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APPLICATION OF THE AMPLITUDE-FREQUENCY FORMULATION TO A NONLINEAR VIBRATION SYSTEM TYPIFIED BY A MASS ATTACHED TO A STRETCHED WIRE

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Abstract. The He's Amplitude-Frequency Formulation is applied to study the periodic solutions of a strongly nonlinear system. This system corresponds to the motion of a mass attached to a stretched wire. The usefulness and effectiveness of the proposed technique is illustrated. The results are compared with exact solutions and those obtained by the harmonic balance show a good accuracy. Approximate frequencies are valid for the complete range of vibration amplitudes. Excellent agreement of the approximate frequencies with the exact one are demonstrated and discussed.

Key words: Nonlinear vibration system; Amplitude-Frequency Formulation; Periodic Solution; Angular frequencies.

1. Introduction.

The study of nonlinear problems is of crucial importance not only in all areas of physics but also in engineering, since most phenomena in our world are essentially nonlinear and are described by nonlinear equations recently many new approaches to nonlinear problems have been proposed, for example, the variational iteration method [16], the homotopy perturbation method [17 – 19], energy balance method [12 – 15] and the parameter-expanding method [11].

To solve nonlinear problems, He proposed an amplitude–frequency formulation for nonlinear oscillators, which was deduced using an ancient Chinese mathematics method and it is now widely used by many authors [3 – 10]. In this paper He's frequency-amplitude formulation is used to solve nonlinear vibration system of conservative single degree of freedom.

Consider the motion of a particle of mass m attached to the centre of a stretched elastic wire [1, 2] and coefficient of stiffness of elastic wire equal to k. The length of the elastic wire when no force is applied to it is $2a$. We assume that the movement of the particle is one-dimensional and this is constrained to move only in the horizontal x direction.

As we can see in Fig. 1, the ends of the wire are fixed a distance $2d$ a part. Length d can be longer or equal to a. If $d = a$, the wire is not stretched for $x = 0$, and there is no tension in each part of it. However, if $d > a$, the wire is stretched for $x = 0$, and the tension in each part of the wire is $k(d - a)$ The equation of motion is given by the following nonlinear differential equation [2]:

$$m\frac{d^2x}{dt^2} + 2kx - \frac{2kax}{\sqrt{d^2 + x^2}} = 0 \ . \tag{1}$$

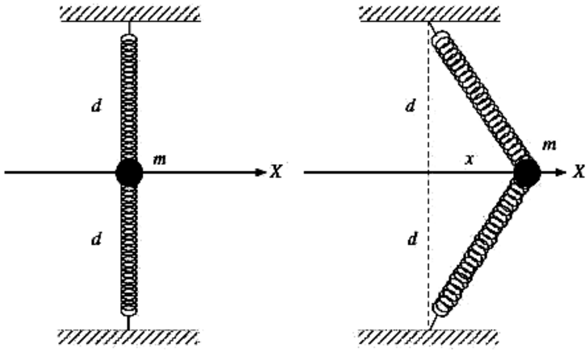

Fig. 1
Mass attached to a stretched wire.

With the initial condition of

$$x(0) = A \ , \ \frac{dx}{dt}(0) = 0 \ .$$

Two dimensionless variables y and τ can be constructed as follows:

$$y = \frac{x}{d} \ , \ \tau = \sqrt{\frac{2k}{m}} t \ . \tag{2}$$

Substituting these dimensionless variables into Eq. (1) gives

$$\frac{d^2 y}{d\tau^2} + y - \frac{\lambda y}{\sqrt{1+y^2}} = 0 \ , \quad 0<\lambda \le 1 \ . \tag{3}$$

With the initial condition of

$$y(0) = A \ , \ \frac{dy}{d\tau}(0) = 0 \ .$$

In Eq. (3) we have defined the following parameters:

$$A = \frac{x_0}{d} \ , \ \lambda = \frac{a}{d} \ , \tag{4}$$

as $0 \le a < d$ it follows that $0 < \lambda \le 1$.

Eq. (3) is an example of a conservative nonlinear oscillatory system in which the restoring force has an irrational form [1, 2] and this system and has the first integral.

$$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + V(y) = E \ge 0 \ , \tag{5}$$

where E is the ''total energy'' of the nonlinear oscillator and the potential function has the irrational form [1]

$$V(y) = \frac{1}{2} y^2 - \lambda\sqrt{1+y^2} + \lambda \ . \tag{6}$$

All the motions corresponding to Eq. (3) are periodic [1]; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A .

