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NOTES ON UNIQUENESS AND VALUE SHARING OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS*

ПРО ЄДИНІСТЬ ТА ПОДІЛ ЗНАЧЕНЬ ДЛЯ МЕРОМОРФНИХ ФУНКЦІЙ ЩОДО ДИФЕРЕНЦІАЛЬНИХ ПОЛІНОМІВ

We study the problem of uniqueness of meromorphic functions concerning differential polynomials, and obtain some results. The results improve earlier results by Li [J. Sichuan Univ. (Natural Science Edition). – 2008. – **45**. – P. 21 – 24] and Dyavanal [J. Math. Anal. and Appl. – 2011. – **374**. – P. 335 – 345].

Вивчається проблема єдиності мероморфних функцій щодо диференціальних поліномів, отримано деякі результати. Ці результати покращують результати, що отримані раніше в роботах Лі [J. Sichuan Univ. (Natural Science Edition). – 2008. – **45**. – P. 21 – 24] та Д'яванала [J. Math. Anal. and Appl. – 2011. – **374**. – P. 335 – 345].

1. Introduction and results. Let f be a nonconstant meromorphic function defined in the whole complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $S(r,f)$ and so on, and these can be found, for instance in [5, 7].

Let f and g be two nonconstant meromorphic functions. If for $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities). When f and g share the value 1 IM, Let z_0 be a 1-points of f of order p, a 1-points of g of order q, we denote by $N_{11}\Big(r, \dfrac{1}{f-1}\Big)$ the counting function of those 1-points of f and g where $p=q=1$ and $\overline{N}_L\Big(r, \dfrac{1}{f-1}\Big)$ is the counting function of those 1-points of both f and g where $p>q$. In the same way, we can define $N_{11}\Big(r, \dfrac{1}{g-1}\Big)$ and $\overline{N}_L\Big(r, \dfrac{1}{g-1}\Big)$. For any constant a, we define

$$\Theta(a,f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}\Big(r, \dfrac{1}{f-a}\Big)}{T(r,f)}.$$

Let f be a nonconstant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}\Big(r, \dfrac{1}{f-a}\Big)$ $\Big($or $\overline{N}_{k)}\Big(r, \dfrac{1}{f-a}\Big)\Big)$ the counting function for zeros of $f-a$ with multiplicity $\le k$ (ignoring multiplicities), and by $N_{(k}\Big(r, \dfrac{1}{f-a}\Big)$ $\Big($or $\overline{N}_{(k}\Big(r, \dfrac{1}{f-a}\Big)\Big)$ the counting function for zeros of $f-a$ with multiplicity at least k (ignoring multiplicities). Set

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \ldots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

We further define

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$$\delta_k(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k\Big(r, \dfrac{1}{f-a}\Big)}{T(r,f)}.$$

Hayman [2] and Clunie [1] proved the following result.

Theorem A. *Let $f(z)$ be a transcendental entire function, $n \geq 1$ be a positive integer, then $f^n f' = 1$ has infinitely many solutions.*

In 1997, Yang and Hua [6] obtained a unicity theorem corresponding to the above result and proved the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem C. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Recently, R. S. Dyavanal [4] improve above results and obtain the following results.

Theorem D. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be a positive integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem E. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be a positive integer satisfying $(n+1)s \geq$ ≥ 7. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

***Remark* 1.1.** If $s = 1$ in Theorem D and Theorem E, respectively, then Theorem D and Theorem E reduces to Theorem B and Theorem C, respectively.

Naturally, one can pose the following question: what can be stated if CM is replaced with IM in the above results.

In 2008, Li [3] prove the following result.

Theorem F. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $n \geq 23$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

In this paper, we shall generalize and improve the above the results and obtain the following two theorems.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be a positive integer satisfying $(n+1)s \geq 24$. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

***Remark* 1.2.** If $s = 1$ in Theorem 1.1, then Theorem 1.1 improves Theorem F.

***Remark* 1.3.** Giving specific values for s in Theorem 1.1, we can get the following interesting cases:

(i) if $s = 1$, then $n \geq 23$,
(ii) if $s = 2$, then $n \geq 11$,
(iii) if $s = 3$, then $n \geq 7$,
(iv) if $s \geq 4$, then $n \geq 5$.

We can conclude that f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be a positive integer satisfying $(n+1)s \geq \geq 13$. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

If $s = 1$ in Theorem 1.2, then Theorem 1.2 reduces to the following result.

