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RATE OF CONVERGENCE FOR THE SZÁSZ – BÉZIER OPERATORS

ШВИДКІСТЬ ЗБІЖНОСТІ ОПЕРАТОРІВ САСА – БЕЗ'Є

We estimate the rate of convergence for functions of bounded variation for the Bézier variant of the Szász operators $S_{n,\alpha}(f, x)$. We study the rate of convergence of $S_{n,\alpha}(f, x)$ for the case $0 < \alpha < 1$.

Знайдено оцінку швидкості збіжності функцій обмеженої варіації для версії Без'є операторів Саса $S_{n,\alpha}(f, x)$. Вивчено швидкості збіжності $S_{n,\alpha}(f, x)$ для $0 < \alpha < 1$.

1. Introduction. For the case where $\alpha \geq 1$ or $0 < \alpha < 1$ and a function f is defined on $[0, \infty)$, the Szász – Bézier operator $S_{n,\alpha}$ is defined by

$$S_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \qquad (1),$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{\infty} s_{n,j}(x)$ with the Szász basis function $s_{n,k}(x) = e^{-nx} \dfrac{(nx)^k}{k!}$, $k = 0, 1, 2, \ldots$. It is well known that for $\alpha = 1$, the operators (1) reduce to the well-known Szász – Mirakyan operators

$$S_n(f, x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right).$$

The rate of convergence for the Szász – Mirakyan operators on functions of bounded variation was first studied by Cheng [1]. Recently, Zeng [2] and Zeng and Zhao [3] estimated the rates of convergence for the Szász – Bézier operators whenever $\alpha \geq 1$. The rates of convergence for the other case of $\alpha \in (0, 1)$ for functions of bounded variation were obtained in [4] and [5], respectively, for the Kantorovich – Bézier operators and Bernstein – Bézier operators. Motivated by this, we extend the results of [1, 3, 6] and study the rate of convergence for the Szász – Bézier operators $S_{n,\alpha}(f, x)$, $0 < \alpha < 1$ for functions of bounded variation.

Our main theorem is stated as follows:

Theorem. *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Let $f(t) = O(t^r)$ for some $r \in N$ as $t \to \infty$. Then for $x \in (0, \infty)$, $0 < \alpha < 1$, there exists a positive constant $M(f, \alpha, x, r)$ such that, for n sufficiently large, we have*

$$\left| S_{n,\alpha}(f, x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right) f(x-) \right| \leq$$

$$\leq \frac{Z(x)}{\sqrt{nx}} |f(x+) - f(x-)| + \frac{1}{\sqrt{2enx}} \varepsilon_n(x) |f(x) - f(x-)| +$$

$$+ \frac{5}{nx} \sum_{k=1}^{n} \Omega_x\left(f_x, \frac{x}{\sqrt{k}}\right) + \frac{M(f, \alpha, x, r)}{n^m},$$

where $Z(x) = \min\{0{,}8\sqrt{(1+3x)} + 0{,}5,\ 1{,}6x^2 + 1{,}3\}$,

$$\Omega_x(f,\lambda) = \sup_{t\in[x-\lambda,\, x+\lambda]} |f(t)-f(x)|, \quad \varepsilon_n(x) = \begin{cases} 1, & \textit{if} \quad x = \dfrac{k'}{n}, \ k' \in N, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$f_x(t) = \begin{cases} f(t)-f(x-), & 0 \le t < x, \\ 0, & t = x, \\ f(t)-f(x+), & x < t < \infty. \end{cases}$$

It is clear that:

(i) $\Omega_x(f,h)$ is monotone nondecreasing with respect to h;

(ii) $\lim\limits_{h\to\infty} \Omega_x(f,h) = 0$ if f is continuous at the point x;

(iii) if f is a function of bounded variation on $[a,b]$ and $V_a^b(f)$ denotes the total variation of f on $[a,b]$, then $\Omega_x(f,h) \le V_{x-h}^{x+h}(f)$.

We recall the Lebesgue – Stieltjes integral representation

$$S_{n,\alpha}(f,x) = \int_0^\infty f(t)\, d_t(K_{n,\alpha}(x,t)),$$

where

$$K_{n,\alpha}(x,t) = \begin{cases} \sum\limits_{k\le nt} Q_{n,k}^{(\alpha)}(x), & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

We also define

$$H_{n,\alpha}(x,t) = \begin{cases} 1 - K_{n,\alpha}(x,t), & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

