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Nonlinear Dynamic Finite Element Analysis of Thick Transversly Flexible Sandwich Panel on Elastic Foundation with Account of Damage Progression in Time. Part 1. Three-Dimensional Formulation and Two-Dimensional Plate Theory

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Нелинейный динамический конечноэлементный анализ гибкой в поперечном направлении толстой многослойной панели на упругом основании с учетом развития повреждения во времени. Сообщение 1. Трехмерная формулировка задачи и двухмерная теория пластин

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С целью развития теории пластин для толстых многослойных панелей, сжимаемых в поперечном направлении, с наружными слоями в виде композитных ламинатных листов предложена упрощенная схема распределения поперечных деформаций по толщине панели. Предполагается, что поперечные деформации ε_{xz}, ε_{yz} *и* ε_{zz} *не изменяются по толщине панели в пределах ее наружных листов и сердцевины, но могут описываться различными функциональными зависимостями от координат в плоскостях разных субламинатов (наружные листы и сердцевина панели). Алгоритм, учитывающий развитие повреждения для динамических задач, используется в расчетной схеме, базирующейся на геометрически нелинейной формулировке, применительно к анализу разрушения многослойной пластины от ударного соприкосновения с грунтом. Модель многослойной пластины характеризуется малым количеством степеней свободы в конечноэлементных расчетах и широким спектром применения: для пластин с тонкими или толстыми наружными слоями (по сравнению с толщиной сердцевины), для случаев сжимаемости или несжимаемости наружных слоев и (или) сердцевины в поперечном направлении.*

Ключевые слова: сжимаемые в поперечном направлении многослойные панели, упругое основание, нелинейная динамика, трехмерное напряженное состояние.

Introduction. Sandwich structures are used in a variety of load-bearing applications. Sandwich plates have a well pronounced zigzag variation of the in-plane displacements in the thickness direction, due to their high thickness-to-length ratios and large difference of values of elastic moduli of the face sheets and the core. Such characteristics of the sandwich plates make it necessary to use a layerwise approach in their analysis, the idea of which is to introduce separate simplifying assumptions regarding the through-thickness variation of displacements, strains or stresses within each face sheet and the core. Many researchers studied the sandwich plate with thick, light-weight, vertically stiff cores and thin rigid face sheets, using discrete-layer (or layerwise) models. Most of the layerwise models of such structures are based on the piecewise linear through the thickness in-plane displacement approximations in addition to constant (through the thickness) transverse displacements [1–9].

The modern cores are usually made of plastic foams and non-metallic honeycombs, like Aramid and Nomex. These cores have properties similar to those used traditionally (for example, metallic honeycombs), but due to their transverse compressibility (i.e., ability of such cores to change height under applied loads) the direct transverse strain ε_{zz} becomes important. Therefore, the models of the sandwich plates with the cores made of plastic foams or non-metallic honeycombs must not exclude the change of height of the core. Frostig and Baruch [10] developed a theory of a sandwich beam with thin face sheets in which account is taken of transverse compressibility of the core, and the longitudinal displacement in the core varies nonlinearly in the thickness direction. In this theory the longitudinal displacement in the face sheets varies linearly in the thickness direction, and the transverse displacement of the face sheets does not vary in the thickness direction, that is, the transverse direct strain ε_{zz} in the face sheets is assumed to be equal to zero in the expression for the strain energy. The transverse shear strain ε_{xz} in the face sheets is also considered to be negligibly small in the expression for the strain energy that is used for variational derivation of the differential equations for the unknown functions. The transverse shear stress in the face sheets can be computed by integration of the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$.

Under certain circumstances, when the face sheets are thick, when the plate is loaded by a concentrated or partially distributed load, or when the plate is on an elastic foundation, taking account of the direct transverse strain ε_{zz} in the face sheets and the transverse shear strain ε_{xz} in the face sheets in the expression for the strain energy allows one to obtain a higher accuracy of the stress computation. Besides, in order to achieve a high accuracy of stress computation in the thick face sheets, a model for such a plate must assume or lead to the nonlinear through-the-thickness variation of the in-plane displacements not only in the core, but also in the face sheets.