***Corollary* 1.1.** *Let $f(z)$ and $g(z)$ be two transcendental entire functions, and let n be a positive integer satisfying $n \geq 12$. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f = tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

2. Some lemmas. For the proof of our result, we need the following lemmas.

Lemma 2.1 (see [2]). *Let f be nonconstant meromorphic function, $a_0, a_1, \ldots, a_n$ be finite complex numbers such that $a_n \neq 0$.Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 (see [2]). *Let $f(z)$ be a nonconstant meromorphic function, k be a positive integer, and let c be a nonzero finite complex number. Then*

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \leq$$

$$\leq \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).$$

Here $N_0\Big(r, \dfrac{1}{f^{(k+1)}}\Big)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f\big(f^{(k)} - c\big) \neq 0$.

Lemma 2.3 (see [3]). *Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f - a_1}\Big) + \overline{N}\Big(r, \frac{1}{f - a_2}\Big) + S(r, f).$$

Lemma 2.4 (see [8]). *Let f be a nonconstant meromorphic function, k, p be two positve integers, then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \leq$$

$$\leq (p + k)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Clearly $\overline{N}\Big(r, \dfrac{1}{f^{(k)}}\Big) = N_1\left(r, \dfrac{1}{f^{(k)}}\right)$.

Lemma 2.5. *Let $f(z)$ and $g(z)$ be two meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and*

$$\Delta = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + (k+2)\Theta(0, f) + (2k+3)\Theta(0, g)+$$

$$+\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 7k + 13, \tag{2.1}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$h(z) = \frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} - \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} + 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \tag{2.2}$$

If z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.2), we see that z_0 is a zero of $h(z)$. Thus, we have

$$N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) = N_{11}\left(r, \frac{1}{g^{(k)}-1}\right) \le \overline{N}\left(r, \frac{1}{h}\right) \le T(r,h) + O(1) \le$$

$$\le N(r,h) + S(r,f) + S(r,g). \tag{2.3}$$

By our assumptions, $h(z)$ have poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g, and those 1-points of $f^{(k)}$ and $g^{(k)}$ whose multiplicities are distinct from the multiplicities of corresponding 1-points of $g^{(k)}$ and $f^{(k)}$ respectively. Thus, we deduce from (2.2) that

$$N(r,h) \le \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) +$$

$$+N_0\left(r, \frac{1}{g^{(k+1)}}\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \tag{2.4}$$

here $N_0\Big(r, \dfrac{1}{f^{(k+1)}}\Big)$ has the same meaning as in Lemma 2.2.

By Lemma 2.2, we have

$$T(r,f) \le \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f), \tag{2.5}$$

$$T(r,g) \le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-c}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g). \tag{2.6}$$

Since $f^{(k)}$ and $g^{(k)}$ share the value 1 IM, we obtain

$$\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) \le$$

$$\le N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + N\left(r, \frac{1}{f^{(k)}-1}\right) \le$$

$$\leq N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+T(r,f^{(k)})+O(1)\leq$$

$$\leq N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+m(r,f)+$$

$$+m\left(r,\frac{f^{(k)}}{f}\right)+N(r,f)+k\overline{N}(r,f)+S(r,f)\leq$$

$$\leq N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+T(r,f)+k\overline{N}(r,f)+S(r,f). \tag{2.7}$$

Noting that, by Lemma 2.4, we get

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)\leq N_{1+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\leq$$

$$\leq (k+1)\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f), \tag{2.8}$$

$$\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)\leq N\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leq N\left(r,\frac{f^{(k)}}{f^{(k+1)}}\right)\leq$$

$$\leq N\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+S(r,f)\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).$$

So, we have

$$\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)\leq (k+1)\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.9}$$

Similarly

$$\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\leq (k+1)\overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+S(r,g). \tag{2.10}$$

We obtain from (2.3) – (2.10) that

$$T(r,g)\leq (2k+3)\overline{N}(r,f)+(2k+4)\overline{N}(r,g)+(k+2)\overline{N}\left(r,\frac{1}{f}\right)+$$

$$+(2k+3)\overline{N}\left(r,\frac{1}{g}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$. Hence

$$T(r,g)\leq \Big\{\big[(7k+14)-(2k+3)\Theta(\infty,f)-(2k+4)\Theta(\infty,g)-$$

$$-(k+2)\Theta(0,f)-(2k+3)\Theta(0,g)-\delta_{k+1}(0,f)-\delta_{k+1}(0,g)\big]+\varepsilon\Big\}T(r,g)+S(r,g) \qquad (2.11)$$

for $r \in I$ and $0 < \varepsilon < \Delta - (7k+13)$. Thus, we obtain from (2.1) and (2.11) that $T(r,g) \le S(r,g)$ for $r \in I$, a contradiction.

Hence, we get $h(z) \equiv 0$; that is

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}.$$

By solving this equation, we obtain

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}. \qquad (2.12)$$

Where a, b are two constants. Next, we consider three cases.

Case 1. $b \ne 0$ and $a = b$.

