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ON THE CONVERGENCE TO INFINITY OF POSITIVE INCREASING FUNCTIONS

ПРО ЗБІЖНІСТЬ ДО НЕСКІНЧЕННОСТІ ДОДАТНО ЗРОСТАЮЧИХ ФУНКЦІЙ

We study conditions for the convergence to infinity of some classes of functions extending the well-known class of regularly varying (RV) functions, such as, for example, O-regularly varying (ORV) functions or positive increasing (PI) functions.

Досліджено умови збіжності до нескінченності деяких класів функцій, що розширюють відомий клас регулярно змінних функцій, таких, як, наприклад, O-регулярно змінних функцій або додатно змінних функцій.

1. Introduction. There is a variety of problems in mathematical as well as in stochastic analysis, where an ordinary or a random function under investigation is assumed to converge to infinity when its argument tends to infinity. An example of such a problem is the question of equivalence of the solutions $X(t)$ and $x(t)$, respectively, of a stochastic differential equation (SDE) and its corresponding ordinary differential equation (ODE) where the ODE is obtained from the SDE after excluding the stochastic differential therein (see [15, 21, 16]). More precisely, under a certain set of assumptions $X(t) \sim x(t)$ almost surely as $t \to \infty$ if $X(t) \to \infty$ almost surely as $t \to \infty$.

This result was derived from a generalization of Karamata's theory of regular variation (see also [12]). Indeed, Karamata's theory provides another area, now in mathematical analysis, in which the convergence to infinity of functions plays a crucial role.

Recall that Karamata [18], in 1930, introduced the notion of *regularly varying* (RV) functions and proved a number of fundamental results for this important class of functions (see also [19]). Later on, these results and their further generalizations developed into a well-established theory having a wide range of applications (cf., e.g., Bingham et al. [6] and Seneta [24]).

Indeed, after the seminal paper [18], it is a variety of generalizations of RV functions that has been introduced and discussed. Certainly a first one to mention is the class of *O-regularly varying* (ORV) *functions* due to Avakumović [3], which has also been studied in numerous papers (see, for example, [20, 14, 1, 4, 2]). In fact, it turns out that, in many asymptotic problems, an important role is played by classes of functions, which generalize RV functions in one way or another.

In the current paper, we continue our earlier work in [7 – 9, 11]. Here now, main attention is paid to the class of (so-called) *positive increasing* (PI) functions (see Definition 2.2 below). These and some related functions have been studied by many authors (cf., e.g., [5, 17, 25, 13, 23], to mention just a few). Along with the classes of measurable ORV and PI functions, we also consider their extensions, i.e., WORV and WPI functions, which may be measurable *or* nonmeasurable functions as well.

The class of PI functions contains all RV functions with positive index. One of the important properties of the latter functions is that they tend to infinity as their argument tends to infinity (see, for example, [6, 24]). We show below that this property retains for all functions, which are both ORV *and* PI functions.

The paper is organized as follows. Section 2 contains all necessary definitions concerning the classes of functions considered in this paper. In Section 3 we prove that the upper limit of a WPI function as well as the upper limits of its limit functions are infinite. Some extra conditions, being sufficient for WPI functions to tend to infinity, are discussed in Section 4, where we separately treat the conditions that either contain *or not* the assumption of measurability. A counterexample, given in Section 5, shows that Theorem 4.1 cannot be improved in general.

2. Definitions and some preliminary results. Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}_+$ the set of positive reals, $\mathbf{N}$ the set of positive integers, and let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Denote by $\mathbb{F}$ the set of all real functions $f = (f(t), t \ge 0)$ and let

$$\mathbb{F}_+ = \bigcup_{a>0} \{f \in \mathbb{F} \colon f(t) > 0, t \in [a, \infty)\}.$$

It is clear that $f \in \mathbb{F}_+$ if and only if $f(t) > 0$ for all sufficiently large t.