Construction of a computational scheme that satisfies these requirements can be approached, for example, with the help of the layerwise laminated plate theory of Reddy [11], which is a generalization of many other displacement-based layerwise theories of laminated plates. In this theory the displacement field in the kth layer is written as

$$u^{(k)}(x,y,z,t) = \sum_{j=1}^{m} u_j^{(k)}(x,y,t)\phi_j^{(k)}(z),$$

$$v^{(\kappa)}(x,y,z,t) = \sum_{j=1}^{m} v_j^{(k)}(x,y,t)\phi_j^{(k)}(z),$$

$$w^{(k)}(x,y,z,t)=\sum_{j=1}^{n} w_j^{(k)}(x,y,t)\psi_j^{(k)}(z),$$

where $u_j^{(k)}(x,y,t)$, $v_j^{(k)}(x,y,t)$, and $w_j^{(k)}(x,y,t)$ are the unknown functions and $\phi_j^{(k)}(z)$ and $\psi_j^{(k)}(z)$ are chosen to be Lagrange interpolation functions of the thickness coordinate, to provide the required continuity of displacements and discontinuity of the transverse strains across the interface between adjacent thickness subdivisions.

This theory allows one to achieve a high accuracy of the transverse stress computation in the composite laminates, but for this purpose it requires a large number of thickness subdivisions of the laminate. This leads to a large number of the unknown functions and degrees of freedom in a finite element model. In effect, the finite element model, based on this generalized layerwise laminated plate theory is equivalent to the three-dimensional finite element model. To reduce the number of the unknown functions in the layerwise model of a laminated plate, one can use the concept of a sublaminate (i.e., make the number of thickness subdivisions less than the number of material layers) and deal with the material properties, averaged through the thickness of a sublaminate. In a model of the sandwich plate, it is natural to choose three sublaminates: the two face sheets and the core. With such a small number of the sublaminates, the nature of assumptions on the through-the-thickness variation of displacements can have a large effect on the accuracy of the computed stresses. Besides, the actual through-the-thickness variation of displacements can depend on the character of applied loads and boundary conditions. Therefore, in a layerwise model of the sandwich plate with only three sublaminates, it is desirable to have a flexibility in the choice of the functions that represent through-the-thickness variation of displacements. Of course, the Lagrange interpolation polynomials, which represent the thickness variation of the displacements within a sublaminate in the Reddy's [11] layerwise theory, can be chosen to be of any desired degree, but such an increase of the degree of the Lagrange interpolation polynomials leads to the increase of the number of the unknown functions.

In the present paper, a computational scheme for analysis of the sandwich plate is constructed in which the simplifying assumptions that lead to a plate-type theory are made with respect to the variation of the transverse strains in the thickness direction of the face sheets and the core of the sandwich plate. The displacements are then obtained by integration of these assumed transverse strains, and the constants of integration are chosen to satisfy the conditions of continuity of the displacements across the borders between the face sheets and the core. In such a method, the required continuity of displacements in the thickness direction is satisfied regardless of the assumed type of through-the-thickness distribution of the transverse strains, and the transverse flexibility of the plate can be taken into account. This leads to a larger number of choices of simplifying assumptions about the variation of strains (and, therefore, displacements) in the thickness direction, and, therefore, allows a better adjustment of the computational scheme to the conditions under which the sandwich plate is analyzed by a layerwise method with only three sublaminates (being the face sheets and the core). The transverse stresses are computed by integration of the pointwise equilibrium equations that leads to satisfaction of conditions of continuity of the transverse stresses across the boundaries between the face sheets and the core and satisfaction of stress boundary conditions on the upper and lower surfaces of the plate.

In the present paper, the model is considered on the basis of the simplest of such assumptions that do not ignore, in the expression for the strain energy, the transverse shear and normal strains in the face sheets. It is assumed here that the transverse strains do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinate in the face sheets and the core. In the post-process stage of analysis, these first approximations of the transverse strains can be improved by substituting the transverse stresses, obtained by integration of the pointwise equations of motion (Appendix) into the strain–stress relations. These improved values of the transverse strains vary in the thickness direction and are sufficiently accurate as compared to those of the known exact solutions, based on the linear three-dimensional theory [12]. In the theory, discussed in this paper, the transverse displacement, obtained by integration of the assumed transverse normal strain, varies linearly in the thickness direction within a sublaminate (therefore, transverse compressibility of the plate is taken into account), and the in-plane displacement, obtained by integration of the assumed transverse shear strains, varies quadratically within the thickness of a sublaminate. The developed theory does not require many degrees of freedom in finite element models, despite its ability to capture the transverse flexibility of the plate and non-linear through-the-thickness variation of the in-plane displacements.

Three-Dimensional Formulation. The sandwich plate is divided into three conventional layers (sublaminates): the two face sheets and the core. Within each sublaminate, the simplifying assumptions of the plate theory are made separately. In the following text, the superscript k denotes the number of a sublaminate: $k=1$ denotes the lower face sheet, $k=2$ denotes the core, and $k=3$ denotes the upper face sheet (Fig. 1).

