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Recent progress in Subset Combinatorics of Groups

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Abstract. We systematize and analyze some results obtained in Subset Combinatorics of G groups after publications the previous surveys [1–4]. The main topics: the dynamical and descriptive characterizations of subsets of a group relatively their combinatorial size, Ramsey-product subsets in connection with some general concept of recurrence in G-spaces, new ideals in the Boolean algebra $\mathcal{P}_G$ of all subsets of a group G and in the Stone-Čech compactification βG of G, the combinatorial derivation.

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1. Introduction

In this paper, we systematize and analyze some results obtained in *Subset Combinatorics of Groups* after publications the surveys [1–4]. The main topics: the descriptive and dynamical characterizations of subsets of a group with respect to their combinatorial size, Ramsey-product subsets in connection with some general concept of recurrence, new ideals in the Boolean algebra $\mathcal{P}_G$ of all subsets of G and in the Stone-Čech compactification βG of G, the combinatorial derivation.

In these investigations, the principal part play ultrafilters on a group G. On one hand, ultrafilters are using as a tool to get some purely combinatorial results. On the other hand, the *Subset Combinatorics of Groups* allows to prove new facts about ultrafilters, in particular, about the Stone-Čech compactification βG of G. In this connection, we recall some basic definitions concerning ultrafilters.

A *filter* $\mathcal{F}$ on a set X is a family of subsets of X such that

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- $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$;
- $A, B \in \mathcal{F} \Longrightarrow A \bigcap B \in \mathcal{F}$;
- $A \in \mathcal{F}$, $A \subseteq C \Longrightarrow C \in \mathcal{F}$.

The family of all filters on X is partially ordered by inclusion. A filter maximal in this ordering is called an *ultrafilter*. A filter $\mathcal{F}$ is an ultrafilter if and only if $X = A \bigcup B$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Now we endow X with the discrete topology and identity the Stone-Čech compactification βX with the set of all ultrafilters on X. An ultrafilter $\mathcal{F}$ is principal if there exists $x \in X$ such that $\mathcal{F} = \{A \subseteq X : x \in A\}$. Otherwise, $\bigcap \mathcal{F} = \emptyset$ and $\mathcal{F}$ is called free. Thus, X is identified with the set of all principal ultrafilters and the set of all free ultrafilter on X is denoted by X^*.

To describe the topology on βX, given any $A \subseteq X$ we denote $\bar{A} = \{\mathcal{F} \in X : A \in \mathcal{F}\}$. Then the set $\{\bar{A} : A \subseteq X\}$ is a base for the topology of X. The characteristic topological property of βX: every mapping $f : X \longrightarrow K$, K is a compact Hausdorff space, can be extended to the continuous mapping $f^{\beta} : \beta X \longrightarrow K$.

Given a filter φ on X, the set $\bar{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$ is closed in βX, and for every non-empty closed subset K of βX, there is a filter φ on X such that $\bar{\varphi} = K$.

Now let G be a discrete group. Using the characteristic property of βG, we can extend the group multiplication on G to the semigroup multiplication on βG in such a way that, for every $g \in G$, the mapping $\beta G \longrightarrow G : p \longmapsto gp$ is continuous and, for every $q \in \beta G$, the mapping $\beta G \longrightarrow \beta G : p \longmapsto pq$ is continuous.

To define the product pq of ultrafilters p and q, we take an arbitrary $P \in p$ and, for each $x \in P$, pick some $Q_x \in q$. Then, $\bigcup_{x \in P} xQ_x$ is a member of pq, and each member of pq contains some subsets of this form.

For properties of the compact right topological semigroup βG and a plenty of its combinatorial application see [5].

2. Diversity of subsets and ultracompanions

Let G be a group with the identity e, $\mathcal{F}_G$ denotes the family of all finite subsets of G. We say that a subset A of G is

- *large* if $G = FA$ for some $F \in \mathcal{F}_G$;
- *small* if $L \setminus A$ is large for every large subset L;

- *extralarge* if $G \setminus A$ is small;
- *thin* if $gA \cap A$ is finite for each $g \in G \setminus \{e\}$;
- *thick* if, for every $F \in \mathcal{F}_G$, there exists $a \in A$ such that $Fa \subseteq A$;
- *prethick* if FA is thick for some $F \in \mathcal{F}_G$;
- *n-thin*, $n \in \mathbb{N}$ if, for every distinct elements $g_0, \dots, g_n \in G$, the set $g_0A \cap \dots \cap g_nA$ is finite;
- *sparse* if, for every infinite subset X of G, there exists a finite subset $F \subset X$ such that $\bigcap_{g \in F} gA$ is finite.

