_0^t\left(\int_0^l e_n(x)\sum_{j=1}^{\infty}G_j(s)\,e_j(x)\,dx\right)dW_n(s)\right]^2=$$

$$=\mathbb{E}\left[\sum_{n=1}^{\infty}\sqrt{q_n}\int_0^t\left(\sum_{j=1}^{\infty}G_j(s)\int_0^l e_n(x)\,e_j(x)\,dx\right)dW_n(s)\right]^2=$$

$$=\mathbb{E}\left[\sum_{n=1}^{\infty}\sqrt{q_n}\int_0^t G_n(s)\,dW_n(s)\right]^2=\sum_{n=1}^{\infty}q_n\,\mathbb{E}\int_0^t G_n^2(s)\,ds=$$

$$=\sum_{n=1}^{\infty}q_n\,\mathbb{E}\int_0^t\left[F_n(s)+\varepsilon\sqrt{q_n}\int_0^s\cos\mu_n(s-r)\,dW_n(r)\right]^2 ds=\mathbb{E}\sum_{n=1}^{\infty}q_n\int_0^t F_n^2(s)\,ds+$$

$$+\mathbb{E}\sum_{n=1}^{\infty}\varepsilon^2 q_n^2\int_0^t\left[\int_0^s\cos^2\mu_n(s-r)\,dr\right]ds=$$

$$=\sum_{n=1}^{\infty}q_n\left[A_n^2\mu_n^2\left(\frac{t}{2}-\frac{1}{4\mu_n}\sin 2\mu_n t\right)+B_n^2\mu_n^2\left(\frac{t}{2}+\frac{1}{4\mu_n}\sin 2\mu_n t\right)-\right.$$

$$\left.-\frac{1}{2}A_nB_n\mu_n(1-\cos 2\mu_n t)\right]+\sum_{n=1}^{\infty}\varepsilon^2 q_n^2\left[\frac{t^2}{4}+\frac{1}{8\mu_n^2}(1-\cos 2\mu_n t)\right].$$

Therefore,

$$Var(E(t))=\sum_{n=1}^{\infty}\varepsilon^2 q_n\left[A_n^2\mu_n^2\left(\frac{t}{2}-\frac{1}{4\mu_n}\sin 2\mu_n t\right)+B_n^2\mu_n^2\left(\frac{t}{2}+\frac{1}{4\mu_n}\sin 2\mu_n t\right)-\right.$$

$$\left.-\frac{1}{2}A_nB_n\mu_n(1-\cos 2\mu_n t)\right]+\sum_{n=1}^{\infty}\varepsilon^4 q_n^2\left[\frac{t^2}{4}+\frac{1}{8\mu_n^2}(1-\cos 2\mu_n t)\right]$$

is obtained.

The diffusion equation with multiplicative noise. Consider the stochastic diffusion equations with zero Dirichlet boundary condition

$$u_t=u_{xx}+\epsilon u\dot{w}_t,\ 0<x<1,$$
$$u(x,0)=f(x), \tag{2}$$

where w_t is a scalar Brownian motion. We take Hilbert space $H = L^2(0,1)$ with an orthonormal basis $e_n = \sqrt{2}\sin(n\pi x)$. We use the method of eigenfunction expansion:

$$u(x,t) = \sum u_n(t)\, e_n(x),$$
$$u_{xx} = \sum u_n(t)\, \ddot{e}_n(x) = \sum - u_n(t)(n\pi)^2 e_n(x).$$

Putting these into the above SPDE (2), with $\lambda_n = (n\pi)^2$, we get

$$\sum \dot{u}_n(t)\, e_n(x) = \sum u_n(t)(-\lambda_n)\, e_n + \epsilon \sum u_n(t)\, e_n(x)\, \dot{w}_t .$$

We further obtain the following system of SODEs:

$$du_n(t) = -\lambda_n u_n(t) + \epsilon u_n(t)\, dw(t), \;\; n = 1, 2, 3, \ldots .$$

Thus

$$u_n(t) = u_n(0)\exp\left(\left(-\lambda_n - \frac{1}{2}\epsilon^2\right) t + \epsilon w(t)\right),$$

where $u(x,0) = f(x) = \Sigma < f(x), e_n(x) > e_n(x) = \sum u_n(0)\, e_n(x)$. Therefore, the final solution is:

$$u(x,t) = \sum a_n e_n(x) \exp(b_n t + \epsilon w_t),$$

with $a_n = < f(x), e_n(x) >$ and $b_n = \left(-\lambda_n - \frac{1}{2}\epsilon^2\right)$.

