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ANISOTROPIC DIFFERENTIAL OPERATORS WITH PARAMETERS AND APPLICATIONS

АНІЗОТРОПНІ ДИФЕРЕНЦІАЛЬНІ ОПЕРАТОРИ З ПАРАМЕТРАМИ ТА ЇХ ЗАСТОСУВАННЯ

In this paper, we study the boundary-value problems for parameter-dependent anisotropic differential-operator equations with variable coefficients. Several conditions for the uniform separability and Fredholmness in Banach-valued L_p-spaces are given. Sharp uniform estimates for the resolvent are established. They imply that the indicated operator is uniformly positive. Moreover, it is also the generator of an analytic semigroup. As an application, the maximal regularity properties of the parameter-dependent abstract parabolic problem and infinite systems of parabolic equations are derived in mixed $L_{\mathbf{p}}$-spaces.

Вивчаються граничні задачі для анізотропних диференціально-операторних рівнянь зі змінними коефіцієнтами, що залежать від параметрів. Наведено кілька умов рівномірної сепарабельності та фредгольмовості в банаховозначних L_p-просторах. Встановлено точні рівномірні оцінки для резольvenти, з яких випливає, що вказаний оператор є рівномірно додатним. Більш того, він є також генератором деякої аналітичної напівгрупи. Як застосування, встановлено властивості максимальної регулярності абстрактної параболічної задачі, що залежить від параметра, та нескінченних систем рівнянь параболічного типу в $L_{\mathbf{p}}$-просторах.

1. Introduction and notations. It is well known that many classes of PDEs, pseudo-DEs and integro-DEs can be expressed as differential-operator equations (DOEs). DOEs have been studied extensively in the literature (see [1–5, 8–11, 13–24, 26–29] and the references therein).

The main aim of the present paper is to discuss the uniform separability properties of BVPs for the following higher order parameter dependent anisotropic DOE:

$$\sum_{k=1}^{n} \varepsilon_k a_k(x) \frac{\partial^{l_k} u}{\partial x_k^{l_k}} + A(x)u + \sum_{|\alpha:l|<1} \prod_{k=1}^{n} \varepsilon_k^{\alpha_k/l_k} A_\alpha(x) D^\alpha u = f(x), \tag{1}$$

where ε_k are small positive parameters, $a_k(x)$ are complex valued continuous functions, $A(x)$ and $A_\alpha(x)$ are operator valued functions, defined for $x \in \Omega$, where Ω is some region in R^n with the operators $A(x)$ and $A_\alpha(x)$, acting in a Banach space E, $u(x)$ and $f(x)$ respectively are a E valued unknown and date functions. The above DOE is a generalized form of the elliptic equation with parameters. In fact, the special case $l_k = 2m$, $k = 1, \ldots, n$, the equation (1) reduces to elliptic equation. Note, the principal part of the corresponding differential operator is non self-adjoint. Nevertheless, the sharp uniform coercive estimate for the resolvent and Fredholmness are established. Note that, maximal regularity properties for higher order anisotropic DOEs were studied, e.g., in [3, 5, 21, 23]. Unlike to these, in the present paper, the nonlocal BVP for parameter depended undegenerate anisotropic equation is studied and uniform separability properties is derived. In application, the maximal regularity properties of mixed problem for the following parabolic equation:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} \varepsilon_k a_k(x) \frac{\partial^{l_k} u}{\partial x^{l_k}} + A(x)u = f(t,x) \tag{2}$$

are obtained. Particularly, the problem (2) occur in atmospheric dispersion of pollutants and evolution models for phytoremediation of metals from soils. Really, if $E = R^3$, $A(x)$ is a 3-dimensional functional matrices, i.e., $A(x) = [a_{ij}(x)]$, $u = (u_1, u_2, u_3)$, $i, j = 1, 2, 3$, then we get the well posedeness of the IVP for the system of parabolic PDE with parameters

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} (-1)^{l_k} \varepsilon_k a_k(x) \frac{\partial^{2l_k} u_i}{\partial x^{2l_k}} + \sum_{j=1}^{3} a_{ij}(x) u_j = f_i(t, x)$$

which arises in phytoremediation process.

Let $L_p(\Omega; E)$ denote the space of all strongly measurable E-valued functions that are defined on the region $\Omega \subset R^n$ with the norm

$$\|f\|_p = \|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

The Banach space E is called a UMD-space if the Hilbert operator $(Hf)(x) = = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$ is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (see., e.g., [6]). UMD-spaces include e.g. L_p-, l_p-spaces and Lorentz spaces L_{pq}, p, $q \in (1, \infty)$.

Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda; \lambda \in \mathbb{C}, |\arg \lambda| \leq \varphi\} \cup \{0\}, \quad 0 \leq \varphi < \pi.$$

Let E_1 and E_2 be two Banach spaces. $B(E_1, E_2)$ denotes the space of bounded linear operators from E_1 into E_2 endowed with the usual uniform operator topology. For $E_1 = E_2$ it denotes by $B(E)$. Now $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$ will denote interpolation spaces defined by the K method [25] (§ 1.3.1).

A linear operator A is said to be φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq M (1 + |\lambda|)^{-1}$$

for all $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, I is an identity operator in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by A_λ. It is known [25] (§ 1.15.1) that there exists fractional powers A^θ of the positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ endowed with graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\| A^\theta u \right\|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

A set $W \subset B(E_1, E_2)$ is called R-bounded (see [6, 8, 26]) if there is a constant $C > 0$ such that for all $T_1, T_2, \ldots, T_m \in W$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^{m} r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is an arbitrary sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$.

The smallest C for which the above estimate holds is called a R-bound of the collection W and is denoted by $R(W)$.

Let $S(R^n; E)$ denote the Schwartz class, i.e., the space of all E-valued rapidly decreasing smooth functions on R^n equipped with its usual topology generated by seminorms. Let Ω be a domain in R^n. $C(\Omega; E)$ and $C^{(m)}(\Omega; E)$ will denote the spaces of E-valued bounded uniformly strongly continuous and m-times continuously differentiable functions on Ω, respectively. Let F denotes the Fourier transformation. A function $\Psi \in C(R^n; B(E))$ is called a Fourier multiplier in $L_p(R^n; E)$ if the map $u \to \Phi u = F^{-1}\Psi(\xi) Fu$, $u \in S(R^n; E)$ is well defined and extends to a bounded linear operator in $L_p(R^n; E)$. The set of all multipliers in $L_p(R^n; E)$ will denoted by $M_p^p(E)$.

Let

$$U_n = \{\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n \colon \beta_k \in \{0, 1\}\}.$$

Definition 1. *A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in C^{(n)}(R^n; B(E))$ the R-boundedness of the set $\left\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\}$ implies that Ψ is a Fourier multiplier in $L_p(R^n; E)$, i.e., $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.*

Let $\Psi_h \in M_p^p(E)$ be a multiplier function dependent of the parameter $h \in Q$. The uniform R-boundedness of the set $\left\{\xi^\beta D^\beta \Psi_h(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\}$, i.e.,

$$\sup_{h \in Q} R\left(\left\{\xi^\beta D^\beta \Psi_h(\xi) : \xi \in R^n \setminus \{0\}, \beta \in U_n\right\}\right) \leq K$$

implies that Ψ_h is a uniform collection of Fourier multipliers.

***Remark* 1.** Note that, if E is UMD-space then e.g., by virtue of [8] (Theorem 3.25) it satisfies the multiplier condition.

Definition 2. *The φ-positive operator A is said to be R-positive in a Banach space E if the set $\left\{A(A + \xi I)^{-1} : \xi \in S_\varphi\right\}$ is R-bounded.*

An operator function $A(x)$ is said to be φ-positive in E uniformly in x if domain $D(A(x))$ of the $A(x)$ is independent of x, $D(A(x))$ is dense in E and $\left\|(A(x) + \lambda I)^{-1}\right\| \leq \dfrac{M}{1 + |\lambda|}$ for any $\lambda \in S_\varphi$, $\varphi \in [0\,,\,\pi)$.

The φ-positive operator $A(x)$, $x \in G$ is said to be uniformly R-positive in a Banach space E if there exists $\varphi \in [0\,,\,\pi)$ such that the set

$$\left\{A(x)(A(x) + \xi I)^{-1} : \xi \in S_\varphi\right\}$$

is uniformly R-bounded, i.e.,

$$\sup_{x \in G} R\left(\left[A(x)(A(x) + \xi I)^{-1}\right] : \xi \in S_\varphi\right) \leq M.$$

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. For $E_1 = E_2 = E$ it is denoted by $\sigma_\infty(E)$.

