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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

ПРО АСИМПТОТИЧНУ ПОВЕДІНКУ РОЗВ'ЯЗКІВ ДИФЕРЕНЦІАЛЬНИХ СИСТЕМ

There are many studies on the asymptotic behavior of solutions of differential equations. In the present paper, we consider another aspect of this problem, namely, the rate of the asymptotic convergence of solutions.

Let $\varphi(t)$ be a scalar continuous monotonically increasing positive function tending to ∞ as $t \to \to \infty$. It is established that if all solutions of a differential system saisfy the inequality:

$$\| x(t; t_0, x_0) \| \le M \frac{\varphi(t_0)}{\varphi(t)} \quad \text{for all} \quad t \ge t_0, \quad x_0 \in \{x : \| x \| \le \alpha\},$$

then the solution $x(t; t_0, x_0)$ of this differential system tends to 0 faster that $M \frac{\varphi(t_0)}{\varphi(t)}$.

Асимптотичній поведінці розв'язків диференціальних рівнянь присвячено чимало досліджень. У даній роботі проблему розглянуто з іншого боку, а саме, з точки зору швидкості асимптотичної збіжності розв'язків.

Нехай $\varphi(t)$ скалярна неперервна монотонно зростаюча додатна функція, що прямує до ∞ при $t \to \infty$. Встановлено, що якщо всі розв'язки диференціальної системи задовольняють нерівність

$$\| x(t; t_0, x_0) \| \le M \frac{\varphi(t_0)}{\varphi(t)} \quad \text{для всіх} \quad t \ge t_0, \quad x_0 \in \{x : \| x \| \le \alpha\},$$

то розв'язок $x(t; t_0, x_0)$ цієї диференціальної системи прямує до 0 швидше, ніж $M \frac{\varphi(t_0)}{\varphi(t)}$.

1. Introduction and preliminaries. Let I denote the interval $a \le t < \infty$, $a \ge 0$, and $\mathbb{R}^n$ denote Euclidean n-space. For $x \in \mathbb{R}^n$, let $\| x \|$ be the Euclidean norm of x. We shall denote by S_α the set of x such that $\| x \| \le \alpha$.

Consider a system of differential equations [1 – 9]

$$\frac{dx}{dt} = X(t, x), \quad X(t, 0) \equiv 0, \tag{1}$$

where $X(t, x)$ is defined on a region in $I \times \mathbb{R}^n$ and continuous in (t, x).

Moreover, suppose that $X(t, x)$ satisfies uniqueness condition of solution. Throughout this paper, a solution passing through a point (t_0, x_0) in $I \times \mathbb{R}^n$ will be denoted by such form as $x(t; t_0, x_0)$. We denote by $C_0(x)$ the family of functions which satisfy locally Lipschitz condition with respect to x, and assume that $\varphi(t)$ is a scalar continuous, monotonically increasing function in I, $\varphi(a) \ge 1$, $\varphi(t) \to +\infty$ as $t \to +\infty$.

We have the following definitions:

Definition 1. *The solution* $x(t) \equiv 0$ *of* (1) *is* φ*-asymptotically stable if given any* $\varepsilon > 0$ *and any* $t_0 \in I$, *there exist* $\delta = \delta(t_0, \varepsilon) > 0$ *such that if* $\| x_0 \| < \delta$, *then* $\| x(t; t_0, x_0) \| < \varepsilon \frac{\varphi(t_0)}{\varphi(t)}$ *for all* $t \ge t_0$.

Definition 2. *The solution* $x(t) \equiv 0$ *of* (1) *is* φ*-uniform asymptotically stable if* δ *in Definition* 1 *is independent of* t_0.

Definition 3. *The solution* $x(t) \equiv 0$ *of* (1) *is* φ*-asymptotically stable in the large if for any* $\alpha > 0$, *there exist* $K(\alpha) > 0$ *such that if* $x_0 \in S_\alpha$, *then* $\| x(t; t_0, x_0) \| < K(\alpha) \dfrac{\varphi(t_0)}{\varphi(t)} \| x_0 \|$ *for all* $t \geq t_0$.

Let $V(t, x)$ be a continuous scalar function defined on an open set S and let $V(t, x) \in C_0(x)$. This function is called Liapunov function [9]. We also define the function:

$$V'_{(1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h} [V(t + h, x + hX(t, x)) - V(t, x)].$$

Let $x(t)$ be a solution of (1) that stays in S. Denote by $V'(t, x(t))$ the upper right-hand derivative of $V(t, x(t))$, i.e.,

$$V'_{(1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h} [V(t + h, x(t + h)) - V(t, x)].$$

We have [9]

$$V'_{(1)}(t, x) = V'(t, x).$$

In the case where $V(t, x)$ has continuous partial derivatives of the first order, it is evident that

$$V'_{(1)}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial t} \cdot X(t, x),$$

where "." denotes the scalar product.