Remark 2.1. In *Topological dynamics*, large subsets are known as syndetic, and a subset is small if and only if it fails to be piecewise syndetic. In [4], the authors use the dynamical terminology.

All above definitions can be unified with usage the following notion [6]. Given a subset A of a group G and an ultrafilter $p \in G^*$, we define a *p-companion* of A by

$$\Delta_p(A) = A^* \cap Gp = \{gp : g \in G,\ A \in gp\}.$$

Then, for every infinite group G, the following statement hold:

- A is large if and only if $\Delta_p(A) \neq \emptyset$ for each $p \in G^*$;
- A is small if and only if, for every $p \in G^*$ and every $F \in \mathcal{F}_G$, we have $\Delta_p(FA) \neq Gp$;
- A is thick if and only if, there exist $p \in G^*$ such that $\Delta_p(A) = Gp$;
- A is thin if and only if, $\Delta_p(A) \leq 1$ for every $p \in G^*$;
- A is n-thin if and only if, $\Delta_p(A) \leq n$ for every $p \in G^*$;
- A is sparse if and only if, $\Delta_p(A)$ is finite for each $p \in G^*$.

Following [1], we say that a subset A of G is *scattered* if, for every infinite subset X of A, there is $p \in X^*$ such that $\Delta_p(X)$ is finite. Equivalently [7, Theorem 1], A is scattered if each subset $\Delta_p(A)$ is discrete in G^*.

Comments. For motivations of above definitions see [1], for more delicate classification of subsets of a group and G-spaces see [2, 8].

3. The descriptive look at the size of subsets of groups

Given a group G, we denote by $\mathbf{P}_G$ and $\mathbf{F}_G$ the Boolean algebra of all subsets of G and its ideal of all finite subsets. We endow $\mathbf{P}_G$ with the topology arising from identification (via characteristic functions) of $\mathbf{P}_G$ with $\{0,1\}^G$. For $K \in F_G$ the sets

$$\{X \in \mathbf{P}_G : K \subseteq X\}, \quad \{X \in \mathbf{P}_G : X \cap K = \emptyset\}$$

form the subbase of this topology.

After the topologization, each family $\mathcal{F}$ of subsets of a group G can be considered as a subspace of $\mathbf{P}_G$, so one can ask about the Borel complexity of $\mathcal{F}$, the question typical in the *Descriptive Set Theory* (see [9]). We ask these questions for the most intensively studied families in *Combinatorics of Groups.*

For a group G, we denote by $\mathbf{L}_G$, $\mathbf{EL}_G$, $\mathbf{S}_G$, $\mathbf{T}_G$, $\mathbf{PT}_G$ the sets of all large, extralarge, small, thick and prethick subsets of G, respectively.

Theorem 3.1. *For a countable group G, we have: $\mathbf{L}_G$ is F_σ, $\mathbf{T}_G$ is G_δ, $\mathbf{PT}_G$ is $G_{\delta\sigma}$, $\mathbf{S}_G$ and $\mathbf{EL}_G$ are $F_{\sigma\delta}$.*

A subset A of a group G is called

- *P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the subsets $\{g_nA : n \in \omega\}$ are pairwise disjoint;
- *weakly P-small* if, for any $n \in \omega$, there exists $g_0, \dots, g_n$ such that the subsets $g_0A, \dots, g_nA$ are pairwise disjoint;
- *almost P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that $g_nA \cap g_mA$ is finite for all distinct n, m;
- *near P-small* if, for every $n \in \omega$, there exists $g_0, \dots, g_n$ such that $g_iA \cap g_jA$ is finite for all distinct $i, j \in \{0, \dots, n\}$.

Every infinite group G contains a weakly P-small set, which is not P-small, see [10]. Each almost P-small subset can be partitioned into two P-small subsets [8]. Every countable Abelian group contains a near P-small subset which is neither weakly nor almost P-small [11].

Theorem 3.2. *For a countable group G, the sets of thin, weakly P-small and near P-small subsets of G are $F_{\delta\sigma}$.*

We recall that a topological space X is *Polish* if X is homeomorphic to a separable complete metric space. A subset A of a topological space

X is *analytic* if A is a continuous image of some Polish space, and A is *coanalytic* if $X \setminus A$ is analytic.