Let $D(\Omega; E)$ denote the class of all E-valued infinite differentiable functions on domain Ω with compact supports. For $E = \mathbb{C}$ it denotes by $D(\Omega)$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-tuples of positive integer, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$ and $|\alpha| = \sum_{k=1}^n \alpha_k$.

Definition 3. *Let $f \in L_p(\Omega; E)$. The function $(D^\alpha f) : \Omega \to E$ is called to be generalized derivative of f on Ω if the following equality*:

$$\int_\Omega D^\alpha f(x) \varphi(x) \, dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \varphi(x) \, dx$$

holds for all $\varphi \in D(\Omega)$.

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E and $l = (l_1, l_2, \ldots, l_n)$.

We let $W_p^l(\Omega; E_0, E)$ denote the space of all functions $u \in L_p(\Omega; E_0)$ possessing generalized derivatives $D_k^{l_k} u = \dfrac{\partial^{l_k} u}{\partial x_k^{l_k}}$ such that $D_k^{l_k} u \in L_p(\Omega; E)$ with the norm

$$\|u\|_{W_p^l(\Omega; E_0, E)} = \|u\|_{L_p(\Omega; E_0)} + \sum_{k=1}^n \left\| D_k^{l_k} u \right\|_{L_p(\Omega; E)} < \infty.$$

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Consider the following parameterized norm:

$$\|u\|_{W_{p,\varepsilon}^l(\Omega; E_0, E)} = \|u\|_{L_p(\Omega; E_0)} + \sum_{k=1}^n \varepsilon_k \left\| D_k^{l_k} u \right\|_{L_p(\Omega; E)} < \infty.$$

If $G_+ = G \times R_+$, $\mathbf{p} = (p, p_1)$, $L_{\mathbf{p}}(G_+; E)$ will be denote the space of all $\mathbf{p}$-summable E-valued functions with mixed norm (see, e.g., [7] for $E = \mathbb{C}$), i.e., the space of all measurable E-valued functions f defined on G for which

$$\|f\|_{L_{\mathbf{p}}(G_+)} = \left(\int_G \left(\int_{R_+} \|f(t,x)\|_E^p \, dt \right)^{p_1/p} dx \right)^{1/p_1} < \infty.$$

Analogously, $W_{\mathbf{p}}^l(G_+; E)$ denotes the E-valued anisotropic Sobolev space with corresponding mixed norm. Let

$$W_{\mathbf{p}}^l(G_+; E_0, E) = W_{\mathbf{p}}^l(G_+; E) \cap L_{\mathbf{p}}(G_+; E_0)$$

endowed with norm

$$\|u\|_{W_{\mathbf{p}}^l(G_+; E_0, E)} = \|u\|_{L_{\mathbf{p}}(G_+; E_0)} + \sum_{k=1}^n \left\| D_k^{l_k} u \right\|_{L_{\mathbf{p}}(G_+; E)} < \infty.$$

2. Background. The embedding theorems in vector valued spaces play a key role in the theory of DOEs. For estimating lower order derivatives we use following embedding theorems from [24].

Theorem $\mathbf{A_1}$. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$ and suppose the following conditions are satisfied*:

(1) *E is a Banach space satisfying the multiplier condition*;

(2) *A is an R-positive operator in E*;

(3) $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $l = (l_1, l_2, \ldots, l_n)$ *are* n*-tuples of nonnegative integer such that* $\kappa = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \leq 1$, $0 \leq \mu \leq 1 - \kappa$, $1 < p < \infty$, $0 < h \leq h_0$, h_0 *is a fixed positive number and* ε_k *are small positive parameters*;

(4) $\Omega \subset R^n$ *is a region such that there exists a bounded linear extension operator from* $W_p^l(\Omega; E(A), E)$ *to* $W_p^l(R^n; E(A), E)$.

Then the embedding $D^\alpha W_p^l(\Omega; E(A), E) \subset L_p\left(\Omega; E\left(A^{1-\kappa-\mu}\right)\right)$ *is continuous and for all* $u \in W_p^l(\Omega; E(A), E)$ *the following uniform estimate holds*:

$$\prod_{k=1}^{n} \varepsilon_k^{\alpha_k/l_k} \|D^\alpha u\|_{L_p(\Omega; E(A^{1-\kappa-\mu}))} \leq h^\mu \|u\|_{W_{p,\varepsilon}^l(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega; E)}.$$

***Remark* 2.** If $\Omega \subset R^n$ is a region satisfying the strong l-horn condition (see [7], § 7), $E = \mathbb{R}$, $A = I$, then for $p \in (1, \infty)$ there exists a bounded linear extension operator from $W_p^l(\Omega) = W_p^l(\Omega; R, R)$ to $W_p^l(R^n) = W_p^l(R^n; R, R)$.

Theorem A$_2$. *Suppose all conditions of Theorem* A_1 *are satisfied for* $0 < \mu \leq 1 - \kappa$. *Moreover, let* Ω *be a bounded region and* $A^{-1} \in \sigma_\infty(E)$. *Then the embedding*

$$D^\alpha W_p^l(\Omega; E(A), E) \subset L_p\left(\Omega; E\left(A^{1-\kappa-\mu}\right)\right)$$

is compact.

Theorem A$_3$. *Suppose all conditions of Theorem* A_1 *satisfied. Let* $0 < \mu \leq 1 - \kappa$. *Then the embedding*

$$D^\alpha W_p^l(\Omega; E(A), E) \subset L_p\left(\Omega; (E(A), E)_{\kappa,p}\right)$$

is continuous and there exists a positive constant C_μ *such that for all* $u \in W_p^l(\Omega; E(A), E)$ *the uniform estimate holds*

$$\prod_{k=1}^{n} \varepsilon_k^{\alpha_k/l_k} \|D^\alpha u\|_{L_p\left(\Omega; (E(A), E)_{\kappa,p}\right)} \leq C_\mu \left[h^\mu \|u\|_{W_{p,\varepsilon}^l(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega; E)}\right].$$

3. Statement of the problem. Consider the nonlocal BVP for the following parameter dependent anisotropic DOE with variable coefficients:

$$\sum_{k=1}^{n} \varepsilon_k a_k(x) D_k^{l_k} u(x) + [A(x) + \lambda] u(x) + \sum_{|\alpha: l| < 1} \prod_{k=1}^{n} \varepsilon_k^{\alpha_k/l_k} A_\alpha(x) D^\alpha u(x) = f(x), \qquad (3)$$

$$\sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_{ik}} \left[\alpha_{kji} D_k^i u(G_{k0}) + \beta_{kji} D_k^i u(G_{kb})\right] = 0, \quad j = 1, 2, \ldots, l_k, \quad k = 1, 2, \ldots, n, \qquad (4)$$

where

$$\sigma_{ik} = \frac{1}{l_k}\left(i + \frac{1}{p}\right), \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad l = (l_1, l_2, \ldots, l_n), \quad |\alpha: l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k},$$

$$G=\{x=(x_1,x_2,\ldots,x_n)\,,\ 0<x_k<b_k\}\,,\quad G_{k0}=(x_1,x_2,\ldots,x_{k-1},0,x_{k+1},\ldots,x_n)\,,$$

$$G_{kb}=(x_1,x_2,\ldots,x_{k-1},b_k,x_{k+1},\ldots,x_n)\,,\quad m_{kj}\in\{0,1,\ldots,l_k-1\}\,,$$

$$x\,(k)=(x_1,x_2,\ldots,x_{k-1},x_{k+1},\ldots,x_n)\,,\quad G_k=\prod_{j\neq k}(0,b_j)\,,\quad j,k=1,2,\ldots,n,$$

$\alpha_{kji}, \beta_{kji}$ are complex numbers, λ is a complex and ε_k are small positive parameters; a_k are complex-valued functions on G, $A\,(x)$ and $A_\alpha\,(x)$ are linear operators in E for $x\in G$. We assume that the domain $D\,(A\,(x))$ of operator valued function $A\,(x)$ is independent of x. So, it will be denote by $D\,(A)$. The same time, the graphical norm $E\,(A\,(x))$ will be denote by $E\,(A)$.

A function $u\in W_p^l\,(G;E\,(A)\,,E,L_{kj})=\left\{u\in W_p^l\,(G;E\,(A)\,,E)\,,L_{kj}u=0\right\}$ satisfying (3) a. e. on G is said to be solution of the problem (3), (4).