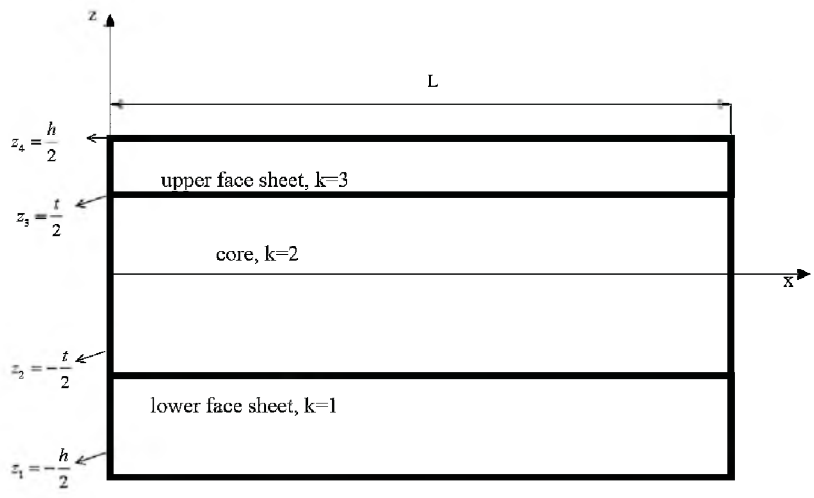

Fig. 1. The coordinate system and notations for the sandwich plate. (Axis z is in the thickness direction, h is a thickness of the whole plate, and t is the core thickness.)

In the subsequent text, both index and non-index notations for the displacements will be used interchangeably without a preliminary notice, the correspondence between them being established as follows: $u_1 = u$, $u_2 = v$, $u_3 = w$.

As energy-conjugate measures of strain and stress, the Green–Lagrange strain tensor and the second Piola–Kirchhoff stress tensor are used. The analysis is limited to the important case of small strains, moderate displacements (of the order of thickness of the plate) and moderate rotations (10–15 degrees). This means that of all the higher order terms in the Green–Lagrange strain–displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{s,i}u_{s,j}) \tag{1}$$

only $u_{3,\alpha}u_{3,\beta}$ $(\alpha, \beta = 1, 2)$ are not negligible compared to $u_{\alpha,i}$ $(\alpha = 1, 2;\ i = 1, 2, 3)$ [11, 13]. Therefore, the strain–displacement relations become

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}(w_{,x})^2, \tag{2}$$

$$\varepsilon_{yy} = v_{,y} + \frac{1}{2}(w_{,y})^2, \tag{3}$$

$$\varepsilon_{xy} = \frac{1}{2}(u_{,y} + v_{,x} + w_{,x}w_{,y}), \tag{4}$$

$$\varepsilon_{xz} = \frac{1}{2}(u_{,z} + w_{,x}), \tag{5}$$

$$\varepsilon_{yz}^{(k)} = \frac{1}{2}(v_{,z} + w_{,y}), \tag{6}$$

$$\varepsilon_{zz} = w_{,z}. \tag{7}$$

Now one needs to find the simplified equations of motion and boundary conditions, such that their accuracy corresponds to the accuracy of the adopted von Karman strain–displacement relations. These equations of motion are used for computation of the transverse stresses in the post-processing stage of the finite element analysis. The equations of motion and boundary conditions, consistent with the von Karman [13] strain–displacement relations (2)–(7), are received using the virtual work principle (see Appendix). The equations of motion are written for each of the three conventional layers: the upper and lower face sheets and the core. The boundary conditions are applied to the upper and lower surfaces of the plate and to the interfaces between the face sheets and the core, with the result that

$$\sigma_{xz}^{(1)} = 0, \qquad \sigma_{yz}^{(1)} = 0, \qquad \sigma_{zz}^{(1)} = -q_l \tag{8}$$

at the lower surface,

$$\sigma_{xz}^{(3)} = 0, \qquad \sigma_{yz}^{(3)} = 0, \qquad \sigma_{zz}^{(3)} = q_u \tag{9}$$

at the upper surface, and

$$\sigma_{xz}^{(1)}(z_2) = \sigma_{xz}^{(2)}(z_2), \qquad \sigma_{yz}^{(1)}(z_2) = \sigma_{yz}^{(2)}(z_2), \qquad \sigma_{zz}^{(1)}(z_2) = \sigma_{zz}^{(2)}(z_2), \tag{10}$$

$$\sigma_{xz}^{(2)}(z_3) = \sigma_{xz}^{(3)}(z_3) \qquad \sigma_{yz}^{(2)}(z_3) = \sigma_{yz}^{(3)}(z_3), \qquad \sigma_{zz}^{(2)}(z_3) = \sigma_{zz}^{(3)}(z_3). \tag{11}$$

at the interfaces.