Using the classical tree technique [9] adopted to groups in [12], we get.

Theorem 3.3. *For a countable group G, the ideal of sparse subsets is coanalytic and the set of P-small subsets is analytic in $\mathbf{P}_G$.*

Given a discrete group G, we identify the Stone-Čech compactification βG with the set of all ultrafilters on G and consider βG as a right-topological semigroup (see Introduction). Each non-empty closed subspace X of βG is determined by some filter φ on G:

$$X = \bigcap\{\overline{\Phi} : \Phi \in \varphi\}, \quad \overline{\Phi} = \{p \in \beta G : \Phi \in p\}.$$

On the other hand, each filter φ on G is a subspace of $\mathbf{P}_G$, so we can ask about complexity of X as the complexity of φ in $\mathbf{P}_G$.

The semigroup βG has the minimal ideal K_G which play one of the key parts in combinatorial applications of βG. By [5], Theorem 1.5, the closure $cl(K_G)$ is determined by the filter of all extralarge subsets of G. If G is countable, applying Theorem 3.1, we conclude that $cl(K_G)$ has the Borel complexity $F_{\sigma\delta}$.

An ultrafilter p on G is called *strongly prime* if $p \notin cl(G^*G^*)$, where G^* is a semigroup of all free ultrafilters on G. We put $X = cl(G^*G^*)$ and choose the filter φ_X which determine X. By [13], $A \in \varphi_X$ if and only if $G \setminus A$ is sparse. If G is countable, applying Theorem 3.3, we conclude that φ_X is coanalitic in $\mathbf{P}_G$.

Let $(g_n)_{n\in\omega}$ be an injective sequence in G. The set

$$\{g_{i_1} g_{i_2} \dots g_{i_n} : 0 \le i_1 < i_2 < \dots < i_n < \omega\}$$

is called an *FP-set*. By the Hindman Theorem 5.8 [5], for every finite partition of G, at least one cell of the partition contains an FP-set. We denote by $\mathbf{FP}_G$ the family of all subsets of G containing some FP-set. A subset A of G belongs to $\mathbf{FP}_G$ if and only if A is an element of some idempotent of βG. By analogy with Theorem 3.3, we can prove that $\mathbf{FP}_G$ is analytic in $\mathbf{P}_G$.

Comments. This section reflects the results from [14].

4. The dynamical look at the subsets of a group

Let G be a group. A topological space X is called a *G-space* if there is the action $X \times G \longrightarrow X : (x, g) \longmapsto xg$ such that, for each $g \in G$, the mapping $X \longrightarrow X : x \longmapsto xg$ is continuous.

Given any $x \in X$ and $U \subseteq X$, we set

$$[U]_x = \{g \in G : xg \in U\}$$

and denote

$$O(x) = \{xg : g \in G\}, T(x) = clO(x),$$

$W(x) = \{y \in T(X) : [U]_x$ is infinite for each neighbourhood U of $y\}$.

We recall also that $x \in X$ is a *recurrent point* if $x \in W(x)$.

Now we identify $\mathcal{P}_G$ with the space $\{0,1\}^G$, endow $\mathcal{P}_G$ with the product topology and consider $\mathcal{P}_G$ as a G-space with the action defined by

$$A \longmapsto Ag,\ Ag = \{ag : a \in A\}.$$

We say that a subset A of G is *recurrent* if A is a recurrent point in $(\mathcal{P}_G, G)$.

All groups in this sections are supposed to be infinite.

Theorem 4.1. *For a subset A of a group G, the following statements hold*

(i) A is finite if and only if $W(A) = \emptyset$;

(ii) A is thick if and only if $G \in W(A)$.

Theorem 4.2. *For a subset A of a group G, the following statements hold*

(i) A is n-thin if and only if $|Y| \le n$ for every $Y \in W(A)$;

(ii) A is sparse if and only if each subset $Y \in W(A)$ is finite;

(iii) A is scattered if and only if, for every subset $B \subseteq A$ there exists $Y \in \mathcal{F}_G$ in the closure of $\{Bb^{b^1} : b \in B\}$.

Let $(g_n)_{n\in\omega}$ be an injective sequence in G. The set

$$FP(g_n)_{n\in\omega} = \{g_{i_1}g_{i_2}\dots g_{i_n} : 0 \le i_1 < i_2 < \dots < i_n < \omega\}$$

is called an FP-set.