By $\mathbb{F}^{(\infty)}$ we denote the set of functions $f \in \mathbb{F}_+$ such that

$$\limsup_{t\to\infty} f(t) = \infty,$$

and by $\mathbb{F}^{\infty}$ we denote the subset of $\mathbb{F}^{(\infty)}$ such that

$$\lim_{t\to\infty} f(t) = \infty.$$

Throughout the paper, “measurability” means measurability in the Lebesgue sense.

For $f \in \mathbb{F}_+$, define its *upper* and *lower limit function*, i.e.,

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

taking values in $[0, \infty]$.

Limit functions are useful for defining and studying various classes of real functions. Note that f^* and f_* are also called the *index functions* of f.

The following properties of limit functions follow directly from the definitions.

Lemma 2.1. *Let $f \in \mathbb{F}_+$. Then*

(i) *for all $c > 0$,*

$$0 \le f_*(c) \le f^*(c) \le \infty;$$

(ii) *for all $c > 0$,*

$$f_*(c) = \frac{1}{f^*(1/c)},$$

where $1/\infty = 0$ and $1/0 = \infty$;

(iii) *for all c_1 and $c_2 > 0$, the following inequalities hold if they do not contain an expression $0 \cdot \infty$ or $\infty \cdot 0$:*

$$f_*(c_1)f_*(c_2) \le f_*(c_1c_2) \le \min\{f_*(c_1)f^*(c_2),\ f_*(c_2)f^*(c_1)\} \le$$

$$\le \max\{f_*(c_1)f^*(c_2),\ f_*(c_2)f^*(c_1)\} \le f^*(c_1c_2) \le f^*(c_1)f^*(c_2);$$

(iv) $f_*(1) = f^*(1) = 1$.

ORV and RV functions. We first recall the definition of ORV functions (see [3, 20], or [1]).

Definition 2.1. *A function $f \in \mathbb{F}_+$ is called a function with O-regular variation in the wide sense* (WORV) *if*

$$0 < f_*(c) \le f^*(c) < \infty \quad \textit{for all} \quad c > 1. \tag{2.1}$$

Correspondingly, a measurable WORV *function is called a function with O-regular variation* (ORV).

The class of all WORV (ORV) functions is denoted by $\mathcal{WORV}$ ($\mathcal{ORV}$).

We say that a measurable function $f \in \mathbb{F}_+$ *varies regularly* [18, 19], or that it is an RV function, if

$$f_*(c) = f^*(c) = \kappa_f(c) \in (0, \infty) \quad \text{for all} \quad c > 0,$$

that is, the limit

$$\kappa_f(c) = \lim_{t\to\infty} \frac{f(ct)}{f(t)}$$

exists and is positive and finite for all $c > 0$.

The class of all RV functions is denoted by $\mathcal{RV}$. It is clear that any RV function belongs to the class $\mathcal{ORV}$.

If $f \in \mathcal{RV}$, then

$$\kappa_f(c) = \kappa(c, \rho) = c^\rho, \quad c > 0, \tag{2.2}$$

for some real number ρ called the *index* of the function f. By $\mathcal{RV}_+$ we mean the class of all RV functions with *positive* index. RV functions such that $\rho = 0$ are called *slowly varying* (SV) functions.

For any RV function f, the following representation holds:

$$f(t) = t^\rho \ell(t), \quad t > 0,$$

where $(\ell(t), t > 0)$ is a slowly varying function.

WPI and PI functions. Here we recall the definitions of WPI and PI functions (see [5, 17, 7, 9, 11]).

Definition 2.2. *A function $f \in \mathbb{F}_+$ is said to be positive increasing in the wide sense* (WPI) *if*

$$f_*(c_0) > 1 \quad \textit{for some} \quad c_0 > 1. \tag{2.3}$$

Correspondingly, a measurable WPI *function is said to be positive increasing* (PI).