We say the problem (3), (4) is L_p-separable, if for all $f\in L_p\,(G;E)$ there exists a unique solution $u\in W_p^l\,(G;E\,(A)\,,E)$ of the problem (3), (4) and a positive constant C depending only on G,p,l,E,A such that the following uniform coercive estimate holds:

$$\sum_{k=1}^{n}\varepsilon_k\left\|D_k^{l_k}u\right\|_{L_p(G;E)}+\|Au\|_{L_p(G;E)}\le C\,\|f\|_{L_p(G;E)}\,.$$

By applying the trace theorem [25] (§ 1.8.2) we have the following theorem.

Theorem A$_4$. *Let m and j be integer numbers, $0\le j\le m-1$, $\theta_j=\dfrac{pj+1}{pm}$, $0<\varepsilon\le 1$, $x_0\in[0,b]$. Then, for $u\in W_p^m\,(0,b;E_0,E)$ the transformations $u\to u^{(j)}\,(x_0)$ are bounded linear from $W_p^m\,(0,b;E_0,E)$ onto $(E_0,E)_{\theta_j,p}$ and the following inequality holds*:

$$\varepsilon^{\theta_j}\left\|u^{(j)}\,(x_0)\right\|_{(E_0,E)_{\theta_j,p}}\le C\left(\left\|\varepsilon u^{(m)}\right\|_{L_p(0,b;E)}+\|u\|_{L_p\ (0,b;E_0)}\right).$$

Proof. By virtue of [25] (§ 1.8.2), for $u\in W_p^m\,(0,b;E_0,E)$ the following inequality holds:

$$\left\|u^{(j)}\,(x_0)\right\|_{(E_0,E)_{\theta_j,p}}\le C\left(\left\|u^{(m)}\right\|_{L_p(0,b;E)}+\|u\|_{L_p\ (0,b;E_0)}\right).$$

Putting $\tilde u\,(x)=u\,(\mu x)$ for $0<\mu<1$ and by applying the above estimate to $\tilde u\,(x)$ we have

$$\mu^j\left\|u^{(j)}\,(x_0)\right\|_{(E_0,E)_{\theta_j,p}}\le$$

$$\le C\left[\mu^m\left(\int_0^b\left\|u^{(m)}\,(\mu x)\right\|_E^p dx\right)^{1/p}+\left(\int_0^b\|u\,(\mu x)\|_{E_o}^p\,dx\right)^{1/p}\right].$$

Substituting $y=\mu x$, in view of $\mu<1$ we get

$$\mu^j\left\|u^{(j)}\,(x_0)\right\|_{(E_0,E)_{\theta_j,p}}\le C\left[\mu^{m-1/p}\left\|u^{(m)}\right\|_{L_p(0,\mu b;E)}+\mu^{-1/p}\,\|u\|_{L_p\ (0,\mu b;E_0)}\right]\le$$

$$\leq C\left[\mu^{m-1/p}\left\|u^{(m)}\right\|_{L_p(0,b;E)}+\mu^{-1/p}\left\|u\right\|_{L_p\ (0,b;E_0)}\right].$$

By chousing $\mu^m=\varepsilon$ we obtain the assertion.

Let

$$G_{kx_0}=\left(x_1,x_2,\ldots,x_{k-1},x_0,x_{k+1},\ldots,x_n\right),\quad x_0\in\left(0,b_k\right),\quad k=1,2,\ldots,n,$$

$$X_k=L_p\left(G_k;E\right),\quad Y_k=W_p^{l(k)}\left(G_k;E\left(A\right),E\right),\quad l^{(k)}=\left(l_1,l_2,\ldots,l_{k-1},l_{k+1},\ldots,l_n\right).$$

By virtue of Theorem A_4 we obtain the following theorem.

Theorem A_5. *Let l_k and j be integer numbers, $\theta_{jk}=\dfrac{1+pj+1}{pl_k}$, $x_{k0}\in\left[0,b_k\right]$, $j=0,1,\ldots,l_k-1$, $k=1,2,\ldots,n$. Then, for any $u\in W_p^l\left(G;E_0,E\right)$ the transformation $u\to D_k^j u\left(G_{kx_0}\right)$ is bounded linear from $W_p^l\left(G;E_0,E\right)$ onto F_{kj} and the following uniform estimate holds*:

$$\varepsilon_k^{\theta_{jk}}\left\|D_k^j u\left(G_{kx_0}\right)\right\|_{F_{kj}}\leq C\left[\left\|u\right\|_{L_p(G;E)}+\varepsilon_k\left\|D_k^{l_k}u\right\|_{L_p(G;E)}+\sum_{j\neq k}\left\|D_j^{l_j}u\right\|_{L_p(G;E)}\right].$$

Proof. It is clear that

$$W_p^l\left(G;E_0,E\right)=W_p^{l_k}\left(0,b_k;Y_k,X_k\right).$$

Then by applying the Theorem A_3 to the space $W_p^{l_k}\left(0,b_k;Y_k,X_k\right)$ we obtain the assertion.

4. BVP for partial DOE with parameters. Let us first consider the BVP for the parameter-dependent DOE with constant coefficients

$$\left(L_\varepsilon+\lambda\right)u=\sum_{k=1}^{n}\varepsilon_k a_k D_k^{l_k}u\left(x\right)+\left(A+\lambda\right)u\left(x\right)=f\left(x\right),\tag{5}$$

$$L_{kj}u=\sum_{i=0}^{m_{kj}}\varepsilon_k^{\sigma_{ik}}\left[\alpha_{kji}D_k^i u\left(G_{k0}\right)+\beta_{kji}D_k^i u\left(G_{kb}\right)\right]=f_{kj},\tag{6}$$

$$j=1,2,\ldots,l_k,\quad k=1,2,\ldots,n,$$

where σ_{ik}, G_{k0} and G_{kb} are defined by (4), a_k are complex numbers, λ is a complex and ε_k are small positive parameters and A is a linear operator in a Banach space E. Let $\omega_{k1},\omega_{k2},\ldots,\omega_{kl_k}$ be the roots of the characteristic equations

$$a_k\,\omega^{l_k}+1=0,\quad k=1,2,\ldots,n.$$

Let $\left[\upsilon_{knj}\right]$ be l_k-dimensional matrix, and $\eta_k=\left|\left[\upsilon_{knj}\right]\right|$ be determinant of matrix $\left[\upsilon_{kij}\right]$, where

$$\upsilon_{kij}=\alpha_{kjm_j}\left(-\omega_{ki}\right)^{l_k},\quad i=1,2,\ldots,d_k,\quad \upsilon_{kij}=\beta_{kjm_j}\omega_{ki}^{l_k},$$

$$i=d_k+1,d_k+2,\ldots,l_k,\quad 0<d_k<l_k,\quad i,j=1,2,\ldots,l_k.$$

Condition 1. Assume the following conditions are satisfied:
(1) E is a Banach space satisfying the multiplier condition;
(2) A is an R-positive operator in E for $\varphi \in [0\,,\,\pi)$;
(3) $a_k \neq 0$, $\left|\alpha_{kjm_j}\right| + \left|\beta_{kjm_j}\right| > 0$, $\eta_k \neq 0$ and

$$|\arg \omega_{kj} - \pi| \leq \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k, \quad |\arg \omega_{kj}| \leq \frac{\pi}{2} - \varphi, \quad j = d_k + 1, \ldots, l_k$$

for $0 < d_k < l_k$, $k = 1, 2, \ldots, n$.
Consider at first, the homogenous BVP

$$(L_\varepsilon + \lambda)\, u = \sum_{k=1}^{n} \varepsilon_k a_k D_k^{l_k} u\,(x) + (A + \lambda)\, u\,(x) = f\,(x)\,, \tag{7}$$

$$L_{kj} u = 0, \quad j = 1, 2, \ldots, l_k. \tag{8}$$

Let $B\,(\varepsilon)$ denote the operator in $L_p\,(G; E)$ generated by BVP (7), (8), i. e., the operator defined as

$$D\,(B\,(\varepsilon)) = W_p^l\,(G; E\,(A)\,, E, L_{kj}) = \Big\{u \in W_p^l\,(G; E\,(A)\,, E)\,, \quad L_{kj} u = 0,$$

$$j = 1, 2, \ldots, l_k, k = 1, 2, \ldots, n,\ B\,(\varepsilon)\, u = \sum_{k=1}^{n} \varepsilon_k a_k D_k^{l_k} u + Au\Big\}.$$

In a similar way as [5] (Theorem 5.1), [18] and [24] we obtain the following theorem.
Theorem $\mathbf{A_6}$. *Let Condition* 1 *be satisfied. Then*:
(a) *problem* (7), (8) *for* $f \in L_p\,(G; E)\,, \lambda \in S_\varphi, \varphi \in [0\,,\,\pi)$ *and sufficiently large* $|\lambda|$ *has a unique solution* u *that belongs to* $W_p^l\,(G; E\,(A)\,, E)$ *and the following coercive uniform estimate holds*:

$$\sum_{k=1}^{n} \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k}\, \varepsilon_k^{i/l_k} \left\|D_k^i u\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq M\, \|f\|_{L_p(G;E)}\,; \tag{9}$$