Given a sequence $(b_n)_{n\in\omega}$ in G, the set

$$\{g_{i_1}g_{i_2}\dots g_{i_n}b_{i_n} : 0 \le i_1 < i_2 < \dots < i_n < \omega\}$$

is called a *(right) piecewise shifted FP*-set [7].

Theorem 4.3. *For a subset A of a group G, the following statements hold*

(i) A is not n-thin if and only if there exist $F \in [G]^{n+1}$ and an injective sequence $(x_n)_{n<\omega}$ in G such that $Fx_n \subseteq A$ for each $n \in \omega$;

(ii) A is not sparse if and only if there exists two injective sequences $(x_n)_{n<\omega}$ and $(y_n)_{n<\omega}$ such that $x_n y_m \in A$ for each $0 \le n \le m < \omega$;

(iii) A is not scattered if and only if A contains a piecewise shifted FP-set;

(iv) A contains a recurrent subset if and only if there exists $x \in A$ and an FP-set Y such that $xY \subseteq A$.

Corollary 4.1. *Every scattered subset of a group G has no recurrent points.*

Remark 4.1. By [4, Theorem 2], every scattered subset A of an amenable group G is absolute null, i.e. $\mu(A) = 0$ for every left invariant Banach measure μ on G. But this statement could not be generalized to subsets with no recurrent points. By [17, Theorem 11.6], there is a subset A of $\mathbb{Z}$ of positive Banach measure such that $(a + B) \setminus A \neq \emptyset$ for any FP-set B. By Theorem 4.3(iv), A has no recurrent subsets.

Remark 4.2. Let G be an arbitrary infinite group. In [15], we constructed two injective sequences $(x_n)_{n\in\omega}$, $(y_n)_{n\in\omega}$ in G such the set $\{x_n y_m : 0 \le n \le m < \omega\}$ is scattered. By Theorem 4.3(ii), this subset is not sparse.

Comments. This section reflects the first part of [15].

5. Ramsey-product subsets and recurrence

In this section, all groups under consideration are supposed to be infinite; a countable set means a countably infinite set.

Let G be a group and let $\overrightarrow{m} = (m_1 \dots, m_k) \in \mathbb{Z}^k$ be a number vector of length $k \in \mathbb{N}$. We say that a subset A of a group G is a *Ramsey $\overrightarrow{m}$-product subset* if every infinite subset X of G contains pairwise distinct elements $x_1, \dots, x_k \in X$ such that,

$$x_{\sigma(1)}^{m_1}\, x_{\sigma(2)}^{m_2} \dots x_{\sigma(k)}^{m_k} \in A$$

for every substitution $\sigma \in S_k$.

Theorem 5.1. *For a group G and a number vector $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$, the following statements hold:*

(i) a subset A of G is a Ramsey $\overrightarrow{m}$-product subset if and only if every infinite subset X of G contains a countable subset Y such that $y_1^{m_1} \dots y_k^{m_k} \in A$ for any distinct elements $y_1, \dots, y_k \in Y$.

(ii) the family $\varphi_{\overrightarrow{m}}$ of all Ramsey $\overrightarrow{m}$-product subsets of G is a filter.

For $t \in \mathbb{Z}$ and $q \in G^*$ we denote by $q^\wedge t$ the ultrafilter with the base $\{x^t : x \in Q\}$, $Q \in q$. Warning: $q^\wedge t$ and q^t are different things. Certainly, $q^\wedge t = q^t$ only if $t \in \{-1, 0, 1\}$.

We remind the reader that, for a filter φ on G, $\overline{\varphi} = \{p \in \beta G : \varphi \subseteq p\}$.

Theorem 5.2. *For every group G and any number vector $\overrightarrow{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$, we have*

$$\overline{\varphi}_{\overrightarrow{m}} = cl\{(q^\wedge m_1) \ \dots \ (q^\wedge m_k) : \ q \in \ G^*\}.$$

Now we consider some special cases of vectors $\vec{m}$.

Proposition 5.1. *For any totally bounded topological group G, any neighborhood U of the identity e of G is a Ramsey $\vec{m}$-product subset for any vector $\vec{m} = (m_1, \dots, m_k)$ such that $m_1 + \dots + m_k = 0$.*

We recall that a *quasi-topological group* is a group G endowed with a topology such that, for any $a, b \in G$ and $\varepsilon \in 1, 1$, the mapping $G \longrightarrow G : x \longmapsto a x^\varepsilon b$, is continuous.