Note that $\mathbb{E}\exp(b_n t + \epsilon w_t) = \exp(b_n t)\,\mathbb{E}\exp(\epsilon w_t) = \exp(b_n t)\exp\left(\frac{1}{2}\epsilon^2 t\right) =$ $= \exp(-\lambda_n t)$. Therefore, we can find out the mean, variance, covariance and correlation of the solution:

$$E(u(x,t)) = \sum a_n e_n(x) \exp(-\lambda_n t),$$
$$Var(u(x,t)) = \mathbb{E}\langle u(x,t) - E(u(x,t)), u(x,t)) - E(u(x,t))\rangle =$$
$$= \sum a_n^2 \exp(-2\lambda_n t)[\exp(\epsilon^2 t) - 1].$$

For $\tau \le t$ we have

$$\mathbb{E}\exp\{\epsilon(w_t + w_\tau)\} = \mathbb{E}\exp\{\epsilon(w_t - w_\tau) + 2\epsilon w_\tau\} =$$
$$= \mathbb{E}\exp\{\epsilon(w_t - w_\tau)\}\,\mathbb{E}\exp\{2\epsilon w_\tau\} =$$
$$= \exp\left\{\frac{1}{2}\epsilon^2(t-\tau)\right\}\exp\{2\epsilon^2\tau\} = \exp\left\{\frac{1}{2}\epsilon^2[(t+\tau) + 2(t \wedge \tau)]\right\}.$$

Therefore, by direct calculation, we can get

$$Cov\,(u(x,t),u(x,\tau))=\sum a_n^2\left\{\exp(b_n(t+\tau)+\frac{1}{2}\epsilon^2((t+\tau)+2(t\wedge\tau))+\right.$$

$$\left.+\exp(-\lambda_n(t+\tau))-\exp\left(-\lambda_n\tau+b_nt+\frac{1}{2}\epsilon^2t\right)-\exp\left(-\lambda_nt+b_n\tau+\frac{1}{2}\epsilon^2\tau\right)\right\}=$$

$$=\sum a_n^2\exp\{-\lambda_n(t+\tau)\}[\exp\{\epsilon^2(t\wedge\tau)\}-1]$$

and

$$Corr\,(u(x,t),u(x,\tau))=\frac{Cov\,(u(x,t),u(x,\tau))}{\sqrt{Var\,(u(x,t))}\sqrt{Var\,(u(x,\tau))}}=$$

$$=\frac{\sum a_n^2\exp\{-\lambda_n(t+\tau)\}[\exp\{\epsilon^2(t\wedge\tau)\}-1]}{\sqrt{\sum a_n^2\exp(-2\lambda_nt)[\exp\{\epsilon^2t)-1]}\sqrt{\sum a_n^2\exp(-2\lambda_n\tau)[\exp\{\epsilon^2\tau)-1]}}\,.$$

5. Lyapunov Exponents. Lyapunov exponents are tools for quantifying growth or decay of linear systems (e. g., PDEs or SPDEs). The following discussions are from [2, 3].

A deterministic PDE system. Let us first look at the following deterministic PDE:

$$\frac{\partial u}{\partial t}=u_{xx}+\alpha u\,, \tag{3}$$

$$u(x,0)=f(x)\,, \tag{4}$$

$$u(x,t)=0,\;\; x\in\partial D, \tag{5}$$

where $D=\{x:0\le x\le 1\}$ and the function $f\in L^2(0,1)$. An orthonormal basis for $L^2(0,1)$ is $\{e_n(x)\}$, $n=0,1,2,\ldots,\partial_{xx}e_j=-\lambda_je_j$. Note that $0\le\lambda_j\uparrow\infty$. We then can write:

$$f=\sum_{j=0}^{\infty}f_je_j\,, \tag{6}$$

where $f_j=\langle f,e_j\rangle$. By using the method of eigenfunction expansion, it is known that the unique solution to the problem is given below:

$$u(x,t)=\sum_{j=0}^{\infty}\exp(t(-\lambda_j+\alpha))f_je_j(x),\; t\ge 0. \tag{7}$$

Theorem 2. Let us fix a non-zero initial condition f. Let j_0 be the smallest integer $j \geq 0$ in the expansion (6) of f such that $f_{j_0} \neq 0$. Then the Lyapunov exponent of the system (3)—(5) exists as a limit and is given by $\lambda^u(f) = -\lambda_{j_0} + \alpha$.