(b) *the operator* $B\,(\varepsilon)$ *is uniformly R-positive in* $L_p\,(G; E)\,.$
Now let

$$F_{kj} = (Y_k, X_k)_{\frac{1+pm_{kj}}{pl_k},p}\,.$$

From Theorems A_5 and A_6 we have the following theorem.
Theorem $\mathbf{A_7}$. *Suppose Condition* 1 *is satisfied. Then for sufficiently large* $|\lambda|$ *with* $|\arg \lambda| \leq \varphi$ *problem* (5), (6) *has a unique solution* $u \in W_p^l\,(G; E\,(A)\,, E)$ *for all* $f \in L_p\,(G; E)$ *and* $f_{kj} \in F_{kj}$. *Moreover, the following uniform coercive estimate holds*:

$$\sum_{k=1}^{n} \sum_{i=0}^{l_k} |\lambda|^{1-i/l_k}\, \varepsilon_k^{i/l_k} \left\|D_k^i u\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq$$

$$\leq M \left(\|f\|_{L_p(G;E)} + \sum_{k=1}^{n} \sum_{j=1}^{l_k} \|f_{kj}\|_{F_{kj}}\right). \tag{10}$$

Consider the BVP (3), (4). Let $\omega_{k1}(x), \omega_{k2}(x), \ldots, \omega_{kl_k}(x)$ denote the roots of the characteristic equations

$$a_k(x)\ \omega^{l_k} + 1 = 0, \quad k = 1, 2, \ldots, n.$$

Let $[\upsilon_{knj}]$ be l_k-dimensional matrix, and $\eta_k(x) = |[\upsilon_{knj}]|$ be determinant of matrix $[\upsilon_{kij}]$, where

$$\upsilon_{kij} = \alpha_{kjm_j}(-\omega_{ki})^{l_k}, \quad i = 1, 2, \ldots, d_k, \quad \upsilon_{kij} = \beta_{kjm_j}\omega_{ki}^{l_k},$$

$$i = d_k + 1, d_k + 2, \ldots, l_k, \quad 0 < d_k < l_k, \quad i, j = 1, 2, \ldots, l_k.$$

Condition 2. Assume:
(1) E is a Banach space satisfying the multiplier condition;
(2) operator valued function $A(x)$ is a uniformly R-positive operator in E for $\varphi \in [0\,,\pi)$;
(3) $a_k \neq 0$, $|\alpha_{kjm_j}| + |\beta_{kjm_j}| > 0$, $\eta_k(x) \neq 0$ and

$$|\arg\omega_{kj} - \pi| \leq \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k, \quad |\arg\omega_{kj}| \leq \frac{\pi}{2} - \varphi, \quad j = d_k + 1, \ldots, l_k,$$

for $x \in G$, $0 < d_k < l_k$, $k = 1, 2, \ldots, n$.

***Remark* 3.** Let $l_k = 2m_k$, $k = 1, 2, \ldots, n$, and $a_k = (-1)^{m_k} b_k(x)$, where b_k are real-valued positive functions and m_k are natural numbers. Then Condition 2 is satisfied for $\varphi \in [0\,,\pi)$.

Theorem 1. *Suppose Condition* 2 *is satisfied and the following hold*:
(1) $a_k(x)$ *are continuous functions on* $\bar{G}$, $a_j(0, x(k)) = a_j(b_k, x(k))$;
(2) $A(x)A^{-1}(\bar{x}) \in C(\bar{G}; B(E))$, $A(0, x(k)) = A(b_k, x(k))$;
(3) $A_\alpha(x)A^{(1-|\alpha:l|-\mu)}(x) \in L_\infty(G; B(E))$ *for* $0 < \mu < 1 - |\alpha\colon l|$.
Then problem (3), (4) *has a unique solution* $u \in W_p^l(G; E(A), E)$ *for* $f \in L_p(G; E)$ *and* $\lambda \in S_\varphi$ *with large enough* $|\lambda|$. *Moreover, the following coercive uniform estimate holds*:

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k} |\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i u\right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C\|f\|_{L_p(G;E)}. \tag{11}$$

Proof. First we will show the uniqueness of the solution. For this aim we use microlocal analysis. Let $D_1, D_2, \ldots, D_N$ be rectangular regions with sides parallel to coordinate planes covering G and let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be a corresponding partition of unity, i.e., $\varphi_j \in C_0^\infty(G)$, $\sigma_j = \operatorname{supp}\varphi_j \subset D_j$ and $\sum_{j=1}^{N}\varphi_j(x) = 1$, where $C_0^\infty(G)$ denotes the space of all infinitely differentiable functions on G with compact support. Now for $u \in W_p^l(G; E(A), E, L_{ki})$, being solution of the equation (3) and $u_j(x) = u(x)\varphi_j(x)$ we get

$$(L_\varepsilon + \lambda)u_j = \sum_{k=1}^{n}\varepsilon_k a_k(x)D_k^{l_k}u_j(x) + (A(x) + \lambda)u_j(x) = f_j(x), \quad L_{ki}u_j = 0, \tag{12}$$

where

$$f_j(x) = f(x)\varphi_j(x) + \sum_{k=1}^{n}\varepsilon_k a_k(x)\sum_{i=0}^{l_k-1}C_i^{l_k}\left(D_k^i u(x)\right)\left(D_k^{l_k-i}\varphi_j(x)\right) -$$

$$-\sum_{|\alpha:l|<1}\prod_{k=1}^{n}\varepsilon_k^{\alpha_k/l_k}\varphi_j(x)A_\alpha(x)D^\alpha u(x),\quad i=1,2,\ldots,l_k. \tag{13}$$

Freezing the coefficients of the equation (12), extending $u_j(x)$ in outside of σ_j we obtain the BVP

$$\sum_{k=1}^{n}\varepsilon_k a_k(x_{0j})D_k^{l_k}u_j(x)+(A(x_{0j})+\lambda)u_j(x)=F_j(x),\quad x\in D_j, \tag{14}$$

$$L_{ki}u_j=0,\quad i=1,2,\ldots,l_k,\quad k=1,2,\ldots,n,$$

where

$$F_j=f_j+[A(x_{0j})-A(x)]u_j+\sum_{k=1}^{n}\varepsilon_k[a_k(x_{0j})-a_k(x)]D_k^{l_k}u_j(x), \tag{15}$$

and C_i^k-are usual coefficients of binomial. By applying Theorem A_6 for all $u\in W_p^l(D_j;E(A),E)$ we obtain the following a priori estimate:

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i u_j\right\|_{L_p(D_j;E)}+\|Au_j\|_{L_p(D_j;E)}\le C\|F_j\|_{L_p(D_j;E)} \tag{16}$$

for problems (14) defined on domains D_j containing the boundary points. In a similar way we obtain the same estimates for domains $D_j\subset G$. By using the representation of F_j, by Theorem A_1, in view of the continuity of coefficients, choosing diameters of supp φ_j sufficiently small we get that for all small δ there is a positive continuous function $C(\delta)$ so that

$$\|F_j\|_{L_p(D_j;E)}\le\|f.\varphi_j\|_{L_p(D_j;E)}+\delta\|u_j\|_{W_{p,\varepsilon}^l(D_j;E(A),E)}+C(\delta)\|u_j\|_{L_p(D_j;E)}. \tag{17}$$

Consequently, from (15)–(17) we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i u_j\right\|_{L_p(D_j;E)}+\|Au_j\|_{L_p(D_j;E)}\le C\|f\|_{L_p(D_j;E)}+$$

$$+\delta\|u_j\|_{W_{p,\varepsilon}^l(D_j;E(A),E)}+M(\delta)\|u_j\|_{L_p(D_j;E)}. \tag{18}$$

Choosing $\varepsilon_k<1$ from (18) we obtain

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i u_j\right\|_{L_p(D_j;E)}+\|Au_j\|_{L_p(D_j;E)}\le$$

$$\le C\left[\|f\|_{L_p(D_j;E)}+\|u_j\|_{L_p(D_j;E)}\right]. \tag{19}$$

Then by using the equality $u(x)=\sum_{j=1}^{N}u_j(x)$ and (19) we get (11). Let O_ε denote the operator generated by problem (3), (4) for $\lambda=0$, i.e.,

$$D(O_\varepsilon)=W_p^l(G;E(A),E,L_{kj}),$$

$$O_\varepsilon u=\sum_{k=1}^{n}\varepsilon_k a_k\left(x\right)D_k^{l_k}u+A\left(x\right)u+\sum_{|\alpha:l|<1}\prod_{k=1}^{n}\varepsilon_k^{\alpha_k}A_\alpha\left(x\right)D^\alpha u.$$