Proposition 5.2. *The closure $\bar{A}$ of any Ramsey $(-1, 1)$-product set A in a quasi-topological group G is a neighborhood of the identity.*

Proposition 5.3. *Let $\vec{m} = (m_1, \dots, m_k)$ be a number vector and $s = m_1 + \dots + m_k$. For any Ramsey $\vec{m}$-product subset A of a group G, the set $\{x^s : x \in G\}$ is contained in the closure of A in any non-discrete group topology on G.*

Proposition 5.4. *Let G be the Boolean group of all finite subsets of $\mathbb{Z}$, endowed with the group operation of symmetric difference. The set*

$$A = G \setminus \{\{x, y\} : x, y \in \mathbb{Z}, 0 \neq x - y \in \{z^3 : z \in \mathbb{Z}\}\}$$

has the following properties:

(i) A is a Ramsey $\vec{m}$-product for any vector $\vec{m} = (m_1, \ldots, m_k) \in (2\mathbb{Z}+1)^k$ of length $k \geq 2$;

(ii) A does not contain the difference BB^{-1} of any large subset B of G;

(iii) A is not a neighborhood of zero in a totally bounded group topology on G.

Now we show how Ramsey $(-1,1)$-product sets arise in some general concept of recurrence on G-spaces.

Let G be a group with the identity e and let X be a G-space with the action $G \times X \longrightarrow X$, $(g,x) \longmapsto gx$. If $X = G$ and gx is the product of g and x then X is called a *left regular G-space.*

Given a G-space X, a family $\mathfrak{F}$ of subset of X and $A \in \mathfrak{F}$, we denote

$$\Delta_{\mathfrak{F}}(A) = \{g \in G : gB \subseteq A \text{ for some } B \in \mathfrak{F}, B \subseteq A\}.$$

Clearly, $e \in \Delta_{\mathfrak{F}}(A)$ and if $\mathfrak{F}$ is upward directed ($A \in \mathfrak{F}$, $A \subseteq C$ imply $C \in \mathfrak{F}$) and if $\mathfrak{F}$ is G-invariant ($A \in \mathfrak{F}$, $g \in G$ imply $gA \in \mathfrak{F}$) then

$$\Delta_{\mathfrak{F}}(A) = \{g \in G : gA \cap A \in \mathfrak{F}\}, \Delta_{\mathfrak{F}}(A) = (\Delta_{\mathfrak{F}}(A))^{-1}.$$

If X is a left regular G-space and $\emptyset \notin \mathfrak{F}$ then $\Delta_{\mathfrak{F}}(A) \subseteq AA^{-1}$.

For a G-space X and a family $\mathfrak{F}$ of subsets of X, we say that a subset R of G is *$\mathfrak{F}$-recurrent* if $\Delta_{\mathfrak{F}}(A) \cap R \neq \emptyset$ for every $A \in \mathfrak{F}$. We denote by $\mathfrak{R}_{\mathfrak{F}}$ the filter on G with the base $\cap\{\Delta_{\mathfrak{F}'}(A) : A \in \mathfrak{F}'\}$, where $\mathfrak{F}'$ is a finite subfamily of $\mathfrak{F}$, and note that, for an ultrafilter p on G, $\mathfrak{R}_{\mathfrak{F}} \in p$ if and only if each member of p is $\mathfrak{F}$-recurrent.

The notion of an $\mathfrak{F}$-recurrent subset is well-known in the case in which G is an amenable group, X is a left regular G-space and $\mathfrak{F} = \{A \subseteq X : \mu(A) > 0$ for some left invariant Banach measure μ on $X\}$. See [16–18] for historical background.

We recall [19] that a filter φ on a group G is *left topological* if φ is a base at the identity e for some (uniquely defined) left translation invariant (each left shift $x \longmapsto gx$ is continuous) topology on G. If φ is left topological then $\overline{\varphi}$ is a subsemigroup of βG [19]. If $G = X$ and a filter φ is left topological then $\varphi = \mathfrak{R}_{\varphi}$.