P r o o f. For a class of initial conditions f we calculate the Lyapunov exponents, which are defined as

$$\lambda^u(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|u(t)\|_{L^2}.$$

By applying (7), we obtain the Lyapunov exponents regarding to PDE system (3)—(5),

$$\lambda^u(f) = \limsup_{t\to\infty} \frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\| .$$

On the one hand,

$$\frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\| \leq$$

$$\leq \frac{1}{t} \log \left(\sum_{j=j_0}^{\infty} \left| \exp(t(-\lambda_{j_0} + \alpha)) f_j \right|^2 \right)^{1/2} = -\lambda_{j_0} + \alpha + \frac{1}{t} \log \|f\| .$$

On the other hand,

$$\frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\| \geq$$

$$\geq \frac{1}{t} \log \left| \exp(t(-\lambda_{j-0} + \alpha)) f_{j_0} \right| = -\lambda_{j_0} + \alpha + \frac{1}{t} \log \left| f_{j_0} \right|.$$

A SPDE system. We now consider the following SPDE

$$dv = (v_{xx} + \beta v)\, dt + \gamma v dw_t, \tag{8}$$

$$v(x, 0) = f(x),\ x \in D, \tag{9}$$

$$v(x, t) = 0,\ x \in \partial D, \tag{10}$$

where w_t is a scalar Brownian motion. The conditions (9) and (10) hold for a.a. $\omega \in \Omega$.

We seek the solution with expansion with respect to the basis $\{e_j\}$ (see the last subsection)

$$v(x, t) = \sum_{j=0}^{\infty} y_j(t)\, e_j(x), \tag{11}$$

where $y_j(t)$, for $j = 0, 1, 2, \ldots$ satisfy the following stochastic ordinary differential equations:

$$dy_j(t) = (-\lambda_j + \beta)\, y_j(t)\, dt + \gamma y_j(t)\, dw_t,$$
$$y_j(0) = f_j.$$

So

$$y_j(t) = \exp(\gamma w_t)\exp\left(\left(-\lambda_j + \beta - \frac{1}{2}\gamma^2\right)t\right) f_j.$$

Thus from (11), we obtain,

$$v(x, t) = \sum_{j=0}^{\infty} \exp(\gamma w_t)\exp\left(\left(-\lambda_j + \beta - \frac{1}{2}\gamma^2\right)t\right) f_j e_j.$$

Observe that

$$v(x, t) = \exp(\gamma w_t)\exp\left(\left((\beta-\alpha)\frac{1}{2}\gamma^2\right)t\right) u(t, x), \tag{12}$$

where $u(t, x)$ is the solution to the above deterministic PDE (3)—(5).

By (12), we can calculate the Lyapunov exponent of the stochastic system (8)—(10) as a function of the Lyapunov exponent of the deterministic system (3)—(5) as follows:

$$\lambda^v(f) = \limsup_{t\to\infty} \frac{1}{t}\log\|v(t)\| =$$
$$= \limsup_{t\to\infty} \frac{1}{t}\log\left\|\exp(\gamma w_t)\exp\left(\left((\beta-\alpha) - \frac{1}{2}\gamma^2\right)t\right) u(t)\right\| =$$
$$= \lambda^u(f) + (\beta-\alpha) - \frac{1}{2}\gamma^2, \text{ a.s.}$$

by the strong law of large number.

Let us state the result in the following theorem.

Theorem 3. Let $f \neq 0$. Then the Lyapunov exponent of the SPDE (8)—(10) almost surely exists as a limit, is non-random and is given in the following formula:

$$\lambda^v(f) = \lambda^u(f) + (\beta-\alpha) - \frac{1}{2}\gamma^2, \text{ a. s.}$$

***Remark** 7.* Let us consider a special case when $\alpha = \beta$. Then by the above theorem, for a fixed initial condition f, the Lyapunov exponent of the stochastic system (8)—(10) is

$$\lambda^v(f) = \lambda^u(f) - \frac{1}{2}\gamma^2,$$

which obviously is smaller than the Lyapunov exponent of the corresponding deterministic system (3)—(5). The result implies that this stochastically perturbed system is more stable than the original deterministic system.