The main objective of this paper is to solve Eq. (3) by applying the first-order Amplitude-Frequency Formulation, and to compare the approximate frequency obtained with the exact one and with another approximate frequency obtained applying the harmonic balance method.

2. Solution method.

Considers the following general nonlinear oscillators in the form:

$$u''(t) + f\left(u(t), u'(t), u''(t)\right) = 0 \ . \tag{7}$$

Oscillation systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation, A . So let us consider such initial conditions

$$u(0) = A, \ u'(0) = 0.$$

According to He's amplitude-frequency formulation [8 – 10], we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$.

Substituting u_1 and u_2 into Eq. (7), we obtain, respectively, the following residuals:

$$R_1 = u_1''(t) + f\left(u_1(t), u_1'(t), u_1''(t)\right) \tag{8}$$

and

$$R_2 = u_2''(t) + f\left(u_2(t), u_2'(t), u_2''(t)\right) . \tag{9}$$

In order to use He's amplitude-frequency formulation [6 – 9], we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1 \cos(t)\, dt \ , \ T_1 = 2\pi \ , \tag{10}$$

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega t)\, dt \ , \ T_2 = \frac{2\pi}{\omega} . \tag{11}$$

Applying He's frequency-amplitude formulation [8 – 10] we have

$$\omega^2 = \frac{{\omega_1}^2 R_{22} - {\omega_2}^2 R_{11}}{R_{22} - R_{11}} \ , \tag{12}$$

where

$$\omega_1 = 1 \ , \ \omega_2 = \omega \ . \tag{13}$$

3. Results and discussion for small amplitudes $(0 \le A << 1)$.

Considering Eq. (3), for small values of A we can write

$$\frac{1}{\sqrt{(1+y^2)}} \approx \left(1 - \frac{1}{2}y^2\right), \quad 0 \le A << 1 \ . \tag{14}$$

We can write Eq. (3) in the form of

$$\frac{d^2 y}{d\tau^2} + y - \lambda y\left(1 - \frac{1}{2}y^2\right) = 0, \ \ 0 < \lambda \le 1 \ , \tag{15}$$

with initial conditions of

$$u(0) = A \ , \ u'(0) = 0 \ .$$

According to He's amplitude-frequency formulation [4 – 6]; we obtain, respectively, the following residuals:

$$R_1 = -\lambda A \cos(t)\left(1 - \frac{1}{2}A^2 \cos^2(t)\right) \tag{16}$$

and

$$R_2 = -A\cos(\omega\tau)\omega^2 + A\cos(\omega\tau) - \lambda A\cos(\omega\tau)\left(1 - \frac{1}{2}A^2\cos^2(\omega\tau)\right). \tag{17}$$

In the equations, the frequency of oscillation is ω and the amplitude of oscillation is A.

In order to use He's amplitude-frequency formulation [6 – 8], we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1 \cos(\tau)\, d\tau = \frac{1}{16}\frac{\lambda A\left(-8\pi + 3A^2\pi\right)}{\pi}\ ,\ T_1 = 2\pi \tag{18}$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega\tau)\, d\tau = \frac{1}{16}\frac{A\left(-8\omega^2\pi + 3\lambda A^2\pi + 8\pi - 8\lambda\pi\right)}{\pi},\ T_2 = \frac{2\pi}{\omega}. \tag{19}$$

Applying He's frequency-amplitude formulation [6 – 8], we, therefore, obtain the first order approximate solution for small amplitudes

$$\omega = \sqrt{1 - \lambda + \frac{3}{8}\lambda A^2}\ . \tag{20}$$

For the first approaching Eq. (20) at $A \to 0$ and for $\lambda \le 1$ we have $\lim_{A\to 0}\omega(A) = \sqrt{(1-\lambda)}$.

4. Results and discussion for large amplitudes (A>>0) .

Eq. (3) can be rewritten in a form that does not contain the square-root expression [2]

$$\left(\frac{d^2 y}{d\tau^2} + y\right)^2 - \frac{\lambda^2 y^2}{(1+y^2)} = 0\ , \tag{21}$$

with initial condition of

$$y(0) = A\ ,\ \frac{dy}{d\tau}(0) = 0\ .$$

According to He's amplitude-frequency formulation [4 – 6]; we obtain, respectively, the following residuals:

$$R_1 = \frac{-\lambda^2 A^2 \cos^2(\tau)}{1 + A^2\cos^2(\tau)} \tag{22}$$

and

$$R_2 = (-A\cos(\omega\tau)\omega^2 + A\cos(\omega\tau))^2 - \frac{\lambda^2 A^2 \cos^2(\omega\tau)}{1 + A^2\cos^2(\omega\tau)}. \tag{23}$$

In the equations, the frequency of oscillation is ω and the amplitude of oscillation is A.