Proposition 5.5. *For every G-space X and any family $\mathfrak{F}$ of subsets of X, the filter $\mathfrak{R}_{\mathfrak{F}}$ is left topological.*

Let X be a G-space and let $\mathfrak{F}$ be a family of subsets of X. We say that a family $\mathfrak{F}'$ of subsets of X is *$\mathfrak{F}$-disjoint* if $A \cap B \notin \mathfrak{F}$ for any distinct $A, B \in \mathfrak{F}'$.

A family $\mathfrak{F}'$ of subsets of X is called $\mathfrak{F}$-*packing large* if, for each $A \in \mathfrak{F}'$, any $\mathfrak{F}$-disjoint family of subsets of X of the form gA, $g \in G$ is finite.

Proposition 5.6. *Let X be a G-space and let $\mathfrak{F}$ be a G-invariant upward directed family of subsets of X. Then $\mathfrak{F}$ is $\mathfrak{F}$-packing large if and only if, for each $A \in \mathfrak{F}$, the set $\triangle_{\mathfrak{F}}(A)$ is a Ramsey (-1,1)-product set.*

Applying Theorem 5.2, we conclude that $\triangle_{\mathfrak{F}}(A)$ contains all ultrafilters of the form $q^{-1}q$, $q \in G^*$, and in the case $X = G$, G is amenable and $\mathfrak{F}$ is the family of all subsets of positive Banach measure, we get Theorem 3.14 from [18].

Comments. The proofs of all above statements can be find in [20, 21].

6. Ideals in $\mathcal{P}_G$ and βG

We recall that a family $\mathcal{I}$ of subsets of a set X is an *ideal* in the Boolean algebra $\mathcal{P}_G$ of all subsets of G if $G \notin \mathcal{I}$ and $A \in \mathcal{I}$, $B \in \mathcal{I}$, $C \subseteq A$ imply $A \cup B \in \mathcal{I}$, $C \in \mathcal{I}$. A family φ of subsets of G is a filter if and only if the family $\{X \setminus A : A \in \varphi\}$ is an ideal.

For an infinite group G, an ideal $\mathcal{I}$ in $\mathcal{P}_G$ is called *left (right) translation invariant* if $gA \in \mathcal{I}$ ($Ag \in \mathcal{I}$) for all $g \in G$, $A \in \mathcal{I}$. If $\mathcal{I}$ is left and right translation invariant then $\mathcal{I}$ is called *translation invariant.* Clearly, each left (right) translation invariant ideal of G contains the ideal $\mathcal{F}_G$ of all finite subsets of G. An ideal $\mathcal{I}$ in $\mathcal{P}_G$ is called a *group ideal* if $\mathcal{F}_G \subseteq \mathcal{I}$ and if $A \in \mathcal{I}$, $B \in \mathcal{I}$ then $AB^{-1} \in \mathcal{I}$.

Now we endow G with the discrete topology and use the standard extension of the multiplication on G to the semigroup multiplication on βG, see Introduction.

It follows directly from the definition of the multiplication in βG that G^*, $\overline{G^*G^*}$ are ideals in the semigroup βG, and G^* is the unique maximal closed ideal in βG. By Theorem 4.44 from [5], the closure $\overline{K(\beta G)}$ of the minimal ideal $K(G)$ of βG is an ideal, so $\overline{K(\beta G)}$ is the smallest closed ideal in βG. For the structure of $\overline{K(\beta G)}$ and some other ideals in βG see [5, Sections 4, 6].

For an ideal $\mathcal{I}$ in $\mathcal{P}_G$, we put

$$\mathcal{I}^\wedge = \{p \in \beta G : G \setminus A \in p \text{ for each } A \in \mathcal{I}\},$$

and use the following observations:

- $\mathcal{I}$ is left translation invariant if and only if $\mathcal{I}^\wedge$ is a left ideal of the semigroup βG ;

- $\mathcal{I}$ is right translation invariant if and only if $(\mathcal{I}^{\wedge})G \subseteq \mathcal{I}^{\wedge}$.

We use also the inverse to $^{\wedge}$ mapping $^{\vee}$. For a closed subset K of βG, we take the unique filter φ on G such that $K = \overline{\varphi}$ and put

$$K^{\vee} = \{G \setminus A : A \in \varphi\}.$$

In this section, all groups under consideration are suppose to be infinite.