2. Lemmas. In the sequel, we shall need the following lemmas:

Lemma 1 [6]. *For all* $x \in (0,\infty)$ *and* $x \in N$ *we have*

$$s_{n,k}(x) < \frac{1}{\sqrt{2e}\sqrt{nx}}.$$

Lemma 2 [2]. *For* $x \in (0,\infty)$ *we have*

$$\left| \sum_{k>nx} s_{n,k}(x) - \frac{1}{2} \right| \le \min\left\{ \frac{0{,}8\sqrt{(1+3x)}+0{,}5}{1+\sqrt{nx}},\ \frac{1{,}6x^2+1{,}3}{1+\sqrt{nx}} \right\}$$

and, for $0 \le t < x$, *we have*

$$\sum_{k\le nt} Q_{n,k}^{(\alpha)}(x) \le \frac{x}{n(t-x)^2}.$$

Lemma 3. *For* $0 < \alpha < 1$ *and* $0 < x < t < \infty$, *we have*

$$H_{n,\alpha}(x) \le \frac{E(\alpha)}{n^m (t-x)^m},$$

where $E(\alpha)$ *is a positive constant depending only on* α.

Proof. Since $0 < x < t < \infty$, we have $\dfrac{|k/n - x|}{|t-x|} \geq 1$ for $k \geq nt$. Thus,

$$H_{n,\alpha}(x) = 1 - K_{n,\alpha}(x,t) =$$

$$= 1 - \sum_{k \leq nt} Q_{n,k}^{(\alpha)}(x) \leq \sum_{k \geq nt} Q_{n,k}^{(\alpha)}(x) \leq \left(\sum_{k \geq nt} \frac{|k/n - x|^{2m/\alpha}}{(t-x)^{2m/\alpha}} s_{n,k}(x) \right)^{\alpha} \leq$$

$$\leq \frac{1}{(t-x)^{2m}} \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2m/\alpha} s_{n,k}(x) \right)^{\alpha}.$$

Put $l = 2m/\alpha$ and suppose that $[l]$ denotes the integral part of l. Following [4] (Lemma 6), choose the numbers $p = \dfrac{2[l]}{2[l]+2-l}$, $q = \dfrac{2[l]}{l-2}$.

For each real, put $\psi_x(t) = t - x$. Note that $\dfrac{2}{p} + \dfrac{2(1+[l])}{q} = \dfrac{2[l]+2-l}{[l]} + \dfrac{l-2}{[l]}(1+l) = l$. The application of Hölder's inequality yields

$$\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2m/\alpha} s_{n,k}(x) \leq \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2} s_{n,k}(x) \right)^{1/p} \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2([l]+1)} s_{n,k}(x) \right)^{1/q} =$$

$$= \left(S_{n,1}(\psi_x^2, x) \right)^{1/p} \left(S_{n,1}(\psi_x^{2([l]+1)}, x) \right)^{1/q}.$$

By using the well-known result $S_{n,1}(\psi_x^r, x) = O(n^{-r})$ as $n \to \infty$ $(r = 1, 2, 3, \ldots)$, we obtain

$$\left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2m/\alpha} s_{n,k}(x) \right)^{\alpha} \leq O\left(n^{-\alpha/p - \alpha([l]+1)/q} \right) = O\left(n^{-m} \right),$$

since

$$-\frac{\alpha}{p} - \frac{\alpha(1+[l])}{q} = -\alpha - \frac{\alpha[l]}{\left(\dfrac{2[l]}{l-2} \right)} = -\alpha - \alpha \frac{l-2}{l} = -\alpha \frac{l}{2} = -m.$$

This completes the proof of Lemma 3.

3. Proof of Theorem. We have

$$f(t) = 2^{-\alpha} f(x+) + (1 - 2^{-\alpha}) f(x-) + g_x(t) + 2^{-\alpha} \Big((f(x+) - f(x-)) \operatorname{sign}^{(\alpha)}(t) \Big) +$$
$$+ \Big(f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-) \Big) \delta_x(t),$$

where

$$\operatorname{sign}^{(\alpha)}(t-x) := \begin{cases} 2^{\alpha} - 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x \end{cases} \quad \text{and} \quad \delta_x(t) = \begin{cases} 1 & \text{if } x = t, \\ 0 & \text{if } x \neq t. \end{cases}$$

Therefore,

$$\left| S_{n,\alpha}(f, x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}} \right) f(x-) \right| \leq$$

$$\leq \left| S_{n,\alpha}(f_x, x) \right| + \left| \frac{f(x+) - f(x-)}{2^{\alpha}} S_{n,\alpha}\left(\operatorname{sign}^{(\alpha)}(t-x), x \right) + \right.$$

$$+\left[f(x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right]S_{n,\alpha}(\delta_x,x)\Bigg|. \qquad (2)$$