In order to use He's amplitude-frequency formulation [6 – 8], we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(\tau)\, d\tau = -\frac{2\lambda^2\left(A\sqrt{1+A^2} - \operatorname{arctan} h\left(\frac{A}{\sqrt{1+A^2}}\right)\right)}{\pi A\sqrt{1+A^2}}\ ,\ T_1 = 2\pi \tag{24}$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega\tau)\, d\tau = -\frac{2}{3}\frac{1}{\pi A\sqrt{1+A^2}}(-2A^3\sqrt{1+A^2}\,\omega^4 -$$

$$-3\lambda^2 \operatorname{arctan} h\left(\frac{A}{\sqrt{1+A^2}}\right) + 4A^3\sqrt{1+A^2}\ \omega^2 - 2A^3\sqrt{1+A^2} + 3A\sqrt{1+A^2}\,\lambda^2),\ T_2 = \frac{2\pi}{\omega}\,. \quad (25)$$

Applying He's frequency-amplitude formulation [6 – 8], we, therefore, obtain the first order approximate solution

$$\omega = \frac{\sqrt{2}}{2A}\sqrt{\frac{2A^3\sqrt{1+A^2} - \sqrt{6A^6\lambda^2 + 6A^4\lambda^2 - 6A^3\sqrt{1+A^2}\,\lambda^2 \operatorname{arctan} h\left(\frac{A}{\sqrt{1+A^2}}\right)}}{A\sqrt{1+A^2}}}\,. \quad (26)$$

It is possible to solve Eq. (3) by applying the harmonic balance method. Following the first-order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (3) would be [2]

$$y = A\cos\omega\tau\,. \quad (27)$$

Substitution of Eq. (27) into Eq. (3) gives

$$-A\cos(\omega\tau)\omega^2 + A\cos(\omega\tau) - \frac{\lambda A\cos(\omega\tau)}{\sqrt{1+A\cos^2(\omega\tau)}} = 0\,. \quad (28)$$

The power-series expansion of $\frac{y}{\sqrt{1+y^2}}$ is [2]

$$\frac{y}{\sqrt{1+y^2}} = y + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!}{2^{n-1}n!(n-1)!}y^{2n+1} \quad (29)$$

Substituting Eq. (29) into Eq. (28) and taking into account Eq. (27) gives

$$-\cos(\omega\tau)\omega^2 + \cos(\omega\tau) - \lambda\cos(\omega\tau) - \lambda\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!}{2^{n-1}n!(n-1)!}A^{2n}\cos^{2n+1}(\omega\tau) = 0\,. \quad (30)$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2n+1}(\omega\tau) = \frac{1}{2^{2n}}\left\{\binom{2n+1}{n}\cos(\omega\tau) + \binom{2n+1}{n-1}\cos(3\omega\tau) + \ldots + \binom{2n+1}{0}\cos(2n+1)(\omega\tau)\right\}. \quad (31)$$

Substituting Eq. (31) into Eq. (30) gives

$$\left[-\omega^2 + 1 - \lambda\sum_{n=0}^{\infty} c_{2n+1}A^{2n}\right]\cos(\omega\tau) + (\text{high order harmonics})=0\,, \quad (32)$$

where the coefficients c_{2n+1} are given by

$$c_1 = 1$$

and

$$c_{2n+1} = (-1)^n \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n!)^2 (n-1)!(n+1)!} \text{ , } n \geq 1 . \tag{33}$$

For the first-order harmonic to be equal to zero, it is necessary to set the coefficient of $\cos(\omega\tau)$ equal to zero in Eq. (32), then

$$\omega = \sqrt{1 - \lambda \sum_{n=0}^{\infty} c_{2n+1} A^{2n}} \text{ .} \tag{34}$$

For the second-order harmonic we obtain [2]

$$\omega = \sqrt{1 - \lambda (f(A))^{-\frac{1}{2}}} \text{ ,} f(A) = 1 + \frac{3}{4} A^2 - \frac{3}{64} A^4 + \frac{13}{512} A^6 + \ldots \text{ .} \tag{35}$$

Table 1. Comparison of He’s frequency-amplitude formulation (Eq. (20)) with harmonic balance frequency (Eq. (35)) for small values of A . (λ =0.5)

Table 1

A	He’s frequency-amplitude formulation	harmonic balance frequency
0.01	0.7071	0.7071
0.05	0.7074	0.7075
0.1	0.7084	0.7085
0.2	0.7124	0.7126
0.3	0.7189	0.7190
0.5	0.7395	0.7369