2. Sufficient conditions.

Theorem 1. *Suppose that there exists a Liapunov function* $V(t, x)$ *defined on* I, $\| x \| < H$, *which satisfies the following conditions:*

(i) $\| x \| \leq V(t, x)$ *and* $V(t, 0) \equiv 0$;

(ii) $V'_{(1)}(t, x) \leq -\lambda(t) V(t, x)$, *where* λ *is a scalar continuous positive function in* I *and* $\int_a^{+\infty} \lambda(t)\, dt = +\infty$.

Then the solution $x(t) \equiv 0$ *of* (1) *is* φ*-asymptotically stable.*

Proof. For any $\varepsilon > 0$ $(\varepsilon < H)$, $t_0 \in I$, we can choose $\delta = \delta(t_0, \varepsilon)$ such that $\| x_0 \| < \delta$ implies $V(t_0, x_0) < \varepsilon$. Let $x(t; t_0, x_0)$ be a solution of (1) such that $\| x_0 \| < \delta$. Applying Theorem 4.1 in [9], by (ii) we have

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi \right) < \varepsilon \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi \right).$$

Denote $\varphi(t) = \exp\left(-\int_a^t \lambda(\xi)\, d\xi \right)$. Because of the feature of the function λ, we can see that φ is continuous monotonically increasing function on I, $\varphi(a) = 1$, $\varphi(t) \to +\infty$ as $t \to +\infty$ and the above estimate leads to

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) \frac{\varphi(t_0)}{\varphi(t)} < \varepsilon \frac{\varphi(t_0)}{\varphi(t)}.$$

Thus, by (ii) we obtain

$$\|x(t; t_0, x_0)\| \le V(t, x(t; t_0, x_0)) < \varepsilon \frac{\varphi(t_0)}{\varphi(t)} \quad \text{for all} \quad t \ge t_0 \quad \text{if} \quad \|x_0\| < \delta.$$

That is, the solution $x(t) \equiv 0$ of (1) is φ-asymptotically stable.

The theorem is proved.

Theorem 2. *Suppose that there exists a Liapunov function* $V(t, x)$ *defined on* I, $\|x\| < H$, *which satisfies the following conditions:*

(i) $\|x\| \le V(t, x) \le b(\|x\|)$, *where* $b(r) \in CIP$ [9, 7];

(ii) $V'_{(1)}(t, x) \le -\lambda(t) V(t, x)$, *where* λ *is the function defined in Theorem* 1.

Then the solution $x(t) \equiv 0$ *of* (1) *is* φ-*uniform asymptotically stable.*

Proof. For a given $\varepsilon > 0$, we can choose $\delta(\varepsilon) > 0$ so that $\delta(\varepsilon) < b^{-1}(\varepsilon)$ and the remainder of the proof can be verified by the same argument as in Theorem 1.

***Corollary* 1.** *If* $\lambda(t) \equiv c$ $(c > 0)$, *then* $x(t) \equiv 0$ *of* (1) *is exponential-asymptotically stable, that is* $\|x(t; t_0, x_0)\| \le \varepsilon e^{-c(t-t_0)}$ *for all* $t \ge t_0$ (*see Definition* 7.8 *in* [9]).

Theorem 3. *Suppose that there exists a Liapunov function* $V(t, x)$ *defined on* $I \times \mathbb{R}^n$ *satisfying the following conditions:*

(i) $\|x\| \le V(t, x)$ *and* $V(t, 0) \equiv 0$;

(ii) $V'_{(1)}(t, \varphi(t)x) \le 0$, *where* $\varphi(t)$ *is a scalar monotonically increasing, differentiable function on* I, $\varphi(a) \ge 1$ *and* $\varphi(t) \to +\infty$ *as* $t \to +\infty$.

Then the solution $x(t) \equiv 0$ *of* (1) *is* φ-*asymptotically stable.*

Proof. For any $\varepsilon > 0$ and a fixed $t_0 \in I$, we can find a number $\delta = \delta(t_0, \varepsilon)$ such that $\|x_0\| < \delta$ implies $V(t_0, \varphi(t_0)x_0) < \varepsilon$. Under assumption that $x(t; t_0, x_0)$ is a solution of (1) satisfying $\|x_0\| < \delta$, we have $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \ge t_0$. Indeed, if there exists $t_1 > t_0$ such that $\|x(t_1; t_0, x_0)\| \ge \varepsilon$, by (i) and (ii) we obtain $\varepsilon \le \|x(t_1; t_0, x_0)\| \le \varphi(t_1)\|x(t_1; t_0, x_0)\| \le V(t_1, \varphi(t_1)x(t_1; t_0, x_0)) \le V(t_0, \varphi(t_0)x_0) < \varepsilon$. This is a contradiction.