We first estimate

$$S_{n,\alpha}\left(\operatorname{sign}^{(\alpha)}(t-x),x\right) = 2^{\alpha}\sum_{k>nx}Q_{n,k}^{(\alpha)}(x)-1+e_n(x)Q_{n,k'}^{(\alpha)}(x) =$$

$$= 2^{\alpha}\sum_{k>nx}\left(J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)\right)-1+\varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x) =$$

$$= 2^{\alpha}\left(\sum_{k>nx}s_{n,k}(x)\right)^{\alpha}-1+\varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x)$$

and

$$S_{n,\alpha}(\delta_x,x) = \varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x).$$

Hence, we have

$$\Bigg|\frac{f(x+)-f(x-)}{2^{\alpha}}S_{n,\alpha}(\operatorname{sign}(t-x),x)+$$

$$+\left[f(x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right]S_{n,\alpha}(\delta_x,x)\Bigg| =$$

$$= \left|\frac{f(x+)-f(x-)}{2^{\alpha}}\left[2^{\alpha}\left(\sum_{k>nx}s_{n,k}(x)\right)^{\alpha}-1\right]+\left[f(x)-f(x-)\right]\varepsilon_n Q_{n,k'}^{(\alpha)}(x)\right|. \qquad (3)$$

By mean value theorem, we have

$$\left|\left(\sum_{j>nx}s_{n,j}(x)\right)^{\alpha}-\frac{1}{2^{\alpha}}\right| = \alpha\left(\zeta_{n,j}(x)\right)^{\alpha-1}\left|\sum_{j>nx}s_{n,j}(x)-\frac{1}{2}\right|,$$

where $\zeta_{n,j}(x)$ lies between $\frac{1}{2}$ and $\sum_{j>nx}s_{n,j}(x)$. In view of Lemma 2, it is observed that, for n sufficiently large, the intermediate point $\varsigma_{n,j}$ is arbitrary close to $\frac{1}{2}$, i.e.,

$$\varsigma_{n,j} = \frac{1}{2+\varepsilon}$$

with an arbitrary small $|\varepsilon|$. Then we have

$$\alpha\left(\zeta_{n,j}(x)\right)^{\alpha-1} \le \alpha(2+\varepsilon)^{1-\alpha}.$$

The latter expression is positive and strictly increasing for $\alpha\in(0,1)$, since

$$\frac{\partial}{\partial\alpha}\alpha(2+\varepsilon)^{1-\alpha} = (2+\varepsilon)^{1-\alpha}[1-\alpha\log(2+\varepsilon)] > 0$$

for sufficiently small $|\varepsilon|$. Thus, it takes maximum value at $\alpha=1$. This implies

$$\alpha\left(\zeta_{n,j}(x)\right)^{\alpha-1} \le 1.$$

Hence,

$$\left|\left(\sum_{j>nx}s_{n,j}(x)\right)^{\alpha}-\frac{1}{2^{\alpha}}\right|\le\frac{Z(x)}{1+\sqrt{nx}}, \qquad Z(x) = \min\left\{0{,}8\sqrt{(1+3x)}+0{,}5,\ 1{,}6x^2+1{,}3\right\}. \qquad (4)$$

We also have

$$Q_{n,k'}^{(\alpha)}(x) = J_{n,k'}^{\alpha}(x) - J_{n,k'+1}^{\alpha}(x) = \alpha\left(\zeta_{n,k'}\right)^{\alpha-1} s_{n,k'}(x),$$

where $J_{n,k'+1}(x) < \zeta_{n,k'}(x) < J_{n,k'}(x)$. Thus, by Lemma 1, we have

$$Q_{n,k'}^{(\alpha)}(x) \le \frac{1}{\sqrt{2enx}}. \tag{5}$$

Combining the estimates of (3) – (5), we have

$$\left|\frac{f(x+)-f(x-)}{2^{\alpha}} S_{n,\alpha}(\operatorname{sign}(t-x), x) + \right.$$

$$\left. + \left[f(x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right) f(x-)\right] S_{n,\alpha}(\delta_x, x)\right| \le$$

$$\le \frac{Z(x)}{1+\sqrt{nx}} |f(x+) - f(x-)| + \frac{1}{\sqrt{2enx}} \varepsilon_n(x) |f(x) - f(x-)|.$$