The class of all PI (WPI) functions is denoted by $\mathcal{PI}$ ($\mathcal{WPI}$). Relation (2.2) implies that $\mathcal{RV}_+ \subset \mathcal{PI}$.

2.1. An example. There are several cases where the limit functions f_* and f^* determine the asymptotics of the original function f. Say, if $f_*(c) = f^*(c)$ for all $c > 0$, then f is regularly varying. If additionally (2.3) holds, then $f(x) \to \infty$ as $x \to \infty$. This is not the case in more general situations. Below we construct a function f depending on two parameters $0 < \theta_1 < 1$ and $\theta_2 > 1$ such that $f_*(c) = \theta_1$ and $f^*(c) = \theta_2$ for $c > 1$. Surprisingly, the asymptotics of f only depends on the specific relation between θ_1

and θ_2. In particular, $\lim f(x) = \infty$ if $\theta_1\theta_2 > 1$. Otherwise, f is bounded. Note that this function f is measurable but $f \notin \mathcal{PI}$.

***Example* 2.1.** Let $0 < \theta_1 < 1 < \theta_2$ be two given numbers. We construct a function f such that

$$f_*(c) = \theta_1 \quad \text{and} \quad f^*(c) = \theta_2 \quad \text{for all} \;\; c > 1.$$

First, choose positive sequences $\{H_n\}$ and $\{t_n\}$ as follows:

$$H_1 = 1, \quad H_{n+1} = \theta_1\theta_2 H_n,$$
$$t_1 = 1, \quad t_{n+1} = n^2 t_n, \quad n \geq 1.$$

For simplicity, let $L_n = [t_n, nt_n)$ and $R_n = [nt_n, t_{n+1})$. Then the function f is given as

$$f(t) = \begin{cases} 1, & 0 \leq t < t_2, \\ H_n, & t \in L_n, \; n \geq 2, \\ \theta_1 H_n, & t \in R_n, \; n \geq 2. \end{cases}$$

Let $c > 1$. If t is sufficiently large and $t \in L_n \cup R_n$, then $ct \in L_n \cup R_n \cup L_{n+1}$. Thus, for $c > 1$ and sufficiently large t,

$$\frac{f(ct)}{f(t)} = \begin{cases} 1, & \text{if} \quad ct, \; t \in L_n, \quad \text{or} \quad ct, \; t \in R_n, \\ \theta_1, & \text{if} \quad ct \in R_n, \quad \text{but} \quad t \in L_n, \\ \theta_2, & \text{if} \quad ct \in L_{n+1}, \quad \text{but} \quad t \in R_n. \end{cases}$$

Therefore, for all $c > 1$,

$$f_*(c) = \theta_1, \quad f^*(c) = \theta_2.$$

Note that

$$\begin{aligned} &\lim f(t) = \infty, & & \text{if} \quad \theta_1\theta_2 > 1, \\ &\liminf f(t) = \theta_1, \quad \limsup f(t) = 1, & & \text{if} \quad \theta_1\theta_2 = 1, \\ &\lim f(t) = 0, & & \text{if} \quad \theta_1\theta_2 < 1. \end{aligned}$$

The rest of the paper is devoted to finding some conditions imposed on limit functions that imply $\limsup f(x) = \infty$ or $\lim f(x) = \infty$. In doing so, the basic tool is the WPI property (2.3).

3. Upper limits of WPI functions. The following result contains a characterization of WPI functions in terms of their lower limit functions. In particular, it shows that $f \in \mathcal{WPI}$ if and only if $f_* \in \mathbb{F}^{(\infty)}$.

Proposition 3.1. *For $f \in \mathbb{F}_+$, the following three conditions are equivalent*:

(a) $f \in \mathcal{WPI}$;

(b) $\limsup_{c\to\infty} f_*(c) = \infty$;

(c) $\limsup_{c\to\infty} f_*(c) > 1$.