6. Impact of Uncertainty. In this section, we first recall some inequalities for estimating solutions of SPDEs, and then we estimate the impact of noises on solutions of the nonlinear Burgers equation.

Differential and integral inequalities. Gronwall inequality: Differential form [4]. Assuming that $y(t) \ge 0$, $g(t)$ and $h(t)$ are integrable, if $\frac{dy}{dt} \le g(t)\,y + h(t)$ for $t \ge t_0$, then

$$y(t) \le y(t_0)\, e^{\int_{t_0}^{t} g(\tau)d\tau} + \int_{t_0}^{t} h(s)[e^{\int_{t_0}^{t} g(\tau)d\tau}]\,ds, \;\; t \ge t_0.$$

In particular, if $\frac{dy}{dt} \le gy + h$ for $t \ge t_0$ with g, h being constants and $t_0 = 0$, we have

$$y(t) \le y(0)\, e^{gt}\, \frac{h}{g}(1 - e^{gt}), \;\; t \ge 0.$$

Note that when constant $g < 0$, then $\lim_{t\to\infty} y(t) = -\frac{h}{g}$.

Gronwall inequality: Integral form [5, 6]. If $u(t)$, $v(t)$ and $c(t)$ are all non-negative, $c(t)$ is differentiable, and $v(t) \le c(t) + \int_0^t u(s)\,v(s)\,ds$ for $t \ge t_0$, then

$$v(t) \le v(t_0)\, e^{\int_{t_0}^{t} u(\tau)d\tau} + \int_{t_0}^{t} c'(s)[e^{\int_{s}^{t} u(\tau)d\tau}]\,ds, \; t \ge t_0.$$

In particular, assuming that $y(t) \ge 0$ and is continuous and $y(t) \le C + K\int_0^t y(s)\,ds$, with C, K being positive constants, for $t > 0$. Then $y(t) \le Ce^{Kt}$, $t \ge 0$.

Sobolev inequalities. We first introduce some common Sobolev spaces. For $k = 1, 2, \ldots$, we define $H^k(0, l) := \{f : f, f', \ldots, f^{(k)} \in L^2(0, l)\}$. Each of these is a Hilbert space with the scalar product

$$\langle u, v \rangle_k = \int_0^l [uv + u'v' + \ldots + u^{(k)} v^{(k)}]\,dx,$$

and the norm

$$\|u\|_k = \sqrt{\langle u, u\rangle_k} = \sqrt{\int_0^l [u^2 + (u')^2 + ... + (u^{(k)})^2]\, dx}.$$

For $k = 1, 2, ...$ and $p \geq 1$, we further define another class of Sobolev spaces $W^{k,p}(D) = \{u : u, Du, ..., D^\alpha u \in L^p(D), |\alpha| \leq k\}$ with norm $\|u\|_{k,p} = \left(\|u\|_{L^p}^p + \right.$ $\left. + \|u'\|_{L^p}^p + ... + \|u^{(k)}\|_{L^p}^p \right)^{\frac{1}{p}}$.

Moreover, $H_0^k(0,l)$ denotes the closure of $C_c^\infty(0,l)$ in $H^k(0,l)$ (i.e., under the norm $\|\cdot\|_k$). It is a sub-Hilbert space in $H^k(0,l)$. Similarly, $W_0^{k,p}(0,l)$ denotes the closure of $C_c^\infty(0,l)$ in $W^{k,p}(0,l)$ (i.e., under the norm $\|\cdot\|_{k,p}$). It is a sub-Hilbert space in $W^{k,p}(0,l)$.