In the laminate coordinate system (x, y, z), whose axes are aligned with the sides of the plate, the stress–strain relations for an orthotropic material have the form [11]

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\ & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\ & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{12}$$

or

$$\{\sigma\} = [\overline{C}]\{\varepsilon\}, \tag{13}$$

where $\overline{C}_{ij}$ are the elastic coefficients, referred to the laminate coordinate system.

In addition, the displacements must be continuous at the interfaces between the faces and the core:

$$u_i^{(1)}\Big|_{z=z_2} = u_i^{(2)}\Big|_{z=z_2}, \qquad u_i^{(2)}\Big|_{z=z_3} = u_i^{(3)}\Big|_{z=z_3}. \tag{14}$$

Two-Dimensional Plate Theory.

Simplifying Assumptions of the Plate Theory. To construct a two-dimensional plate theory, simplifying assumptions regarding a distribution of the transverse strain components in the thickness direction are made. It is assumed that within the face sheets and the core the transverse strains do not depend on the z-coordinate, but they can be different functions of coordinates x, y, and time t in different face sheets and the core:

$$\begin{cases} \varepsilon_{xz}^{(k)} = \varepsilon_{xz}^{(k)}(x, y, t), \\ \varepsilon_{yz}^{(k)} = \varepsilon_{yz}^{(k)}(x, y, t), \\ \varepsilon_{zz}^{(k)} = \varepsilon_{zz}^{(k)}(x, y, t), \\ (k = 1, 2, 3), \end{cases} \tag{15}$$

where the superscript k denotes the number of a sublaminate: $k=1$ denotes the lower face sheet, $k=2$ denotes the core, and $k=3$ denotes the upper face sheet. An accuracy of the theory, based on these assumptions, is studied in [12].

The assumed transverse strains of equations (15), together with displacements of the middle surface of the plate

$$u_0(x,y,t) \equiv u^{(2)}\big|_{z=0},$$
$$v_0(x,y,t) \equiv v^{(2)}\big|_{z=0}, \tag{16}$$
$$w_0(x,y,t) \equiv w^{(2)}\big|_{z=0}$$

are the unknown functions of the problem that will be computed by the finite element method. Therefore, all displacements, strains and stresses must be expressed in terms of these functions.

Displacements in Terms of the Unknown Functions. In order to obtain expressions for the displacements in terms of the unknown functions $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, u_0, v_0, and w_0, the strain–displacement relations (5)–(7) are integrated with the following result:

$$w^{(1)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)z_2 + \varepsilon_{zz}^{(1)}(x,y,t)(z - z_2) \quad (z_1 \le z \le z_2), \tag{17}$$

$$w^{(2)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)z, \tag{18}$$

$$w^{(3)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)z_3 + \varepsilon_{zz}^{(3)}(x,y,t)(z - z_3) \quad (z_3 \le z \le z_4), \tag{19}$$

$$u^{(1)} = u_0 + (2\varepsilon_{xz}^{(2)} - w_{0,x})z_2 - \frac{1}{2}\varepsilon_{zz,x}^{(2)} z_2^2 + (2\varepsilon_{xz}^{(1)} - w_{0,x} - \varepsilon_{zz,x}^{(2)} z_2)(z - z_2) - $$
$$- \frac{1}{2}\varepsilon_{zz,x}^{(1)}(z - z_2)^2 \qquad (z_1 \le z \le z_2), \tag{20}$$

$$u^{(2)} = u_0 + (2\varepsilon_{xz}^{(2)} - w_{0,x})z - \frac{1}{2}\varepsilon_{zz,x}^{(2)} z^2 \qquad (z_2 \le z \le z_3), \tag{21}$$

$$u^{(3)} = u_0 + (2\varepsilon_{xz}^{(2)} - w_{0,x})z_3 - \frac{1}{2}\varepsilon_{zz,x}^{(2)} z_3^2 +$$
$$+ (2\varepsilon_{xz}^{(3)} - w_{0,x} - \varepsilon_{zz,x}^{(2)} z_3)(z - z_3) - \frac{1}{2}\varepsilon_{zz,x}^{(3)}(z - z_3)^2 \qquad (z_3 \le z \le z_4), \tag{22}$$

$$v^{(1)} = v_0 + (2\varepsilon_{yz}^{(2)} - w_{0,y})z_2 - \frac{1}{2}\varepsilon_{zz,y}^{(2)} z_2^2 + (2\varepsilon_{yz}^{(1)} - w_{0,y} -$$

$$-\varepsilon_{zz,y}^{(2)} z_2)(z - z_2) - \frac{1}{2}\varepsilon_{zz,y}^{(1)}(z - z_2)^2 \qquad (z_1 \le z \le z_2), \tag{23}$$