Proof. If $f \in \mathcal{WPI}$, then condition (2.3) and Lemma 2.1 imply that

$$\limsup_{c\to\infty} f_*(c) \geq \limsup_{m\to\infty} f_*(c_0{}^m) \geq \limsup_{m\to\infty} (f_*(c_0))^m = \infty.$$

Therefore the implication (a) $\Longrightarrow$ (b) is proved. On the other hand, the implications (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (a) are trivial.

Lemma 2.1 (i) and Proposition 3.1 imply that the lower and upper limit functions belong to the class $\mathbb{F}^{(\infty)}$ if $f \in \mathcal{WPI}$. Below we show that WPI functions themselves belong to the class $\mathbb{F}^{(\infty)}$.

Lemma 3.1. *If $f \in \mathcal{WPI}$, then $f \in \mathbb{F}^{(\infty)}$.*

Proof. Let $f \in \mathcal{WPI}$, that is, condition (2.3) holds. This implies that there are constants $c_0 > 1, t_0 > 0$, and $r > 1$ such that $f(t) > 0$ for $t \geq t_0$, and

$$\frac{f(c_0 t)}{f(t)} \geq r \quad \text{for all} \quad t \geq t_0. \tag{3.1}$$

Thus

$$\frac{f(c_0^m t)}{f(t)} = \frac{f(c_0^m t)}{f(c_0^{m-1} t)} \cdots \frac{f(c_0 t)}{f(t)} \geq r^m$$

and

$$f(c_0^m t) \geq r^m f(t) \tag{3.2}$$

for all $t \geq t_0$ and all $m \in \mathbf{N}_0$. This implies that

$$\limsup_{t\to\infty} f(t) \geq \limsup_{m\to\infty} f(c_0^m t_0) \geq f(t_0) \lim_{m\to\infty} r^m = \infty.$$

4. Conditions for the convergence to infinity of WPI and PI functions. In view of Lemma 3.1, a natural problem is to find conditions under which WPI functions tend to infinity. We discuss some appropriate conditions below. The following result does not require measurability of the function f.

Proposition 4.1. *Let $f \in \mathcal{WPI}$ and $c_0 > 1$, $t_0 > 0$, and $r > 1$ be the constants from inequality* (3.1). *If there exists a $T_0 \geq t_0$ such that*

$$\varepsilon_0 = \inf_{T_0 \leq \theta \leq c_0 T_0} f(\theta) > 0, \tag{4.1}$$

then $f \in \mathbb{F}^{\infty}$.

Proof. Inequality (3.2) implies that, for all $t \geq T_0$,

$$f(t) \geq f\left(\frac{t}{c_0^{\mathfrak{m}(t)}}\right) r^{\mathfrak{m}(t)},$$

where

$$\mathfrak{m}(t) = \max\left\{n \in \mathbf{N}_0 \colon c_0^n T_0 \leq t\right\}.$$

It is clear that $\mathfrak{m}(t)$ is the integer part of $\log_{c_0}(t/T_0)$ and

$$\frac{t}{c_0^{\mathfrak{m}(t)}} \in [T_0, c_0 T_0]$$

for all $t \geq T_0$. Thus

$$f(t) \geq \varepsilon_0 r^{\mathfrak{m}(t)}, \quad t \geq T_0. \tag{4.2}$$

Since $\lim_{t\to\infty} \mathfrak{m}(t) = \infty$ and $r > 1$, we have

$$\lim_{t\to\infty} r^{\mathfrak{m}(t)} = \infty.$$

Taking condition (4.1) into account, we obtain

$$\liminf_{t\to\infty} f(t) \geq \varepsilon_0 \lim_{t\to\infty} r^{\mathfrak{m}(t)} = \infty.$$

***Remark* 4.1.** One can substitute any of the following two conditions for (4.1):

(i) for all sufficiently large $s > 0$, there exists $T = T(s) \geq s$, such that

$$\inf_{T \leq t \leq c_0 T} f(t) > 0, \tag{4.3}$$

or

(ii) $\liminf_{t\to\infty} f(t) > 0$.