We denote by Sm_G, Sc_G, Sp_G the families of all small, scattered and sparse subsets of a group G. These families are translation invariant ideals in $\mathcal{P}_G$ (see [6, Proposition 1]), and for every group G, the following inclusions are strict [6, Proposition 12]

$$Sp_G \subset Sc_G \subset Sm_G.$$

We say that a subset A of G is *finitely thin* if A is n-thin for some $n \in \mathbb{N}$. The family FT_G of all finitely thin subsets of G is a translation invariant ideal in $\mathcal{P}_G$ which contains the ideal $< T_G >$ generated by the family of all thin subsets of G. By [22, Theorem 1.2] and [23, Theorem 3], if G is either countable or Abelian and $|G| < \aleph_\omega$ then $FT_G =< T_G >$. By [23, Example 3], there exists an Abelian group G of cardinality $\aleph_\omega$ such that $< T_G >\subset FT_G$.

Theorem 6.1. *For every group G, we have $Sm_G^{\wedge} = \overline{K(\beta G)}$.*

This is Theorem 4.40 from [5] in the form given in [24, Theorem 12.5].

Theorem 6.2. *For every group G, $Sp_G^{\wedge} = \overline{G^*G^*}$.*

This is Theorem 10 from [13].

6.1. Between $\overline{G^*G^*}$ and G^*.

Theorem 6.3. *For every group G, the following statements hold:*

(i) if $\mathcal{I}$ is a left translation invariant ideal in $\mathcal{P}_G$ and $\mathcal{I} \neq \mathcal{F}_G$ then there exists a left translation invariant ideal $\mathcal{J}$ in $\mathcal{P}_G$ such that $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ and $\mathcal{J} \subset Sp_G$;

(ii) if $\mathcal{I}$ is a right translation invariant ideal in $\mathcal{P}_G$ and $\mathcal{I} \neq \mathcal{F}_G$ then there exists a right translation invariant $\mathcal{J}$ in $\mathcal{P}_G$ such that $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$;

(iii) if G is either countable or Abelian and $\mathcal{I}$ is a translation invariant ideal in $\mathcal{P}_G$ such that $\mathcal{I} \neq \mathcal{F}_G$ then there exists a translation invariant ideal $\mathcal{J}$ in $\mathcal{P}_G$ such that $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ and $\mathcal{J} \subset Sp_G$.

Theorem 6.4. *For every group G, the following statements hold:*

(i) if L is a closed left ideal in βG such that $L \subset G^*$ then there exists a closed left ideal L' of βG such that $L \subset L' \subset G^*$, $\overline{G^*G^*} \subset L'$;

(ii) if R is a closed subset of G^* such that $R \neq G^*$ and $RG \subseteq R$ then there exists a closed subset R' of G^* such that $R \subset R' \subset G^*$, $R'G \subseteq R$;

(iii) if G is either countable or Abelian and I is a closed ideal in βG such that $I \subset G^*$ then there exists a closed ideal I' in βG such that $I \subset I' \subset G^*$, $\overline{G^*G^*} \subset I$.

For a cardinal κ, S_κ denotes the group of all permutations of κ.

Theorem 6.5. *For every infinite cardinal κ, there exists a closed ideal I in βS_κ such that*

(i) $S^*_\kappa S^*_\kappa \subset I$;

(ii) if M is a closed ideal in βS_κ and $I \subseteq M \subseteq G^*$ then either $M = I$ or $M = S^*_\kappa$.

Theorem 6.6. *For every group G, we have $FT_G \subset Sp_G$ so $\overline{G^*G^*} \subset FT^{\wedge}_G$.*

For subsets X, Y of a group G, we say that the product XY is an *n-stripe* if $|X| = n$, $n \in \mathbb{N}$ and $|Y| = \omega$. It is easy to see that a subset A of G is n-thin if and only if A has no $(n+1)$-stripes. Thus, $p \in FT^{\wedge}_G$ is and only if each member $P \in p$ has an n-stripe for every $n \in \mathbb{N}$.

We say that XY is an *(n,m)-rectangle* if $|X| = n$, $|Y| = m$, $n, m \in \mathbb{N}$. We say that a subset A of G *has bounded rectangles* if there is $n \in \mathbb{N}$ such that A has no (n,n)-rectangles (and so (n,m)-rectangles for each $m > n$).

We denote by BR_G the family of all subsets of G with bounded rectangles.