Subcase 1.1. $b = -1$. Then we deduce from (2.12) that $f^{(k)}(z)g^{(k)}(z) \equiv 1$.

Subcase 1.2. $b \ne -1$. Then we get from (2.12) that

$$\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)}-1}$$

so

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \dfrac{1}{1+b}}\right) \le \overline{N}\left(r, \frac{1}{f^{(k)}}\right). \qquad (2.13)$$

From (2.13) and (2.8), we get

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \dfrac{1}{1+b}}\right) \le (k+1)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

By Lemma 2.2, we have

$$T(r,g) \le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - \dfrac{1}{b+1}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) \le$$

$$\le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + k\overline{N}(r,f) + (k+1)\overline{N}\left(r, \frac{1}{f}\right) + S(r,f) + S(r,g) \le$$

$$\le (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r, \frac{1}{f}\right) +$$

$$+(2k+3)\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g).$$

That is $T(r,g) \le (7k+14-\Delta)T(r,g) + S(r,g)$ for $r \in I$.

Thus, by (2.1), we obtain that $T(r,g) \le S(r,g)$ for $r \in I$, a contradiction.

Case 2. $b \neq 0$ and $a \neq b$.

Subcase 2.1. $b = -1$. Then we obtain from (2.12) that

$$f^{(k)} = \frac{a}{-g^{(k)} + a + 1}.$$

Therefore

$$\overline{N}\left(r, \frac{a}{-g^{(k)} + a + 1}\right) = \overline{N}(r, f^{(k)}) = \overline{N}(r, f).$$

By Lemma 2.2, we have

$$T(r,g) \leq \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - (a+1)}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g) \leq$$

$$\leq (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r, \frac{1}{f}\right) +$$

$$+(2k+3)\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g).$$

Using the argument as in case 1, we get a contradiction.

Subcase 2.2. $b \neq -1$. Then we get from (2.12) that

$$f^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2\left(g^{(k)} + \dfrac{a-b}{b}\right)}.$$

Therefore

$$\overline{N}\left(r, \frac{1}{g^{(k)} + \dfrac{a-b}{b}}\right) = \overline{N}\left(r, f^{(k)} - \left(1 + \frac{1}{b}\right)\right) = \overline{N}(r,f).$$

By Lemma 2.2, we get

$$T(r,g) \leq \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} + \dfrac{a-b}{b}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) + S(r,g) \leq$$

$$\leq (2k+3)\overline{N}(r,f) + (2k+4)\overline{N}(r,g) + (k+2)\overline{N}\left(r, \frac{1}{f}\right) +$$

$$+(2k+3)\overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g).$$

Using the argument as in case 1, we get a contradiction.

Case 3. $b = 0$. From (2.12), we obtain

$$f = \frac{1}{a} g + P(z), \tag{2.14}$$

where $P(z)$ is a polynomial. If $P(z) \neq 0$, then by Lemma 2.3, we have

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-P}\right) + S(r,f) \leq$$

$$\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f). \tag{2.15}$$

From (2.14), we obtain $T(r,f) = T(r,g) + S(r,f)$. Hence, substituting this into (15), we get

$$T(r,f) \leq \left\{3 - \Big[\Theta(\infty,f) + \Theta(0,f) + \Theta(0,g)\Big] + \varepsilon\right\} T(r,f) + S(r,f),$$

where

$$0 < \varepsilon < 1 - \delta_{k+1}(0,f) + 1 - \delta_{k+1}(0,g) + (2k+2)\big[1 - \Theta(\infty,f)\big] +$$

$$+ (2k+4)\big[1 - \Theta(\infty,g)\big] + \big[1 - \Theta(0,f)\big] + 2\big[1 - \Theta(0,g)\big].$$

Therefore $T(r,f) \leq [7k + 14 - \Delta] T(r,f) + S(r,f)$.

That is $\big[\Delta - (7k+13)\big] T(r,f) < S(r,f)$.

Hence, by (2.1), we deduce that $T(r,f) \leq S(r,f)$ for $r \in I$, a contradiction.

Therefore, we deduce that $P(z) \equiv 0$, that is

$$f = \frac{1}{a} g. \tag{2.16}$$

If $a \neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 IM, we deduce from (2.16) that $g^{(k)} \neq 1$. That is $\overline{N}\Big(r, \dfrac{1}{g^{(k)} - 1}\Big) = 0$.

Next, we can deduce a contradiction as in case 1. Thus, we get that $a = 1$, that is $f \equiv g$.

Lemma 2.5 is proved.