$$v^{(2)} = v_0 + (2\varepsilon_{yz}^{(2)} - w_{0,y})z - \frac{1}{2}\varepsilon_{zz,y}^{(2)} z^2 \qquad (z_2 \le z \le z_3), \tag{24}$$

$$v^{(3)} = v_0 + (2\varepsilon_{yz}^{(2)} - w_{0,y})z_3 - \frac{1}{2}\varepsilon_{zz,y}^{(2)} z_3^2 + (2\varepsilon_{yz}^{(3)} - w_{0,y} - \varepsilon_{zz,y}^{(2)})(z - z_3) -$$

$$-\frac{1}{2}\varepsilon_{zz,y}^{(3)}(z - z_3)^2 \qquad (z_3 \le z \le z_4). \tag{25}$$

It can be verified easily that these expressions for the displacements are continuous across the boundaries between the faces and the core, that is, at $z = z_2$ and $z = z_3$.

Strains in Terms of the Unknown Functions. Expressions for the in-plane strains $\varepsilon_{xx}^{(k)}$, $\varepsilon_{xy}^{(k)}$, and $\varepsilon_{yy}^{(k)}$ in terms of the unknown functions are obtained by substituting expressions (17)–(25) for displacements in terms of the unknown functions into the strain–displacement relations (2), (3) and (4). The transverse strains $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, and $\varepsilon_{zz}^{(k)}$ are the unknown functions themselves.

Extended Hamilton's Principle, Written for This Specific Problem. In order to derive either differential equations for the unknown functions with boundary conditions, or the finite element formulation, one can use the extended Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T - \Pi)dt + \int_{t_1}^{t_2}\delta' W_{nc} dt = 0, \tag{26}$$

where T is a kinetic energy of the system, Π is a total potential energy of the system, and $\delta' W_{nc}$ is a virtual work of external non-conservative forces. Therefore, the extended Hamilton's principle for the sandwich plate on an elastic foundation can be written as follows:

$$\delta\int_{t_1}^{t_2} [(\text{kinetic energy of plate}) - (\text{strain energy of plate}) -$$

$$- (\text{strain energy of elastic foundation}) -$$

$$- (\text{potential energy of plate in gravity field})]dt +$$

$$+\int_{t_1}^{t_2}(\text{virtual work of damping forces})dt +$$

$$+\int_{t_1}^{t_2}(\text{virtual work of surface forces})dt = 0. \tag{27}$$

In order to derive the differential equations for the unknown functions with the boundary conditions, or to obtain a finite element formulation, all terms in the Hamilton's principle (27) need to be written in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(3)}$, and $\varepsilon_{zz}^{(3)}$.

Kinetic Energy of the Sandwich Plate. Considering that the mass density of the face sheets is constant, kinetic energy of the lower face sheet, core and the upper face sheet can be written as follows:

$$T^{(k)} = \frac{1}{2}\rho^{(k)}\iiint\limits_{(V^{(k)})}\begin{Bmatrix}\dot{u}^{(k)}\\ \dot{v}^{(k)}\\ \dot{w}^{(k)}\end{Bmatrix}^T\begin{Bmatrix}\dot{u}^{(k)}\\ \dot{v}^{(k)}\\ \dot{w}^{(k)}\end{Bmatrix}dV \qquad (k = 1, 2, 3), \tag{28}$$

where dots over the letters denote partial derivatives with respect to time. The displacements in equation (28) are expressed in terms of the unknown functions by formulas (17)–(25). The kinetic energy of the sandwich plate is the sum of kinetic energies of the face sheets and the core:

$$T = T^{(1)} + T^{(2)} + T^{(3)}. \tag{29}$$

Strain Energy of the Sandwich Plate. The face sheets of a sandwich plate are made from composite laminates, which are built up of fiber-reinforced plies. The orientation of the fibers can vary from ply to ply, and, therefore, values of the stiffness coefficients $\overline{C}_{ij}$ in the Hooke's law (referred to the laminate coordinate system) can vary from ply to ply in the face sheets. Let us introduce the following notation for a stiffness coefficient in the Hooke's law for a ply of the lower face sheet, in the laminate coordinate system:

$${}^{\alpha}\overline{C}_{ij}^{(1)}, \tag{30}$$

where the right superscript (1) denotes that a stiffness coefficient is associated with the first sublaminate (i.e., the lower face sheet), the left superscript α is a number of a ply in a lower face sheet, subscripts i and j denote a position of the stiffness coefficient in the stiffness matrix. The stiffness matrix with components ${}^{\alpha}\overline{C}_{ij}^{(1)}$ will be denoted as $[\overline{C}_{\alpha}^{(1)}]$. So, the strain energy of a lower face sheet's ply with a number α is