Theorem 6.7. *For a group G, the following statements hold:*

(i) BR_G is a translation invariant ideal in $\mathcal{P}_G$;

(ii) $BR^{\wedge}_G$ is a closed ideal in βG and $p \in BR^{\wedge}_G$ if and only if each member $P \in p$ has an (n,n)-rectangle for every $n \in \mathbb{N}$;

(iii) $BR_G \subset FT_G$.

6.2. Between $\overline{K(G)}$ and $\overline{G^*G^*}$.

Theorem 6.8. *For a group G, the following statements hold:*

(i) $Sc_G^\wedge = cl\{\epsilon p : \epsilon \in G^*,\ p \in \beta G,\ \epsilon\epsilon = \epsilon\}$;

(ii) $Sc_G^\wedge$ *is an ideal in* βG *and* $p \in Sc_G^\wedge$ *if and only if each member of* p *contains a piecewise shifted* FP*-set;*

(iii) $Sc_G^\wedge$ *is the minimal closed ideal in* βG *containing all idempotents of* G^*.

For a group G, we put $I_{G,n} = G^*$, $I_{G,n+1} = \overline{G^* I_{G,n}}$ and note that $I_{G,n}$ is an ideal in βG.

Theorem 6.9. *For every group G and $n \in \omega$, we have*

(i) $I_{G,n+1} \subset I_{G,n}$

(ii) $Sc_G^\wedge \subset I_{G,n}$.

For a natural number n, we denote by $(G^*)^n$ the product of n copies of n. Clearly, $\overline{(G^*)^{n+1}} \subseteq \overline{(G^*)^n}$. and $\overline{(G^*)^n} \subseteq I_{G,n}$.

Theorem 6.10. *For every group G and $n \in \omega$, we have*

(i) $\overline{(G^*)^{n+1}} \subset \overline{(G^*)^n}$;

(ii) $Sc_G^\wedge \subset \overline{(G^*)^n}$.

Comments. This section is an extract from [25].

7. The combinatorial derivation

Let G be a group with the identity e. For a subset A of G, we denote

$$\triangle(A) = \{g \in G : |gA \bigcap A = \infty|\},$$

observe that $(\triangle(A))^{-1} = \triangle(A)$, $\triangle(A) \subseteq AA^{-1}$, and say that the mapping

$$\triangle : \mathcal{P}_G \longrightarrow \mathcal{P}_G,\quad A \longmapsto \triangle(A)$$

is the *combinatorial derivation.*

Theorem 7.1. *For an infinite group G and a subset A of G, the following statements hold*

(1) A is finite if and only if $\triangle(A) = \emptyset$;
(2) $\triangle(A) = \{e\}$ if and only if A is infinite and thin;
(3) if A is thick then $\triangle(A) = G$;
(4) if A is prethick then $\triangle(A)$ is large.

Theorem 7.2. *Every infinite group G contains a subset A such that $G = AA^{-1}$ and $\triangle(A) = \{e\}$.*

Theorem 7.3. *Let A be a subset of an infinite group G such that $A = A^{-1}$. Then there exist two thin subsets X, Y of G such that $\triangle(X \bigcup Y) = A$.*

We consider also the inverse to $\triangle$, multivalued mapping ∇ defined by

$$\nabla(A) = \{B \subseteq G : \triangle(B) = A\}.$$

For a family F of subsets of a group G, we say that $\mathcal{F}$ is $\triangle$-complete (∇-complete) if $\triangle(A) \in \mathcal{F}$ ($\nabla(A) \subseteq \mathcal{F}$) for each $A \in \mathcal{F}$.

Theorem 7.4. *For every infinite group G, the following statements hold*

(1) the families of all small and sparse subsets of G is ∇-complete;

(2) if an ideal $\mathcal{I}$ in $\mathcal{P}_G$ is $\triangle$-complete and ∇-complete then $\mathcal{I} = \mathcal{P}_G$;

(3) If $\mathcal{I}$ is a group ideal in $\mathcal{P}_G$, $\mathcal{I} \neq \mathcal{P}_G$, then $\mathcal{I}$ is $\triangle$-complete and $\mathcal{I}$ is contained in the ideal of all small subsets of G.

Comments. More information on combinatorial derivation in [26–28]. In particular, Theorem 6.2 from [26] shows that the trajectory $A \longrightarrow \triangle(A) \longrightarrow \triangle^2(A) \longrightarrow \dots$ of a subset A of G could be surprisingly complicated: stabilizing, increasing, decreasing, periodic or chaotic. Also [26] contains some parallels between the combinatorial and topological derivations.

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