Standard abbreviations $L^2 = L^2(D)$, $H_0^k = H_0^k(D)$, $k = 1, 2, ...$, are used for the common Sobolev spaces in fluid mechanics, with $< \cdot , \cdot >$ and $\|\cdot\|$ denoting the usual (spatial) scalar product and norm, respectively, in $L^2(D)$:

$$< f, g > := \int_D fg dxdy, \quad \|f\| := \sqrt{< f, f >} = \sqrt{\int_D f(x, y)\, dxdy}\,.$$

C a u c h y - S c h w a r z   i n e q u a l i t y. In the space $L^2(D)$ of square-integrable functions defined on a domain $D \subset \mathbb{R}^n$:

$$\left| \int_D f(x)\, g(x)\, dx \right| \leq \sqrt{\int_D f^2(x)\, dx}\, \sqrt{\int_D g^2(x)\, dx}\,.$$

H ö l d e r   i n e q u a l i t y. In the space $L^r(D)$ of functions defined on a domain $D \subset \mathbb{R}^n$:

$$\left| \int_D f(x)\, g(x)\, dx \right| \leq \left(\int_D |f(x)|^p\, dx \right)^{\frac{1}{p}} \left(\int_D |g(x)|^q\, dx \right)^{\frac{1}{q}}.$$

M i n k o w s k i   i n e q u a l i t y. In the space $L^p(D)$ of functions defined on a domain $D \subset \mathbb{R}^n$:

$$\left(\int_D |f(x) \pm g(x)|^p\, dx \right)^{\frac{1}{p}} \leq \left(\int_D |f(x)|^p\, dx \right)^{\frac{1}{p}} + \left(\int_D |g(x)|^p\, dx \right)^{\frac{1}{p}}.$$

P o i n c a r é   i n e q u a l i t y [4]. For $g \in H_0^1(D)$,

$$\|g\|^2 = \int_D g^2(x, y)\, dxdy \le \frac{|D|}{\pi} \int_D |\nabla g|^2\, dxdy = \frac{|D|}{\pi} \|\nabla g\|^2 ,$$

where $|D|$ is the Lebesgue measure of the domain D.

For $u \in W_0^{1,p}(D), 1 \le p < \infty$ and $D \subset \mathbb{R}^n$ a bounded domain $\|u\|_p \le C\|\nabla u\|_p$, where C is a positive constant depending only on the domain D.

Let $u \in W^{1,p}(D), 1 \le p < \infty$ and $D \subset \mathbb{R}^n$ a bounded convex domain. Let $S \subset D$ be a measurable subset, and define the spatial average of u over S by $u_S = \frac{1}{|S|}\int_D u dx$ (here $|S|$ is the volume or Lebesgue measure of S). Then $\| u - u_S \|_p \le C \|\nabla u\|_p$, where C is a positive constant depending only on the domain D and S.

A g m o n   i n e q u a l i t y [4]. Let $D \subset \mathbb{R}^n$. There exists a constant C depending only on domain D such that

$$\|u\|_{L^\infty(D)} \le C \| u \|^{\frac{1}{2}}_{H^{\frac{n-1}{2}}(D)} \| u \|^{\frac{1}{2}}_{H^{\frac{n+1}{2}}(D)}, \quad \text{for } n \text{ odd},$$

$$\|u\|_{L^\infty(D)} \le \| u \|^{\frac{1}{2}}_{H^{\frac{n-2}{2}}(D)} \| u \|^{\frac{1}{2}}_{H^{\frac{n+2}{2}}(D)}, \quad \text{for } n \text{ even}.$$

In particular, for $n = 1$ and $u \in H^1(0, l)$,

$$\|u\|_{L^\infty(0,l)} \le C \| u \|^{\frac{1}{2}}_{L^2(0,l)} \| u \|^{\frac{1}{2}}_{H^1(0,l)}.$$

Moreover, for $n = 1$ and $u \in H_0^1(0, l)$,

$$\|u\|_{L^\infty(0,l)} \le C \| u \|^{\frac{1}{2}}_{L^2(0,l)} \| u_x \|^{\frac{1}{2}}_{L^2(0,l)}.$$

Stochastic Burgers equation. We now consider the Burgers equation with additive noise forcing as in [7]:

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u + \sigma \dot{W}_t,$$

$$u(0, t) = 0,\ u(l, t) = 0,\ u(x, 0) = u_0(x),$$

where W_t is a Brownian motion, with covariance Q, taking values in the Hilbert space $L^2(0, l)$ with the usual scalar product $\langle \cdot, \cdot \rangle$. We assume that the trace $Tr(Q)$ is finite. So $\dot{W}_t$ is noise colored in space but white in time.