It is clear that

$$\|u\|_{L_p(G;E)}=\frac{1}{|\lambda|}\left\|\left(O_\varepsilon+\lambda\right)u-O_\varepsilon u\right\|_{L_p(G;E)}\leq$$

$$\leq\frac{1}{|\lambda|}\left[\left\|\left(O_\varepsilon+\lambda\right)u\right\|_{L_p(G;E).}+\|O_\varepsilon u\|_{L_p(G;E)}\right].$$

Hence, by using the definition of $W_p^l\left(G;E\left(A\right),E\right)$ and applying Theorem A_1 we obtain

$$\|u\|_p\leq\frac{C}{|\lambda|}\left[\left\|\left(O_\varepsilon+\lambda\right)u\right\|_{L_p(G;E)}+\|u\|_{W_{p,\varepsilon}^l(G;E(A),E)}\right].$$

From the above estimate we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i u\right\|_{L_p(G;E)}+\|Au\|_{L_p(G;E)}\leq C\left\|\left(O_\varepsilon+\lambda\right)u\right\|_{L_p(G;E)}. \tag{20}$$

The estimate (20) implies that uniqueness of solution of the problem (3), (4). It implies that the operator $O_\varepsilon+\lambda$ has a bounded inverse in its rank space. We need to show that this rank space coincides with the space $L_p\left(G;E\right)$, i.e., we have to show that for all $f\in L_p\left(G;E\right)$ there is a unique solution of the problem (3), (4). We consider the smooth functions $g_j=g_j\left(x\right)$ with respect to a partition of unity $\varphi_j=\varphi_j\left(y\right)$ on the region G that equals 1 on supp φ_j, where supp $g_j\subset D_j$ and $|g_j\left(x\right)|<1$. Let us construct for all j the functions u_j that are defined on the regions $\Omega_j=G\cap D_j$ and satisfying problem (3), (4). The problem (3), (4) can be expressed as

$$\sum_{k=1}^{n}\varepsilon_k a_k\left(x_{0j}\right)D_k^{l_k}u_j\left(x\right)+A_\lambda\left(x_{0j}\right)u_j\left(x\right)=g_j\left\{f+\left[A\left(x_{0j}\right)-A\left(x\right)\right]u_j+\right.$$

$$\left.+\sum_{k=1}^{n}\varepsilon_k\left[a_k\left(x_{0j}\right)-a_k\left(x\right)\right]D_k^{l_k}u_j-\sum_{|\alpha:l|<1}\prod_{k=1}^{n}\varepsilon_k^{\alpha_k/l_k}A_\alpha\left(x\right)D^\alpha u_j\right\},\quad x\in D_j, \tag{21}$$

$$L_{ki}u_j=0,\quad j=1,2,\ldots,N.$$

Consider operators $O_{j\lambda}\left(\varepsilon\right)=O_j\left(\varepsilon\right)+\lambda$ in $L_p\left(D_j;E\right)$ that are generated by BVPs (14), i. e.,

$$D\left(O_j\left(\varepsilon\right)\right)=W_p^l\left(D_j;E\left(A\right),E,L_{ki}\right),\quad i=1,2,\ldots,l_k,\quad k=1,2,\ldots,n,$$

$$O_{j\lambda}\left(\varepsilon\right)u=\sum_{k=1}^{n}\varepsilon_k a_k\left(x_{0j}\right)D_k^{l_k}u_j\left(x\right)+\left[A\left(x_{0j}\right)+\lambda\right]u_j\left(x\right),\quad x\in D_j,\quad j=1,\ldots,N.$$

By virtue of Theorem A_6, the local operators $O_{j\lambda}$ have inverses $O_{j\lambda}^{-1}$ for $|\arg\lambda|\leq\varphi$ and for sufficiently large $|\lambda|$. Moreover, the operators $O_{j\lambda}^{-1}$ are bounded from $L_p\left(D_j;E\right)$ to $W_p^l\left(D_j;E\left(A\right),E\right)$ and for $f\in L_p\left(D_j;E\right)$ we have the following uniform estimate:

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\,\varepsilon_k^{i/l_k}\left\|D_k^i O_{j\lambda}^{-1}f\right\|_{L_p(D_j;E)}+\left\|AO_{j\lambda}^{-1}f\right\|_{L_p(D_j;E)}\leq C\,\|f\|_{L_p(D_j;E)}\,. \tag{22}$$

Extending solutions u_j of problems (21) zero on outside of supp φ_j and using the substitutions $u_j = O_{j\lambda}^{-1}\upsilon_j$ we obtain the operator equations

$$\upsilon_j = K_{j\lambda}\upsilon_j + g_j f, \quad j = 1, 2, \ldots, N, \tag{23}$$

where $K_{j\lambda} = K_{j\lambda}(\varepsilon)$ are bounded linear operators in $L_p(D_j; E)$ defined by

$$K_{j\lambda} = K_{j\lambda}(\varepsilon) = g_j\Bigg\{f + [A(x_{0j}) - A(x)]\,O_{j\lambda}^{-1} +$$

$$+\sum_{k=1}^{n}\varepsilon_k\,[a_k(x_{0j}) - a_k(x)]\,D_k^{l_k}O_{j\lambda}^{-1} - \sum_{|\alpha:\,l|<1}\prod_{k=1}^{n}\varepsilon_k^{\alpha_k/l_k}A_\alpha(x)\,D^\alpha O_{j\lambda}^{-1}\Bigg\}.$$

In fact, due to smoothness of the coefficients of the expression $K_{j\lambda}$ and in view of the estimate (22), for sufficiently large $|\lambda|$ there is a sufficiently small $\delta > 0$ such that

$$\left\|[A(x_{0j}) - A(x)]\,O_{j\lambda}^{-1}\upsilon_j\right\|_{L_p(D_j;E)} \leq \delta\,\|\upsilon_j\|_{L_p(D_j;E)}\,,$$

$$\sum_{k=1}^{n}\varepsilon_k\left\|[a_k(x_{0j}) - a_k(x)]\,D_k^{l_k}O_{j\lambda}^{-1}\upsilon_j\right\|_{L_p(D_j;E)} \leq \delta\,\|\upsilon_j\|_{L_p(D_j;E)}\,.$$

Moreover, from the assumption (2) and by Theorem A_1 we obtain that for all $\delta > 0$ there is a constant $C(\delta) > 0$ such that

$$\sum_{|\alpha:l|<1}\prod_{k=1}^{n}\varepsilon_k^{\alpha_k/l_k}\left\|A_\alpha(x)\,D^\alpha O_{j\lambda}^{-1}\upsilon_j\right\|_{L_p(D_j;E)} \leq \delta\,\|\upsilon_j\|_{W_p^l(D_j;E(A),E)} + C(\delta)\,\|\upsilon_j\|_{L_p(D_j;E)}\,.$$

Hence, for $|\arg\lambda| \leq \varphi$ with sufficiently large $|\lambda|$ there is a $\gamma \in (0,1)$ such that $\|K_{j\lambda}\| < \gamma$. Consequently, equations (23) for all j have a unique solution $\upsilon_j = [I - K_{j\lambda}]^{-1} g_j f$. Moreover,

$$\|\upsilon_j\|_{L_p(D_j;E)} = \left\|[I - K_{j\lambda}]^{-1} g_j f\right\|_{L_p(D_j;E)} \leq \|f\|_{L_p(D_j;E)}\,.$$

Thus, $[I - K_{j\lambda}]^{-1} g_j$ are bounded linear operators from $L_p(G; E)$ to $L_p(D_j; E)$. Thus, the functions

$$u_j = U_{j\lambda}f = O_{j\lambda}^{-1}\,[I - K_{j\lambda}]^{-1}\,g_j f$$

are solutions of (21). Consider the following linear operator $U = U_\varepsilon$ in $L_p(G; E)$ defined by

$$D(U) = W_p^l(G; E(A), E, L_{kj}), \quad j = 1, 2, \ldots, l_k, \quad k = 1, 2, \ldots, n,$$

$$Uf = \sum_{j=1}^{N}\varphi_j(y)\,U_{j\lambda}f = O_{j\lambda}^{-1}\,[I - K_{j\lambda}]^{-1}\,g_j f, \quad j = 1, 2, \ldots, N.$$