Lemma 2.6 (see [9]). *Let f and g be two nonconstant entire functions*, $n \geq 1$. *If $f^n f' g^n g' = 1$, then $f(z) = c_2 e^{-cz}$, $g(z) = c_1 e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

3. Proof of Theorem 1.1. Let $F = \dfrac{f^{n+1}}{n+1}$ and $G = \dfrac{g^{n+1}}{n+1}$. Then $F' = f^n f'$ and $G' = g^n g'$ share the value 1 IM.

Consider

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f^{n+1}}\right) \leq \frac{1}{s(n+1)} N\left(r, \frac{1}{F}\right) \leq \frac{1}{s(n+1)} \left[T\left(r, \frac{1}{F}\right) + O(1)\right].$$

Therefore

$$\Theta(0,F) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \dfrac{1}{F}\right)}{T(r,F)} \geq 1 - \frac{1}{s(n+1)}, \tag{3.1}$$

$$\delta_{k+1}(0,F)=1-\overline{\lim_{r\to\infty}}\frac{N_{k+1}\left(r,\dfrac{1}{F}\right)}{T(r,F)}\geq 1-\overline{\lim_{r\to\infty}}\frac{(k+1)\overline{N}\left(r,\dfrac{1}{F}\right)}{T(r,F)}\geq 1-\frac{k+1}{s(n+1)}. \qquad (3.2)$$

Similarly

$$\Theta(0,G)\geq 1-\frac{1}{s(n+1)}, \qquad (3.3)$$

$$\Theta(\infty,F)\geq 1-\frac{1}{s(n+1)}, \qquad (3.4)$$

$$\Theta(\infty,G)\geq 1-\frac{1}{s(n+1)}, \qquad (3.5)$$

$$\delta_{k+1}(0,G)\geq 1-\frac{k+1}{s(n+1)}. \qquad (3.6)$$

From (3.1) – (3.6), we get

$$\Delta=(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+(k+2)\Theta(0,f)+(2k+3)\Theta(0,g)+$$

$$+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\geq 7k+14-\frac{9k+14}{s(n+1)}. \qquad (3.7)$$

Since $s(n+1)\geq 24$, for $k=1$, we obtain $\Delta>20$ from (3.7). Hence by Lemma 2.5, we get either $F'G'\equiv 1$ or $F\equiv G$.

Consider the case $F'G'\equiv 1$, that is

$$f^nf'g^ng'\equiv 1. \qquad (3.8)$$

Suppose that f has a pole z_0 (with order $p\geq s$ say). Then z_0 is a zero of g (with order $m\geq s$ say). By (3.8), we get

$$nm+m-1=np+p+1.$$

That is, $(m-p)(n+1)=2$, which is impossible since $n\geq 2$ and m, p are positive integers. Therefore, we conclude that f and g are entire functions. From Lemma 2.6, we get $f(z)=c_2e^{-cz}$, $g(z)=c_1e^{cz}$, where c, c_1 and c_2 are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$.

Next we consider another case $F\equiv G$. This gives $f^{n+1}=g^{n+1}$. So $f=tg$ for a constant t such that $t^{n+1}=1$.

Theorem 1.1 is proved.

4. Proof of Theorem 1.2. Since f and g are entire functions, we have $N(r,f)=N(r,g)=0$. Proceeding as in the proof Theorem 1.1 and applying Lemma 2.5 we shall obtain that Theorem 1.2 holds.

5. One open question.

Question 1. Can the condition $(n+1)s\geq 24$ in Theorem 1.1 be further relaxed?

1. *Cluine J.* On a result of Hayman // J. London Math. Soc. – 1967. – **42**. – P. 389 – 392.
2. *Hayman W. K.* Picard value of meromorphic functions and their derivatives // Ann. Math. – 1959. – **70**. – P. 9 – 24.
3. *Li J. D.* Uniqueness of meromorphic functions sharing one value IM // J. Sichuan Univ. (Nat. Sci. Ed.). – 2008. – **45**, № 1. – P. 21 – 24.
4. *Dyavanal R. S.* Uniqueness and value-sharing of differential polynomials of meromorphic functions // J. Math. Anal. and Appl. – 2011. – **374**. – P. 335 – 345.
5. *Yang L.* Distribution theory. – Berlin: Springer-Verlag, 1993.
6. *Lahiri I.* Weighted sharing and Uniqueness of meromorphic functions // Nagoya Math. J. – 2001. – **161**. – P. 193 – 206.
7. *Yi H. X., Yang C. C.* Uniqueness theory of meromorphic functions. – Beijing: Sci. Press, 1995.
8. *Zhang Q. C.* Meromorphic function that share one small function with its derivative // J. Inequal. Pure and Appl. Math. – 2005. – **6**, № 4. – Art. 116.
9. *Yang C. C., Hua X.* Uniqueness and value sharing of meromorphic functions // Ann. Acad. Sci. Fenn. Math. – 1997. – **22**. – P. 395 – 406.

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