We next estimate $S_{n,\alpha}(g_x, x)$ as follows:

$$S_{n,\alpha}(f_x, x) = \int_0^{\infty} f_x(t) d_t\left(K_{n,\alpha}(x,t)\right) =$$

$$= \left(\int_{I_1} + \int_{I_2} + \int_{I_3} + \int_{I_4}\right) f_x(t) d_t\left(K_{n,\alpha}(x,t)\right) = E_1 + E_2 + E_3 + E_4 \quad \text{say}, \tag{6}$$

where $I_1 = \left[0, x - \frac{x}{\sqrt{n}}\right]$, $I_2 = \left[x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right]$, $I_3 = \left[x + \frac{x}{\sqrt{n}}, 2x\right]$, and $I_4 = [2x, \infty)$. We first estimate E_2. Noting that $f_x(x) = 0$, we have

$$|E_2| \le \int_{x - x/\sqrt{n}}^{x + x/\sqrt{n}} |g_x(t) - g_x(x)| d_t(K_{n,\alpha}(x,t)) \le \Omega_x\left(f_x, \frac{x}{\sqrt{n}}\right) \le$$

$$\le \frac{x}{nx} \sum_{k=1}^{n} \Omega_x\left(f_x, \frac{x}{\sqrt{k}}\right). \tag{7}$$

We next estimate E_1. Writing $y = x - \frac{x}{\sqrt{n}}$ and using Lebesgue – Stieltjes integration by parts, we have

$$|E_1| = \left|\int_0^y f_x(t) d_t\left(K_{n,\alpha}(x,t)\right)\right| \le \int_0^y \Omega_x(f_x, x-t) d_t K_{n,\alpha}(x,t) =$$

$$= \Omega_x(f_x, x-y) K_{n,\alpha}(x,y) + \int_0^y \hat{K}_{n,\alpha}(x,t) d_t(-\Omega_x(f_x, x-t)),$$

where $\hat{K}_{n,\alpha}(x,t)$ is the normalized form of $K_{n,\alpha}(x,t)$. Since $\hat{K}_{n,\alpha}(x,t) \le K_{n,\alpha}(x,t)$ on $(0,\infty)$, by Lemma 2 it follows that

$$|E_1| \le \Omega_x(f_x, x-y) \frac{x}{n(x-y)^2} + \frac{x}{n} \int_0^y \frac{1}{(x-t)^2} d_t(-\Omega_x(f_x, x-t)).$$

Integrating by parts the last term, we have

$$\int_0^y \frac{1}{(x-t)^2} d_t(-\Omega_x(f_x, x-t)) = -\frac{\Omega_x(f_x, x-t)}{(x-t)^2}\Bigg|_0^{y+} + \int_0^y \Omega_x(f_x, x-t) \frac{2dt}{(x-t)^3}.$$

Hence, by replacing the variable t in the last integral by $x-\dfrac{x}{\sqrt{u}}$, we get

$$|E_1| \le \frac{2}{nx}\sum_{k=1}^{n}\Omega_x\left(f_x, \frac{x}{\sqrt{k}}\right). \tag{8}$$

Using the similar method for the estimation of E_3, we get

$$|E_3| \le \frac{2}{nx}\sum_{k=1}^{n}\Omega_x\left(f_x, \frac{x}{\sqrt{k}}\right). \tag{9}$$

Finally, by assumption, we have the estimate

$$|f_x(t)| \le Mt^r \le M\left(\frac{t-x}{x}\right)^r \quad \text{for} \quad t \ge 2x.$$

Now

$$|E_4| = \left|\int_{2x}^{\infty} f_x(t)\,K_{n,\alpha}(x,t)\right| \le \int_{2x}^{\infty}|f_x(t)|d_t K_{n,\alpha}(x,t) \le$$

$$\le Mx^{-r}\int_{0}^{\infty}(t-x)^r\,d_t\,K_{n,\alpha}(x,t) \le -Mx^{-r}\int_{2x}^{\infty}(t-x)^r\,d_t\big(1-K_{n,\alpha}(x,t)\big) =$$

$$= -Mx^{-r}\int_{2x}^{\infty}(t-x)^r\,d_t\big(H_{n,\alpha}(x,t)\big) \le -Mx^{-r}\int_{0}^{\infty}(t-x)^r\,d_t\big(1-K_{n,\alpha}(x,t)\big) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r H_{n,\alpha}(x,t)\Big|_{2x}^{R} + \int_{2x}^{R}H_{n,\alpha}(x,t)\,d_t(t-x)^r\right) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r H_{n,\alpha}(x,t)\Big|_{2x}^{R} + \int_{2x}^{R}H_{n,\alpha}(x,t)\,r(t-x)^{r-1}\,dt\right) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r\,\frac{E(\alpha)}{n^m(t-x)^m}\Bigg|_{2x}^{R} + \frac{rE(\alpha)}{n^m}\int_{2x}^{R}(t-x)^{r-m-1}\,dt\right) =$$

$$= M\,\frac{E(\alpha)}{n^m x^m} + \frac{r\,E(\alpha)}{n^m(m-r)x^{m-r}}, \quad m > r. \tag{10}$$

Combining the estimates of (2) – (10), we obtain the required result.

This completes the proof of the theorem.

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