Each of the latter two conditions implies condition (4.1) and thus is sufficient to show that a WPI function f tends to infinity.

In turn, condition (4.3) holds if, for example, for all sufficiently large $s > 0$ there exists $T \geq s$ such that the positive function f is continuous on the interval $[T, c_0 T]$.

***Remark* 4.2.** Inequality (4.2) shows that any WPI function cannot grow slower than a power function depending on the point T_0 given in condition (4.1).

Proposition 4.1 yields the following result.

***Corollary* 4.1.** *Let $f \in \mathcal{WPI}$. If there is a $T > 0$ such that f is continuous on the interval $[T, \infty)$, then $f \in \mathbb{F}^\infty$.*

In fact, the function f in Corollary 4.1 belongs to $\mathcal{PI}$. The following result treats this case, too.

Theorem 4.1. *If $f \in \mathcal{ORV} \cap \mathcal{PI}$, then $f \in \mathbb{F}^\infty$.*

***Remark* 4.3.** Since $\mathcal{ORV} \cap \mathcal{PI} = \mathcal{ORV} \cap \mathcal{WPI}$, Theorem 4.1 means that the ORV property, like condition (4.1) in Proposition 4.1, is also an extra condition under which a WPI function tends to infinity. Note that the ORV property requires both the WORV condition (2.1) and measurability of the corresponding function.

***Remark* 4.4.** Since $\mathcal{RV}_+ \subset \mathcal{ORV} \cap \mathcal{PI}$, Theorem 4.1 implies that any RV function with *positive* index tends to infinity (see [6, 24]).

We need an auxiliary result to prove Theorem 4.1.

Lemma 4.1. *Let $f \in \mathcal{ORV} \cap \mathcal{PI}$. Then*

$$\lim_{n\to\infty} \frac{f(a_n t_n)}{f(t_n)} = \infty \tag{4.4}$$

for all sequences of positive numbers $\{a_n\}$ and $\{t_n\}$ such that

$$\lim_{n\to\infty} a_n = \infty \quad \text{and} \quad \lim_{n\to\infty} t_n = \infty.$$

Proof. Let $\{a_n\}$ and $\{t_n\}$ be sequences of positive numbers such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} t_n = \infty$. Moreover let $c_0 > 1$, $t_0 > 0$ and $r > 1$ be the constants from inequality (3.1) corresponding to the function f.

Using $\{a_n\}$ and c_0, we construct a sequence $\{m_n\} \subset \mathbf{N}$ such that, for some $n_0 \in \mathbf{N}$,

$$c_0^{m_n} \le a_n < c_0^{m_n+1}, \quad n \ge n_0.$$

Since $\lim_{n\to\infty} a_n = \infty$, we get $\lim_{n\to\infty} m_n = \infty$.

It is clear that, for all $n \ge 1$,

$$\frac{f(a_n t_n)}{f(t_n)} = \frac{f(a_n t_n)}{f(c_0^{m_n} t_n)} \prod_{k=1}^{m_n} \frac{f(c_0^k t_n)}{f(c_0^{k-1} t_n)}.$$

This together with (3.1) implies that

$$\frac{f(a_n t_n)}{f(t_n)} \ge \frac{f(a_n t_n)}{f(c_0^{m_n} t_n)}\, r^{m_n}$$

for all large n.