It is clear from the constructions $U_{j\lambda}$and from the estimate (22) that the operators $U_{j\lambda}$ are bounded linear from $L_p(G;E)$ to $W_p^l(D_j;E(A),E)$ and for $|\arg\lambda|\leq\varphi$ with sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k\left\|D_k^iU_{j\lambda}f\right\|_{L_p(D_j;E)}+\|AU_{j\lambda}f\|_{L_p(D_j;E)}\leq C\|f\|_{L_p(G;E)}. \tag{24}$$

Therefore, U is a bounded linear operator in $L_p(G;E)$. By contraction of solution operators $U_{j\lambda}$ of local equations (21), acting $O_\varepsilon+\lambda$ to $u=\sum_{j=1}^{N}\varphi_jU_{j\lambda}f$ gives

$$(O_\varepsilon+\lambda)u=\sum_{j=1}^{N}(O_\varepsilon+\lambda)(\varphi_jU_{j\lambda}f)=$$

$$=\sum_{j=1}^{N}[\varphi_j(O_\varepsilon+\lambda)(U_{j\lambda}f)+\Phi_{j\lambda}f]=\sum_{j=1}^{N}\varphi_jg_jf+\sum_{j=1}^{N}\Phi_{j\lambda}f=f+\sum_{j=1}^{N}\Phi_{j\lambda}f,$$

where $\Phi_{j\lambda}=\Phi_{j\lambda}(\varepsilon)$ are bounded linear operators defined by

$$\Phi_{j\lambda}f=\left\{\sum_{k=1}^{n}\varepsilon_ka_k\sum_{i=0}^{l_k-1}C_i^{l_k}\left(D_k^iU_{j\lambda}f\right)D_k^{l_k-i}\varphi_j+\right.$$

$$\left.+\sum_{|\alpha:l|<1}A_\alpha\prod_{k=1}^{n}\varepsilon_k^{\alpha_k/l_k}\sum_{i=0}^{\alpha_k-1}C_i^{\alpha_k}\left(D_k^i(U_{j\lambda}f)\right)D_k^{\alpha_k-i}\varphi_j\right\}.$$

Indeed, from Theorem A_1, the estimate (24) and from the expression $\Phi_{j\lambda}$ we obtain that the operators $\Phi_{j\lambda}$ are bounded linear from $L_p(G;E)$ to $L_p(G;E)$ and for $|\arg\lambda|\leq\varphi$ with sufficiently large $|\lambda|$ there is an $\delta\in(0,1)$ such that $\|\Phi_{j\lambda}\|<\delta$. Therefore, there exists a bounded linear invertible operator $\left(I+\sum_{j=1}^{N}\Phi_{j\lambda}\right)^{-1}$, i.e., we infer for all $f\in L_p(G;E)$ that the BVP (3), (4) has a unique solution

$$u(x)=(O_\varepsilon+\lambda)^{-1}f=\sum_{j=1}^{N}\varphi_jO_{j\lambda}^{-1}[I-K_{j\lambda}]^{-1}g_j\left(I+\sum_{j=1}^{N}\Phi_{j\lambda}\right)^{-1}f.$$

***Result* 1.** Theorem 1 implies that the resolvent $(O_\varepsilon+\lambda)^{-1}$ satisfies the following anisotropic type uniform sharp estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k}|\lambda|^{1-i/l_k}\varepsilon_k^{i/l_k}\left\|D_k^i(O_\varepsilon+\lambda)^{-1}\right\|_{B(L_p(G;E))}+\left\|A(O_\varepsilon+\lambda)^{-1}\right\|_{B(L_p(G;E))}\leq C$$

for $|\arg\lambda|\leq\varphi$ and $\varphi\in[0,\pi)$.

Theorem 2. *Let all conditions of Theorem* 1 *hold and* $A^{-1}\in\sigma_\infty(E)$. *Then the operator* O_ε *is Fredholm from* $W_p^l(G;E(A),E)$ *into* $L_p(G;E)$.

Proof. Theorem 1 implies that the operator $O_\varepsilon+\lambda$ for sufficiently large $|\lambda|$ has a bounded inverse $(O_\varepsilon+\lambda)^{-1}$ from $L_p(G;E)$ to $W_p^l(G;E(A),E)$, that is the operator $Q_\varepsilon+\lambda$ is Fredholm from $W_p^l(G;E(A),E)$ into $L_p(G;E)$. Then, by Theorem A_2 and the perturbation theory of linear operators we obtain that the operator O_ε is Fredholm from $W_p^l(G;E(A),E)$ into $L_p(G;E)$.

***Example* 1.** Now, let us consider a special case of (3), (4). Let $E=\mathbb{C}$, $l_1=2$ and $l_2=4$, $n=2$, $G=(0,1)\times(0,1)$ and $A=a(x,y)>0$, i. e., consider the problem

$$L_\varepsilon u=-\varepsilon_1 a_1\frac{\partial^2 u}{\partial x^2}+\varepsilon_2 a_2\frac{\partial^4 u}{\partial y^4}+b\varepsilon_1^{1/2}\varepsilon_2^{1/4}\frac{\partial^2 u}{\partial x\partial y}+au=f(x,y),$$

$$\sum_{i=0}^{m_{1j}}\varepsilon_1^{\sigma_{i1}}\left[\alpha_{ji}u_x^{(i)}(0,y)+\sum_{i=0}^{m_{1j}}\beta_{ji}u_x^{(i)}(1,y)\right]=0,\quad j=1,2, \tag{25}$$

$$\sum_{i=0}^{m_{2j}}\varepsilon_2^{\sigma_{i2}}\left[\alpha_{ji}u_y^{(i)}(0,y)+\sum_{i=0}^{m_{1j}}\beta_{ji}u_y^{(i)}(1,y)\right]=0,\quad j=1,2,3,4,$$

where ε_1 and ε_2 are positive parameters, $a_k=a_k(x,y)$, $k=1,2$ are real-valued functions on G and

$$\sigma_{i1}=\frac{1}{2}\left(i+\frac{1}{p}\right),\quad \sigma_{i2}=\frac{1}{4}\left(i+\frac{1}{p}\right),\quad m_{1j}\in\{0,1\},\quad m_{2j}\in\{0,1,2,3\},$$

$$a_k\neq 0,\quad \left|\alpha_{kjm_j}\right|+\left|\beta_{kjm_j}\right|>0,\quad \eta_k\neq 0,$$

$$a,a_k>0,\quad a,a_1,a_2\in C\left(\bar{G}\right),\quad b\in L_\infty(G),\quad a(0,y)=a(1,y),\quad a(x,0)=a(x,1),$$

$$a_k(0,y)=a_k(1,y),\quad a_k(x,0)=a_k(x,1),\quad x,y\in G,\quad k=1,2.$$

***Result* 2.** Theorem 1 implies that for each $f\in L_p(G)$ and sufficiently large a the problem (25) has a unique solution $u\in W_p^l(G)$ satisfying the uniform coercive estimate

$$\varepsilon_1\left\|D_x^2u\right\|_{L_p(G)}+\varepsilon_2\left\|D_y^{[4]}u\right\|_{L_p(G)}+\|u\|_{L_p(G)}\le C\|f\|_{L_p(G)}.$$

***Example* 2.** Consider the following BVP for the system of anisotropic PDEs with variable coefficients

$$\sum_{k=1}^{n}(-1)^{m_k}\varepsilon_k b_k(x)D_k^{2m_k}u_m(x)+(d_m(x)+\lambda)u_m(x)=f_m(x),$$

$$\sum_{i=0}^{m_{kj}}\alpha_{kji}\varepsilon_k^iD_k^iu_m(G_{k0})+\sum_{i=0}^{m_{kj}}\beta_{kji}\varepsilon_k^iD_k^iu_m(G_{kb})=0,\quad k=1,2,\ldots,n,$$

$$j=1,2,\ldots,2m_k,\quad m=1,2,\ldots,\nu,$$

where b_k are positive continuous function on G, $E=\mathbb{C}^\nu$, λ is a complex, ε_k, $k=1,2,\ldots,n$, are positive parameters and $d_m(x)>0$, $m=1,2,\ldots,\nu$.