Taking

$$F(u)=\frac{1}{2}\int_0^l u^2dx=\frac{1}{2}\langle u,u\rangle$$

and applying the Ito's formula, we obtain

$$\frac{1}{2}d\,\|u\|^2=\langle u,\sigma dW_t\rangle+\left[\langle u,vu_{xx}-uu_x\rangle+\frac{1}{2}\sigma^2 lTr\,(Q)\right]dt.$$

Thus

$$\frac{d}{dt}\mathbb{E}\|u\|^2=2\langle u,vu_{xx}-uu_x\rangle+\sigma^2 lTr\,(Q)=-2v\,\|u_x\|^2+\sigma^2 lTr\,(Q)\,.$$

By the Poincare inequality $\|u\|^2\le c\,\|u_x\|^2$ for some positive constant depending only on the interval (0, *l*), we have

$$\frac{d}{dt}\mathbb{E}\|u\|^2\le-\frac{2v}{c}\|u\|^2+\sigma^2 lTr\,(Q).$$

Then using the Gronwall inequality, we finally get

$$\mathbb{E}\|u\|^2\le\mathbb{E}\|u_0\|^2e^{-\frac{2v}{c}t}+\frac{1}{2}c\sigma^2 lTr\,(Q)[1-e^{-\frac{2v}{c}t}\,].$$

Note that the first term in this estimate involves the initial data, and the second term involves the noise intensity σ as well as the trace of the noise covariance.

We finally consider the Burgers equation with multiplicative noise forcing $\partial_t u+u\partial_x u=v\partial_x^2 u+\sigma u\dot{w}_t$, with the same boundary condition and initial condition as above, where w_t is a scalar Brownian motion (e. g., with covariance $Q=1$ and the trace $Tr\,(Q)=1$). So $\dot{W}_t$ is noise homogeneous in space but white in time.

By the Ito's formula, we obtain

$$\frac{1}{2}d\,\|u\|^2=\langle u,\sigma udw_t\rangle+\left[\langle u,vu_{xx}-uu_x\rangle+\frac{1}{2}\sigma^2\|u\|^2\right]dt.$$

Thus

$$\frac{d}{dt}\mathbb{E}\|u\|^2=2\langle u,vu_{xx}-uu_x\rangle+\sigma^2\|u\|^2=-2v\,\|u_x\|^2+\sigma^2\|u\|^2\le\left(\sigma^2-\frac{2v}{c}\right)\|u\|^2.$$

Therefore,

$$\mathbb{E}\|u\|^2\le\mathbb{E}\|u_0\|^2e^{\left(\sigma^2-\frac{2v}{c}\right)t}.$$

Note here that the multiplicative noise affects the mean energy growth or decay rate, while the additive noise affects the mean energy upper bound.

Likelihood for staying bounded. By the Chebyshev inequality, we can estimate the likelihood of solution orbits staying inside or outside a bounded domain in Hilbert space $H = L^2(0, l)$. Taking the bounded domain as a ball centered at the origin with radius $\delta > 0$. For example, for the above Burgers equation with multiplicative noise, we have

$$\mathbb{P}(\omega : \|u\| \geq \delta) \leq \frac{1}{\delta^2} \mathbb{E}\|u\|^2 \leq \frac{\mathbb{E}\|u_0\|^2}{\delta^2} e^{\left(\sigma^2 - \frac{2\nu}{c}\right)t}$$

and

$$\mathbb{P}(\omega : \|u\| < \delta) = 1 - \mathbb{P}(\omega : \|u\| \geq \delta) \geq 1 - \frac{\mathbb{E}\|u_0\|^2}{\delta^2} e^{\left(\sigma^2 - \frac{2\nu}{c}\right)t}.$$

I would like to thank Hongbo Fu and Jiarui Yang for helpful comments.

Розглянуто деякі методи представлення розв'язків стохастичних диференціальних рівнянь у частинних похідних, зокрема у задачах кореляції оцінки, експоненти Ляпунова та впливу шумів. Методи придатні для розуміння передбачуваності у просторово розподілених системах з невизначеністю моделі, наприклад, у фізиці, геофізиці та біологічних науках.

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Поступила 08.12.08