Since $f \in \mathcal{ORV}$, the integral representation of ORV functions (see [1]) implies that there exist measurable bounded functions α and β such that

$$f(t) = \Phi(t) \exp\left\{ \int_{t_0}^{t} \beta(u) \frac{du}{u} \right\} \tag{4.5}$$

for all sufficiently large t, where $\Phi = \exp \circ \alpha$. This, for sufficiently large n, implies that

$$\frac{f(a_n t_n)}{f(c_0^{m_n} t_n)} = \frac{\Phi(a_n t_n)}{\Phi(c_0^{m_n} t_n)} \exp\left\{ \int_{c_0^{m_n} t_n}^{a_n t_n} \beta(u) \frac{du}{u} \right\} \ge$$

$$\ge \frac{\Phi(a_n t_n)}{\Phi(c_0^{m_n} t_n)} \exp\left\{ -B \ln\left(\frac{a_n}{c_0^{m_n}} \right) \right\} \ge K\, c_0^{-B},$$

where

$$K = \frac{\liminf_{t\to\infty} \Phi(t)}{2 \limsup_{t\to\infty} \Phi(t)} > 0 \quad \text{and} \quad B = |\inf_{t\in[t_0,\infty)} \beta(t)| < \infty.$$

Therefore

$$\liminf_{n\to\infty} \frac{f(a_n t_n)}{f(t_n)} \ge \left(K\, c_0^{-B} \right) \liminf_{n\to\infty} r^{m_n} = \infty,$$

whence relation (4.4) follows.

Lemma 4.1 is proved.

***Proof of Theorem* 4.1.** Assume the converse, that is, let $f(t)$ *not* tend to ∞ as $t \to \infty$. Then there is a sequence of positive numbers $\{u_n\}$ and a number $p \in [0, \infty)$ such that $u_n \to \infty$ and $\lim_{n\to\infty} f(u_n) = p$.

If $p \in (0, \infty)$, there exists a sequence of natural numbers $\{n_k\}$ such that, with $s_k = u_{n_k}$, $k \ge 1$, we have $\lim_{k\to\infty} s_k = \infty$ and

$$\lim_{k\to\infty} s_{k+1}/s_k = \infty \quad \text{and} \quad \lim_{k\to\infty} f(s_{k+1})/f(s_k) = 1.$$

If $p = 0$, there is a sequence of natural numbers $\{n_k\}$ such that, with $s_k = u_{n_k}$, $k \geq 1$, we have $\lim_{k\to\infty} s_k = \infty$ and

$$\lim_{k\to\infty} s_{k+1}/s_k = \infty \quad \text{and} \quad \lim_{k\to\infty} f(s_{k+1})/f(s_k) = 0.$$

In both cases, this is a contradiction to Lemma 4.1, which completes the proof of $f \in \mathbb{F}^{\infty}$.

5. Counterexamples. In this section, we discuss two examples highlighting the sufficiency conditions of our preceding results.

5.1. Measurability. Analyzing the proof of Theorem 4.1, we see that the WORV condition (2.1) can be weakened. Indeed, Theorem 4.1 remains true for measurable functions f for which the integral representation (4.5) holds for sufficiently large t, where the measurable function α is bounded and the measurable function β is bounded from below and locally bounded from above. This rises the interesting question whether or not one can drop the measurability in Theorem 4.1 and instead add the following "one-sided" WORV condition to the WPI condition (2.3):

$$f_*(c) > 0 \quad \text{for all} \quad c > 1,$$

or, more stronger,

$$f_*(c) > 1 \quad \text{for all} \quad c > 1$$

(implying the WPI condition (2.3)).

The following result shows that this is *not* the case.

Proposition 5.1. *There exists a nonmeasurable function $f \in \mathbb{F}_+$ such that*

$$\lim_{t\to\infty} \frac{f(ct)}{f(t)} = \infty \quad \textit{for all} \quad c > 1, \tag{5.1}$$

but

$$\liminf_{t\to\infty} f(t) = 0. \tag{5.2}$$

Proof. Let $\mathbf{H}$ be the Hamel basis (see, for example, [16, 6]), that is, a set of real numbers such that every real number $x \neq 0$ can uniquely be represented as a finite linear combination of elements of $\mathbf{H}$ with rational coefficients, i.e.,

$$x = \sum_{i=1}^{n(x)} r_i(x) b_i(x),$$

where $n(x) \in \mathbf{N}$, $r_i(x) \in \mathbf{Q} \setminus \{0\}$ and $b_i(x) \in \mathbf{H}$.