Table 2. Comparison of He’s frequency-amplitude formulation (Eq. (20)) with harmonic balance frequency (Eq. (35)) for small values of A . (λ =1.0)

Table2

A	He’s frequency-amplitude formulation	harmonic balance frequency
0.01	0.0061	0.0063
0.05	0.0306	0.0316
0.1	0.0612	0.0630
0.2	0.1225	0.1249
0.3	0.1837	0.1844
0.5	0.3062	0.2935

The exact frequency can then be derived as follows [2]:

$$\omega_{exact} = \frac{\pi}{2} \left[\int_0^1 \frac{A du}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{1+A^2} - \sqrt{1+A^2u^2})}} \right]^{-1} \text{ .} \tag{36}$$

Now we are going to obtain an asymptotic representation for large amplitudes. We consider the expression for the exact frequency ω_{exact} Eq. (36) and we do the change $A = 1/B$. For large amplitudes $A \to \infty$ we have $B \to 0$. Taking this into account, and doing the power-series expansion of the result for small values of B , we obtain [2]

$$\omega_{exact} = \frac{\pi}{2} \left[\int_0^1 \frac{du}{\sqrt{1 - u^2 - 2B\lambda(\sqrt{1+B^2} - \sqrt{B^2+u^2})}} \right]^{-1} \tag{37}$$

and

$$\omega_{exact} \approx \frac{\pi}{2}\left[\int_0^1 \left(\frac{1}{\sqrt{1-u^2}} - \frac{\lambda B}{(1+u)\sqrt{1-u^2}} + \frac{3\lambda^2 B^2}{2(1+u)^2\sqrt{1-u^2}} + \ldots\right) du\right]^{-1}. \tag{38}$$

The power-series expansion for the exact frequency for small values of B (large values of A) is:

$$\omega_{exact} \approx \frac{\pi}{2}\left(\frac{\pi}{2} + \lambda B + \lambda^2 B^2 + \ldots\right)^{-1} \approx 1 - \frac{2\lambda}{\pi A} - \frac{2(\pi-2)\lambda^2}{\pi^2 A^2} + \ldots =$$

$$= 1 - \frac{0.63662\lambda}{A} - \frac{0.23134\lambda^2}{A^2} + \ldots\ . \tag{39}$$

Table 3

Comparison of He's frequency-amplitude formulation frequency (Eq. (26)) with exact frequency (Eq. (39)) for large amplitude ($\lambda = 0.5$).

A	He's frequency-amplitude formulation	Exact frequency	Error (%)
5	0.9398	0.9340	0.6176
6	0.9495	0.9453	0.4362
7	0.9565	0.9533	0.3305
8	0.9898	0.9593	0.2631
9	0.9618	0.9639	0.2171
10	0.9694	0.9676	0.1842

Table 4

Comparison of He's frequency-amplitude formulation frequency (Eq. (26)) with exact frequency (Eq. (39)) for large amplitude ($\lambda = 1$).

A	He's frequency-amplitude formulation	Exact frequency	Error (%)
5	0.8755	0.8634	1.3937
6	0.8961	0.8875	0.9706
7	0.9109	0.9043	0.0715
8	0.9221	0.9168	0.7283
9	0.9308	0.9264	0.5755
10	0.9377	0.9340	0.3984

The method of He's frequency-amplitude formulation is capable of producing analytical approximation to the solution to the nonlinear system, valid even for the case where the amplitude are not small but we see that harmonic balance solution is valid only for small amplitude.

5. Conclusions.

He's frequency-amplitude formulation has been used to solve nonlinear vibration system typified by a mass attached to a stretched wire. With the procedure, the analytical approximate frequency and the corresponding periodic solution, valid for small as well as large amplitudes of oscillation, can be obtained. The method, which is proved to be a powerful mathematical tool to the search for angular frequencies of nonlinear vibration systems, can be easily extended to any nonlinear equation, and the present letter can be used as paradigms for many other applications in searching for periodic solutions, limit cycles or other approximate solutions for real-life physics problems. We think that the method have great potential which still needs further development.

Р Е З Ю М Е . Амплітудно-частотний підхід Хе застосовано до вивчення періодичних розв'язків сильно нелінійних систем. Проілюстровано корисність і ефективність запропонованої методики. Результати порівняно з точними розв'язками і розв'язками, отриманими на основі енергетичного балансу. Порівняння показало добру точність. Наближено обчислені частоти виявилися вірними у всьому діапазоні амплітуд коливань. Продемонстровано і обговорено узгодженість між наближеними та точними значеннями частот.

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