***Result* 3.** Let $b_k, d_m \in C\left(\bar{G}\right), b_k \neq 0, \left|\alpha_{kjm_j}\right| + \left|\beta_{kjm_j}\right| > 0, \eta_k \neq 0$ and $b_j\left(G_{k0}\right) = b_j\left(G_{kb}\right),$ $d_m\left(G_{k0}\right) = d_m\left(G_{kb}\right).$ Then, Theorem 1 implies that for each $f \in L_p\left(G; \mathbb{C}^\nu\right)$ and for all $\lambda \in S\left(\varphi\right)$ with sufficiently large $|\lambda|$ the above problem has a unique solution $u \in W_p^l(G; \mathbb{C}^\nu)$ satisfying the uniform coercive estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{2m_k} |\lambda|^{1-i/2m_k} \varepsilon_k^{i/2m_k} \left\|D_k^i u\right\|_{L_p(G;\mathbb{C}^\nu)} \leq C \left\|f\right\|_{L_p(G;\mathbb{C}^\nu)}.$$

5. Abstract Cauchy problem for parabolic equation with small parameters. Consider now mixed BVP for the following parabolic equation with small parameters, i. e.,

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} \varepsilon_k a_k\left(x\right) D_k^{l_k} u + A\left(x\right) u = f\left(t, x\right), \tag{26}$$

$$\sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_{ki}} \left[\alpha_{kji} D_k^i u\left(t, G_{k0}\right) + \beta_{kji} D_k^i u\left(t, G_{kb}\right)\right] = 0, \quad j = 1, 2, \ldots, l_k, \tag{27}$$

$$u\left(0, x\right) = 0, \quad \sigma_{ki} = \frac{1}{l_k}\left(i + \frac{1}{p}\right), \quad t \in R_+, \quad x \in G,$$

where $A\left(x\right)$ is an operator function in a Banach space E for $x \in G$, a_k are complex valued functions, ε_k are small positive parameters, G, G_{k0} and G_{kb} are domains defined in the problem (3), (4).

In this section, we obtaın the existence and uniqueness of the maximal regular solution of problem (26), (27) in mixed $L_{\mathbf{p}}$-norms.

Let O_ε denote differential operator generated by (3), (4) for $\lambda = 0$.

Theorem 3. *Let all conditions of Theorem* 1 *are hold for* $A_\alpha = 0$ *and* $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. *Then*:

(a) *the operator* O_ε *is an R-positive in* $L_p\left(G; E\right)$;

(b) *the operator* O_ε *is a generator of an analytic semigroup.*

Proof. Really, by virtue of Theorem 1 we obtain that for $f \in L_p\left(G; E\right)$ the BVP (3), (4) have a unique solution expressing in the form

$$u\left(x\right) = \left(O_\varepsilon + \lambda\right)^{-1} f = \sum_{j=1}^{N} \varphi_j O_{j\lambda}^{-1} \left[I - K_{j\lambda}\right]^{-1} g_j \left(I + \sum_{j=1}^{N} \Phi_{j\lambda}\right)^{-1} f,$$

where $O_{j\lambda} = O_j\left(\varepsilon\right) + \lambda$ are local operators generated by BVPs with constant coefficients of type (7), (8) and $K_{j\lambda} = K_{j\lambda}\left(\varepsilon\right), \Phi_{j\lambda} = \Phi_{j\lambda}\left(\varepsilon\right)$ are uniformly bounded operators defined in the proof of the Theorem 1. By virtue of Theorem A_6 operators $O_j\left(\varepsilon\right)$ are R-positive. Then by using the above representation and by virtue of Kahane's contraction principle, product and additional properties of the collection of R-bounded operators (see, e.g., [8], Lemma 3.5, Proposition 3.4) we obtain the assertions.

Theorem 4. *Let all conditions of Theorem* 3 *hold. Then for* $f \in L_{\mathbf{p}}\left(G_+; E\right)$ *problem* (26), (27) *has a unique solution* $u \in W_{\mathbf{p}}^{1,l}\left(G_+; E\left(A\right), E\right)$ *and the following uniform coercive estimate holds*:

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p}}(G_+;E)} + \sum_{k=1}^{n} \varepsilon_k \left\|D_k^{l_k} u\right\|_{L_{\mathbf{p}}(G_+;E)} + \left\|Au\right\|_{L_{\mathbf{p}}(G_+;E)} \leq C \left\|f\right\|_{L_{\mathbf{p}}(G_+;E)}.$$

Proof. The problem (26), (27) can be expressed as the following Cauchy problem:

$$\frac{du}{dt} + O_{\varepsilon} u(t) = f(t), \quad u(0) = 0. \tag{28}$$

The Theorem 3 implies that the operator O_{ε} is R-positive and also is a generator of an analytic semigroup in $F = L_p(G; E)$. Then by virtue of [1] or [26] (Theorem 4.2) we obtain that for all $f \in L_{p_1}((R_+); F)$ problem (28) has a unique solution $u \in W_{p_1}^1((0,1); D(O), F)$ and the following uniform estimate holds:

$$\left\| \frac{du}{dt} \right\|_{L_{p_1}(R_+;F)} + \|O_{\varepsilon} u\|_{L_{p_1}(R_+;F)} \leq C \|f\|_{L_{p_1}(R_+;F)}. \tag{29}$$

Since $L_{p_1}(0,1;F) = L_{\mathbf{p}}(G_+; E)$, by Theorem1 we have $\|O_{\varepsilon} u\|_{L_{p_1}(R_+;F)} = D(O_{\varepsilon})$. This relation and the estimate (29) implies the assertion.

6. BVPs for quasielliptic PDE with small parameters. In this section, maximal regularity properties of anisotropic PDE with small parameters are studied. Maximal regularity properties for PDEs have been studied, e.g., in [8] for smooth domains and in [12] for nonsmooth domains. In this section, consider the following BVP with small parameters:

$$Lu = \sum_{k=1}^{n} \varepsilon_k a_k(x) D_k^{l_k} u(x,y) + \sum_{|\alpha| \leq 2m} a_{\alpha}(y) D_y^{\alpha} u(x,y) +$$

$$+ \sum_{|\beta:l|<1} \prod_{k=1}^{n} \varepsilon_k^{\alpha_k/l_k} b_{\beta}(x,y) D_y^{\beta} u(x,y) + \lambda u(x,y) = f(x,y), \quad x \in G, \quad y \in \Omega, \tag{30}$$

$$L_{kj} u = \sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_{ki}} \left[\alpha_{kji} D_k^i u(G_{k0}, y) + \beta_{kji} D_k^i u(G_{kb}, y) \right] = 0, \quad y \in \Omega, \tag{31}$$

$$j = 1, 2, \ldots, l_k, \quad x(k) \in G_k,$$

$$B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D_y^{\beta} u(x,y) \mid_{y \in \partial\Omega} = 0, \quad x \in G, \quad j = 1, 2, \ldots, m, \tag{32}$$

where $D_j = -i\dfrac{\partial}{\partial y_j}$, α_{kji}, β_{kji} are complex number, λ is a complex and ε_k are small positive parameter, $y = (y_1, \ldots, y_{\mu}) \in \Omega \subset R^{\mu}$ and

$$\sigma_{ki} = \frac{1}{l_k}\left(i + \frac{1}{p}\right), \quad G = \{x = (x_1, x_2, \ldots, x_n), \ 0 < x_k < b_k\},$$

$$G_{k0} = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n), \quad G_{kb} = (x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n),$$

$$m_{kj} \in \{0, 1, \ldots, l_k - 1\}, \quad \left|\alpha_{kjm_j}\right| + \left|\beta_{kjm_j}\right| > 0, \quad j = 1, 2, \ldots, l_k,$$

$$x(k) = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad G_k = \prod_{j \neq k} (0, b_j), \quad j, k = 1, 2, \ldots, n.$$

Let $\omega_{kj} = \omega_{kj}(x)$, $j = 1, 2, \ldots, l_k$, $k = 1, 2, \ldots, n$, denote the roots of the equations

$$a_k(x)\,\omega^{l_k} + 1 = 0.$$

Let Q_ε denote the operator generated by BVP (30)–(33). Let

$$F = B\left(L_{\mathbf{p}}\left(\tilde{\Omega}\right)\right), \quad \tilde{\Omega} = G \times \Omega.$$

Theorem 5. *Let the following conditions be satisfied*:

(1) $a_\alpha \in C\left(\bar{\Omega}\right)$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in [L_\infty + L_{r_k}](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geq p_1$, $p_1 \in (1, \infty)$, $2m - k > \dfrac{l}{r_k}$ *and* $b_\beta \in L_\infty(\tilde{\Omega})$;

(2) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for each* j, β, $m_j < 2m$, $p \in (1, \infty)$;