Note that $(n(x), x \in \mathbf{R})$ is a nonmeasurable and subadditive function, that is

$$n(x+y) \leq n(x) + n(y) \quad \text{for all} \quad x, y \in \mathbf{R}, \tag{5.3}$$

(see, for example, [6] or [22]). Moreover, for all fixed $n \geq 1$ and all fixed, but different $b_1, \ldots, b_n \in \mathbf{H}$,

$$\text{the set } M_n = \left\{ \sum_{i=1}^{n} r_i b_i;\ r_1, \ldots, r_n \in \mathbf{Q} \setminus \{0\} \right\} \text{ is dense in } \mathbf{R}. \tag{5.4}$$

Let

$$h(x) = x^2 - n(x), \quad x > 0, \quad \text{and} \quad f(t) = \exp\{h(\ln t)\}, \quad t > 0.$$

It is clear that $f \in \mathbb{F}_+$. Moreover, inequality (5.3) implies that

$$h(x+u) - h(x) = 2xu + u^2 - (n(x+u) - n(x)) \geq 2xu - n(u)$$

for all $x > 0$ and $u > 0$. Hence

$$\lim_{x\to\infty} \big(h(x+u) - h(x)\big) = \infty \quad \text{for all} \quad u > 0.$$

This implies that

$$\lim_{t\to\infty} \frac{f(ct)}{f(t)} = \lim_{t\to\infty} \exp\{h(\ln t + \ln c) - h(\ln t)\} =$$

$$= \exp\left\{ \lim_{t\to\infty} (h(\ln t + \ln c) - h(\ln t)) \right\} = \infty$$

for all $c > 1$. This proves relation (5.1).

On the other hand, according to (5.4), there exists a sequence $\{x_k\}$ such that

$$x_k \in (k-1, k) \cap M_{k^3}, \quad k \geq 1.$$

It is clear that

$$h(x_k) < k^2 - n(x_k) = k^2 - k^3 \quad \text{for all} \quad k \geq 1,$$

whence

$$\liminf_{x\to\infty} h(x) = -\infty.$$

This implies (5.2) and thus completes the proof of Proposition 5.1.

5.2. Upper limit functions. The sufficient condition of Theorem 4.1 for $f \in \mathbb{F}^\infty$ is expressed in terms of the lower limit function f_*. Note that the upper limit function, in turn, is not an appropriate tool here. The following example exhibits a bounded measurable function f such that $f^*(c) = \infty$ for all $c > 0$. One may compare this function with $g(x) = e^x$ for which the upper limit function is nearly the same, i.e., $g^*(c) = \infty$ for $c > 1$, but g grows to infinity very fast.

***Example* 5.1.** Let $B = \{1!, 2!, 3!, \ldots\}$ and $A = \mathbf{R}_+ \setminus B$. Put

$$f(t) = \mathbb{1}_A(t) + \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{1}_{\{n!\}}(t).$$

In other words,

$$f(t) = \begin{cases} 1, & t \in A, \\ \dfrac{1}{n}, & t = n! \quad \text{for some } n. \end{cases}$$

For any $c > 0$, the case where both $ct = n!$ and $t = m!$ does not occur if t is sufficiently large. Then, for $c > 0$ and sufficiently large t,

$$\frac{f(ct)}{f(t)} = \begin{cases} 1, & ct \in A, \quad t \in A, \\ \dfrac{1}{n}, & ct = n! \quad \text{for some } n, \quad \text{but} \quad t \in A, \\ n, & ct \in A, \quad \text{but} \quad t = n! \quad \text{for some } n. \end{cases}$$

Therefore $f_*(c) = 0$ and $f^*(c) = \infty$ for all $c > 0$. However $0 < f(t) \leq 1$.

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