$$U_{\alpha}^{(1)} = \frac{1}{2}\iiint\limits_{(V_{\alpha}^{(1)})}\{\varepsilon^{(1)}\}^T[\overline{C}_{\alpha}^{(1)}]\{\varepsilon^{(1)}\}dV, \tag{31}$$

where $V_{\alpha}^{(1)}$ is volume of a ply with number α, of the lower face sheet, and the column-matrix of strains $\{\varepsilon^{(1)}\}$ is defined as follows:

$$\{\varepsilon^{(1)}\} \equiv [\varepsilon_{xx}^{(1)} \quad \varepsilon_{yy}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad 2\varepsilon_{yz}^{(1)} \quad 2\varepsilon_{xz}^{(1)} \quad 2\varepsilon_{xy}^{(1)}]^{T}. \tag{32}$$

Unlike the material coefficients ${}^{\alpha}\overline{C}_{ij}^{(1)}$, the strains do not have a subscript α, which denotes a number of a ply of the lower face sheet, because assumptions about through-the-thickness variation of strains are made for the whole lower face sheet, not for each individual ply of the lower face sheet. Therefore, each strain in the lower face sheet, as a function of z-coordinate, is represented with a single expression through the thickness of the lower face sheet.

The strain energy $U^{(1)}$ of the whole lower face sheet is a sum of strain energies of the plies of the lower face sheet:

$$U^{(1)} = \sum_{\alpha=1}^{n} U_{\alpha}^{(1)}. \tag{33}$$

Similarly, one can write an expression for the strain energy of the upper face sheet, $U^{(3)}$. The core of the sandwich plate is considered to be a homogeneous orthotropic medium. But the failure in the core can be distributed non-uniformly in the thickness direction. As a result of this, in the presence of failure, the coefficients $\overline{C}_{ij}$ of the stress–strain relation of the core can vary in the thickness direction. To take account of this, the core is nominally divided into layers parallel to the $x-y$-plane, such that within each layer the coefficients of the stress–strain relation can be considered approximately constant in the thickness direction. Thus, the core is treated as a laminated plate, the same way as the face sheets. The strain energy of the sandwich plate is the sum of the strain energies of the core and the face sheets

$$U = U^{(1)} + U^{(2)} + U^{(3)}. \tag{34}$$

Potential Energy of the Sandwich Plate in the Gravity Field. The potential energy of the sandwich plate in the gravity field, Π_g, is equal to the sum of potential energies of the lower face, $\Pi_g^{(1)}$, the core, $\Pi_g^{(2)}$, and the upper face, $\Pi_g^{(3)}$:

$$\Pi_g = \Pi_g^{(1)} + \Pi_g^{(2)} + \Pi_g^{(3)}, \tag{35}$$

where

$$\Pi_g^{(k)} = \rho^{(k)} g \int_0^B \int_0^L \int_{z_k}^{z_{k+1}} w^{(k)} dz dx dy. \tag{36}$$

Strain Energy of Elastic Foundation. The strain energy of the elastic foundation, modeled as a Winkler foundation, is defined by the expression

$$U_f = \frac{1}{2}\int_0^B \int_0^L s(x)\left(w^{(1)}\Big|_{z=z_1}\right)^2 dxdy, \tag{37}$$

where $s(x)$ is a modulus of the foundation.

Virtual Work of Surface Forces. It is assumed that the upper and lower surfaces of the plate are loaded by distributed forces in the transverse direction (along z-axis). Let $q_u(x,y,t)$ and $q_l(x,y,t)$ be forces per unit area in the transverse direction, acting on the plate's upper and lower surfaces respectively. Then the virtual work $\delta' W$ of these forces is

$$\delta' W_s = \int_0^B \int_0^L q_u(x,y,t)\delta w^{(3)}\Big|_{z=z_4} dxdy + \int_0^B \int_0^L q_l(x,y,t)\delta w^{(1)}\Big|_{z=z_1} dxdy. \tag{38}$$

Virtual Work of Damping Forces. The damping force per unit volume will be denoted as $\vec{\Phi}$. It will be considered, as it is generally accepted, that the damping force is proportional to the velocity. Then, for the kth sublaminate, one can write the following expression for the column-matrix of components of the damping force per unit volume:

$$\begin{Bmatrix} \Phi_x^{(k)} \\ \Phi_y^{(k)} \\ \Phi_z^{(k)} \end{Bmatrix} = -\beta^{(k)}\rho^{(\kappa)}\frac{\partial}{\partial t}\begin{Bmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{Bmatrix}, \tag{39}$$

where $\beta^{(k)}$ and $\rho^{(k)}$ are, respectively, a damping parameter and mass density of the kth sublaminate. The virtual work of the damping force in the kth sublaminate can be written as follows:

$$\delta' W_d^{(k)} = \int_0^B \int_0^L \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \delta u^{(k)} \\ \delta v^{(k)} \\ \delta w^{(k)} \end{Bmatrix}^T \begin{Bmatrix} \Phi_x^{(k)} \\ \Phi_y^{(k)} \\ \Phi_z^{(k)} \end{Bmatrix} dzdxdy. \tag{40}$$