(3) *for* $y \in \bar{\Omega}$, $\xi \in R^\mu$, $\eta \in S(\varphi_1)$, $\varphi_1 \in \left[0, \dfrac{\pi}{2}\right)$, $|\xi| + |\eta| \neq 0$ *let*

$$\eta + \sum_{|\alpha|=2m} a_\alpha(y)\,\xi^\alpha \neq 0;$$

(4) *for each* $y_0 \in \partial\Omega$ *the local BVPs in local coordinates corresponding to* y_0

$$\eta + \sum_{|\alpha|=2m} a_\alpha(y_0)\,D^\alpha \vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0)\,D^\beta \vartheta(y) = h_j, \quad j = 1, 2, \ldots, m,$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, \ldots, h_m) \in R^m$ *and for* $\xi' \in R^{\mu-1}$ *with* $|\xi'| + |\eta| \neq 0$;

(5) $a_k \in C\left(\bar{G}\right)$, $a_k(x) \neq 0$, $\left|\alpha_{kjm_j}\right| + \left|\beta_{kjm_j}\right| > 0$, $\eta_k(x) \neq 0$ *and*

$$|\arg \omega_{kj} - \pi| \leq \frac{\pi}{2} - \varphi, \quad j = 1, 2, \ldots, d_k,$$

$$|\arg \omega_{kj}| \leq \frac{\pi}{2} - \varphi, \quad \varphi \in \left[0, \frac{\pi}{2}\right),$$

$$j = d_k + 1, \ldots, l_k, \quad 0 < d_k < l_k, \quad k = 1, 2, \ldots, n, \quad x \in G.$$

Then:

(a) *problem* (30)–(33) *has a unique solution* $u \in W_{\mathbf{p}}^{l,2m}(\tilde{\Omega})$ *for* $f \in L_p(\tilde{\Omega})$ *and* $\lambda \in S_\varphi$ *with large enough* $|\lambda|$. *Moreover, the following coercive uniform estimate holds*:

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k} |\lambda|^{1-i/l_k}\,\varepsilon_k^{i/l_k}\left\|D_k^{l_k}u\right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \sum_{|\beta|=2m}\left\|D_y^\beta u\right\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \|u\|_{L_{\mathbf{p}}(\tilde{\Omega})} \leq C\,\|f\|_{L_{\mathbf{p}}(\tilde{\Omega})};$$

(b) *for* $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$ *there exists a resolvent* $(Q_\varepsilon + \lambda)^{-1}$ *and*

$$\sum_{k=1}^{n}\sum_{i=0}^{l_k} |\lambda|^{1-i/l_k}\,\varepsilon_k^{i}\left\|D_k^{i}(Q_\varepsilon + \lambda)^{-1}\right\|_F + \left\|A(Q_\varepsilon + \lambda)^{-1}\right\|_F \leq M;$$

(c) *the problem* (30)–(33) *is Fredholm in* $L_{\mathbf{p}}(\tilde{\Omega})$ *for* $\lambda = 0$.

Proof. Let $E = L_{p_1}(\Omega)$. Then by [8] (Theorem 3.6), part (1) of Condition 2 is satisfied. Consider the operator A which is defined by

$$D(A) = W_{p_1}^{2m}(\Omega; B_j u = 0), \quad Au = \sum_{|\beta| \leq 2m} a_\beta(y) D^\beta u(y).$$

For $x \in \Omega$ we also consider operators

$$A_\alpha(x) u = b_\alpha(x, y) D^\alpha u(y), \quad |\alpha : l| < 1.$$

The problem (30) – (33) can be rewritten as the form of (3), (4), where $u(x) = u(x, .)$ and $f(x) = f(x, .)$ are functions with values in $E = L_{p_1}(\Omega)$. From [8] (Theorem 8.2) problem

$$\eta u(y) + \sum_{|\beta| \leq 2m} a_\beta(y) D^\beta u(y) = f(y),$$

$$B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1, 2, \ldots, m,$$

has a unique solution for $f \in L_{p_1}(\Omega)$ and $\arg \eta \in S(\varphi_1)$, $|\eta| \to \infty$. Moreover, the operator A is R-positive in L_{p_1}, i.e., all conditions of the Theorem 1 hold.

7. Cauchy problem for infinite systems of parabolic equation with small parameters. Consider the infinity systems of BVP for the anisotropic PDE with parameters

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} \varepsilon_k a_k(x) \frac{\partial^{l_k} u_m}{\partial x_k^{l_k}} + \sum_{j=1}^{\infty} (d_j(x) + \lambda) u_m +$$

$$+ \sum_{|\alpha : l| < 1} \sum_{j=1}^{\infty} \prod_{k=1}^{n} \varepsilon_k^{\alpha_k / l_k} d_{\alpha j m}(x) D^\alpha u_j = f_m(t, x), \quad m = 1, 2, \ldots, \infty, \tag{33}$$

$$\sum_{i=0}^{m_{kj}} \varepsilon_k^{\sigma_{ki}} \left[\alpha_{kji} D_k^{(i)} u(t, G_{k0}) + \sum_{i=0}^{m_{kj}} \beta_{kji} D_k^{(i)} u(t, G_{kb}) \right] = 0, \quad j = 1, 2, \ldots, l_k, \tag{34}$$

$$u(0, x) = 0, \quad x \in G, \quad t \in (0, \infty), \quad x(k) \in G_k, \quad j = 1, 2, \ldots, l_k,$$

where a_k, d_k, $d_{\alpha jm}$ are complex valued functions, ε_k are small positive parameters and α_{kji}, β_{kji} are complex numbers. Let

$$\sigma_{ki} = \frac{1}{l_k}\left(i + \frac{1}{p}\right), \quad G = \{x = (x_1, x_2, \ldots, x_n),\ 0 < x_k < b_k\},$$

$$G_{k0} = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n),$$

$$G_{kb} = (x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n), \quad m_{kj} \in \{0, 1, \ldots, l_k - 1\},$$

$$x(k) = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad G_k = \prod_{j \neq k} (0, b_j), \quad j, k = 1, 2, \ldots, n,$$

$$D(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \quad du = \{d_m u_m\}, \quad m = 1, 2, \ldots, \infty,$$

$$l_q(D) = \left\{ u \colon u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{1/q} < \infty \right\},$$

$$j = 1, 2, \ldots, l_k, \quad k = 1, 2, \ldots, n.$$

Let $V = V(\varepsilon)$ denote the operator in $L_p(G; l_q)$ generated by problem (34), (35). Let

$$G_+ = (0, \infty) \times G, \quad B = B(L_p(G; l_q)).$$

Theorem 6. *Let* $p \in (1, \infty)$, $a_k \in C(\bar{G})$, $a_i(0, x(k)) = a_i(b_k, x(k))$, $a_k(x) \neq 0$, $|\alpha_{kjm_j}| + |\beta_{kjm_j}| > 0$, $\eta_k(x) \neq 0$ *and* $|\arg \omega_{kj} - \pi| \leq \frac{\pi}{2} - \varphi$, $|\arg \omega_{kj}| \leq \frac{\pi}{2} - \varphi$, $j = 1, 2, \ldots, l_k$, $\varphi \in \varphi \in \left[0, \frac{\pi}{2}\right)$, $x \in G$, $d_m \in C(\bar{G})$, $d_{\alpha jm} \in L_\infty(G)$ *such that*

$$\max_{\alpha} \sup_{m} \sum_{j=1}^{\infty} d_{\alpha jm}(x)\, d_j^{-(1-|\alpha\colon l|-\mu)}(x) < M \quad \textit{for all} \quad x \in G \quad \textit{and} \quad 0 < \mu < 1 - |\alpha \colon l|.$$

Then for $f(t, x) = \{f_m(t, x)\}_1^\infty \in L_p(G; l_q)$, $|\arg \lambda| \leq \varphi$ *and sufficiently large* $|\lambda|$ *the problem* (34), (35) *has a unique solution* $u = \{u_m(t, x)\}_1^\infty$ *that belongs to the space* $W_{\mathbf{p}}^{1,l}(G_+, l_q(D), l_q)$ *and the following coercive uniform estimate holds*:

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}(G_+; l_q)} + \sum_{k=1}^{n} \varepsilon_k \left\| D_k^{l_k} u \right\|_{L_{\mathbf{p}}(G_+; l_q)} + \|Au\|_{L_{\mathbf{p}}(G_+; l_q)} \leq C \|f\|_{L_{\mathbf{p}}(G_+; l_q)}.$$

Proof. Let $E = l_q$, A and $A_\alpha(x)$ be infinite matrices, such that

$$A = [d_m \delta_{mj}], \quad A_\alpha(x) = [d_{\alpha jm}(x)], \quad m, j = 1, 2, \ldots, \infty.$$

It is clear that the operator A is R-positive in l_q. The problem (34), (35) can be rewritten in the form (26), (27). Then, from Theorem 4 we obtain that the assertion.

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