The virtual work of the damping forces in the whole sandwich plate is the sum of the virtual works in the face sheets and the core:

$$\delta' W_d = \delta' W_d^{(1)} + \delta' W_d^{(2)} + \delta' W_d^{(3)}. \tag{41}$$

The extended Hamilton's principle, written in terms of the unknown functions, can be used for deriving either differential equations and boundary conditions for the unknown functions, or it can be used for a finite element formulation. In the following section, a finite element formulation for the sandwich plate in cylindrical bending will be developed.

Appendix.

Pointwise Equilibrium Equations Variationally Consistent with the von Karman Strain–Displacement Relations. In the equations of this Appendix, the upper superscripts (k), which denote the numbers of the sublaminates, will not be used, because these equations have a very general character and their validity is not limited to the layerwise plate theory, presented in this paper.

In order to derive pointwise equations of motion, consistent with the von Karman strain–displacement relations, let us substitute variations of the von Karman strain–displacement relations, written with the us of index notations,

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta}) \qquad (\alpha, \beta = 1, 2), \tag{A.1}$$

$$\varepsilon_{i3} = \frac{1}{2}(u_{i,3} + u_{3,i}) \qquad (i = 1, 2, 3), \tag{A.2}$$

into the virtual work principle

$$\iiint_{(V)} \sigma_{ij}\delta\varepsilon_{ij} dV = \iiint_{(V)} (\bar{F}_i - \rho\ddot{u}_i)\, \delta u_i \, dV + \iint_{(S)} \bar{t}_i \delta u_i \, dS, \tag{A.3}$$

where $\bar{F}_i$ are components of the body force per unit volume, ρ is density, and $\bar{t}_i$ are components of the surface traction. The variations of these strains of equations (A.1) and (A.2) have the form

$$\delta\varepsilon_{\alpha\beta} = \frac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha} + u_{3,\alpha}\delta u_{3,\beta} + u_{3,\beta}\delta u_{3,\alpha}) \qquad (\alpha = 1, 2;\ \beta = 1, 2), \tag{A.4}$$

$$\delta\varepsilon_{i3} = \frac{1}{2}(\delta u_{i,3} + \delta u_{3,i}) \qquad (i = 1, 2, 3). \tag{A.5}$$

Expression $\sigma_{ij}\delta\varepsilon_{ij}$ can be presented in the form

$$\begin{gathered} \sigma_{ij}\delta\varepsilon_{ij} = \sigma_{\alpha\beta}\delta\varepsilon_{\alpha\beta} + 2\sigma_{\alpha 3}\delta\varepsilon_{\alpha 3} + \sigma_{33}\delta\varepsilon_{33} \\ (\alpha = 1, 2;\ \beta = 1, 2;\ i = 1, 2, 3;\ j = 1, 2,3). \end{gathered} \tag{A.6}$$

Substitution of Eqs. (A.4) and (A.5) into Eq. (A.6) produces the result

$$\begin{gathered} \sigma_{ij}\delta\varepsilon_{ij} = \sigma_{ij}\delta u_{i,j} + \sigma_{\alpha\beta}u_{3,\alpha}\delta u_{3,\beta} \\ (\alpha = 1, 2;\ \beta = 1, 2;\ i = 1, 2, 3;\ j = 1, 2,3). \end{gathered} \tag{A.7}$$

If one substitutes expression (A.7) into the left-hand side of equation (A.3), one receives

$$\iiint\limits_{(V)} \sigma_{ij}\delta\varepsilon_{ij}dV = \iint\limits_{(S)} [\sigma_{\alpha j}n_j\delta u_\alpha + (\sigma_{3j}n_j + \sigma_{\alpha\beta}u_{3,\alpha}n_\beta)\delta u_3]dS -$$

$$-\iiint\limits_{(V)} \{\sigma_{\alpha j,j}\delta u_\alpha + [\sigma_{3j,j} + (\sigma_{\alpha\beta}u_{3,\alpha})_{,\beta}]\delta u_3\}dV \tag{A.8}$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ i = 1, 2, 3;\ j = 1, 2,3),$$

where n_1, n_2, and n_3 are components of the outward unit normal vector to the surface. The substitution of expression (A.8) into the virtual work principle (A.3) yields

$$0 = \iiint\limits_{(V)} \sigma_{ij}\delta\varepsilon_{ij}dV - \iiint\limits_{(V)} (\bar{F}_i - \rho\ddot{u}_i)\delta u_i dV - \iint\limits_{(S)} \bar{t}_i\delta u_i dS =$$

$$= \iint\limits_{(S)} [(\sigma_{\alpha j}n_j - \bar{t}_\alpha)\delta u_\alpha dS + (\sigma_{3j}n_j + \sigma_{\alpha\beta}u_{3,\alpha}n_\beta - \bar{t}_3)\delta u_3]dS -$$

$$-\iiint\limits_{(V)} \{(\sigma_{\alpha j,j} + \bar{F}_\alpha - \rho\ddot{u}_\alpha)\delta u_\alpha + [\sigma_{3j,j} + (\sigma_{\alpha\beta}u_{3,\alpha})_{,\beta} + \bar{F}_3 - \rho\ddot{u}_3]\delta u_3\}dV$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ j = 1, 2, 3). \tag{A.9}$$

If one equates to zero the coefficients of variations of displacements, one obtains the equations of motion

$$\sigma_{\alpha j,j} + \bar{F}_\alpha = \rho\ddot{u}_\alpha; \qquad \sigma_{3j,j} + (\sigma_{\alpha\beta}u_{3,\alpha})_{,\beta} + \bar{F}_3 = \rho\ddot{u}_3 \tag{A.10}$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ j = 1, 2, 3)$$

and natural boundary conditions

$$\sigma_{\alpha j}n_j = \bar{t}_\alpha; \qquad \sigma_{3j}n_j + \sigma_{\alpha\beta}u_{3,\alpha}n_\beta = \bar{t}_3 \qquad \text{at} \quad S_\sigma \tag{A.11}$$

$$(\alpha = 1, 2;\ \beta = 1, 2;\ j = 1, 2, 3),$$

where S_σ is a part of the surface on which displacement constraints are not imposed. Equations of motion (A.10) in expanded form are

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \bar{F}_x = \rho\ddot{u}, \tag{A.12}$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \bar{F}_y = \rho\ddot{v}, \tag{A.13}$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \frac{\partial}{\partial x}(\sigma_{xx}w_{,x} + \sigma_{yx}w_{,y}) +$$

$$+\frac{\partial}{\partial y}(\sigma_{xy}w_{,x}+\sigma_{yy}w_{,y})+\bar{F}_z=\rho\ddot{w}. \tag{A.14}$$

The boundary conditions (A.11) in expanded form are

$$\sigma_{xx}n_x+\sigma_{xy}n_y+\sigma_{xz}n_z=\bar{t}_x, \tag{A.15}$$

$$\sigma_{yx}n_x+\sigma_{yy}n_y+\sigma_{yz}n_z=\bar{t}_y, \tag{A.16}$$

$$\sigma_{zx}n_x+\sigma_{zy}n_y+\sigma_{zz}n_z+\sigma_{xx}w_{,x}n_x+\sigma_{yy}w_{,y}n_y+$$

$$+\sigma_{xy}(w_{,x}n_y+w_{,y}n_x)=\bar{t}_z. \tag{A.17}$$

In the postprocessing stage of the finite element analysis, the computation of the transverse stresses is done with the use of the pointwise equations of motion (A.12), (A.13) and (A.14), variationally consistent with the von Karman strain–displacement relations (2)–(7).

Резюме

Із метою розвитку теорії пластин для товстих багатошарових панелей, що стискувані у поперечному напряму, із зовнішніми шарами у вигляді композитних ламінатних листів запропоновано спрощену схему розподілу поперечних деформацій по товщині панелі. Припускається, що поперечні деформації ε_{xz}, ε_{yz} та ε_{zz} не змінюються по товщині панелі в інтервалі її зовнішніх листів і серцевини, але можуть описуватися різними функціональними залежностями від координати в площинах різних субламінатів (зовнішні листи і серцевина панелі). Алгоритм, що ураховує розвиток пошкодження для динамічних задач, використовується в розрахунковій схемі, що базується на геометрично нелінійному формулюванні, стосовно до аналізу руйнування багатошарової пластини від ударного стикання з грунтом. Модель багатошарової пластини характеризується малою кількістю степеней вільності у скінченноелементних розрахунках та широким використанням: для пластин із тонкими або товстими зовнішніми шарами (у порівнянні з товщиною серцевини), для випадків стисливості або нестисливості зовнішніх шарів та (або) серцевини в поперечному напряму.

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