On the other hand, conditions (i), (ii) imply:

$$\varphi(t)\|x(t; t_0, x_0)\| \le V(t, \varphi(t)x(t; t_0, x_0)) \le V(t_0, \varphi(t_0)x_0) < \varepsilon.$$

Thus, we have $\|x(t; t_0, x_0)\| < \dfrac{\varepsilon}{\varphi(t)} \le \varepsilon \dfrac{\varphi(t_0)}{\varphi(t)}$ for all $t \ge t_0$ if $\|x_0\| < \delta$.

This shows that the solution $x(t) \equiv 0$ of (1) is φ-asymptotically stable.

Theorem 4. *Suppose that there exists a Liapunov function* $V(t, x)$ *defined on* $I \times \mathbb{R}^n$ *which satisfies the following conditions:*

(i) $V(t, 0) \equiv 0$;

(ii) $\varphi(t)\|x\| \le V(t, x)$, *where* $\varphi(t)$ *is a continuous monotonically increasing function on* I, $\varphi(a) \ge 1$ *and* $\varphi(t) \to +\infty$ *as* $t \to +\infty$;

(iii) $V'_{(1)}(t, x) \le 0$.

Then the solution $x(t) \equiv 0$ *of* (1) *is* φ-*asymptotically stable.*

Proof. The proof can be given by the same idea as in the proof of Theorem 3.

Theorem 5. *Suppose that there exists a Liapunov function* $V(t, x)$ *defined on* $I \times \mathbb{R}^n$ *which satisfies the following conditions:*

(i) $\|x\| \le V(t, x) \le K(\alpha)\|x\|$ *for* $x \in S_\alpha$, $K(\alpha)$ *is a positive number;*

(ii) $V'_{(1)}(t, x) \le -\lambda(t)V(t, x)$, *where* λ *is the function defined in Theorem* 1.

Then the solution $x(t) \equiv 0$ *of* (1) *is* φ*-asymptotically stable in the large.*

The proof can be given by the same idea as in the proof of Theorem 11.6 in [9].

Example. Consider the equation

$$\begin{aligned} x' &= -\frac{1}{2\sqrt{t}}x, \\ y' &= -\frac{1}{2\sqrt{t}}y - \frac{e^{2\sqrt{t}}}{t}x^2 y, \quad t \ge 1. \end{aligned} \tag{2}$$

Let $V(t, x, y) = \left(x^2 + y^2\right)^{\frac{1}{2}}$, then $V(t, x, y)$ is a Liapunov function defined on $\{1 \le t < \infty\} \times R^2$, which satisfies condition (i) of Theorem 5; condition (ii) is also satisfied because:

$$V'_{(2)}(t, x, y) = \frac{1}{2}\left(2xx' + 2yy'\right)\left(x^2 + y^2\right)^{-\frac{1}{2}} =$$

$$= \left(-\frac{1}{2\sqrt{t}}x^2 - \frac{1}{2\sqrt{t}}y^2 - \frac{1}{\sqrt{t}}\frac{e^{2\sqrt{t}}}{t}x^2y^2\right)\left(x^2 + y^2\right)^{-\frac{1}{2}} \le$$

$$\le -\frac{1}{2\sqrt{t}}\left(x^2 + y^2\right)\left(x^2 + y^2\right)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{t}}\left(x^2 + y^2\right)^{\frac{1}{2}} = -\frac{1}{2\sqrt{t}}V(t, x, y).$$

Then, for a given $\alpha > 0$ we have $\| z(t; t_0, z_0) \| \le \frac{e^{\sqrt{t_0}}}{e^{\sqrt{t}}} \| z_0 \|$ for all $t \ge t_0 \ge 1$, $z_0 \in S_\alpha$, where $z(t; t_0, z_0) = \operatorname{colon}\left(x(t; t_0, x_0), y(t; t_0, y_0)\right)$, and $z_0 = \operatorname{colon}\left(x_0, y_0\right)$.

Thus, the solution $x(t) \equiv 0$ of equation (2) is $e^{\sqrt{t}}$-asymptotically stable in the large.

3. Converse theorems on φ-asymptotic stability. Let us begin with converse theorems on φ-asymptotic stability of linear systems. Consider the system

$$\frac{dx}{dt} = A(t)x, \tag{3}$$

where $A(t)$ is continuous $n \times n$ matrix on I.

Theorem 6. *Suppose that there exists* $K \ge 1$ *satisfying the following condition:*

$$\| x(t; t_0, x_0) \| \le K \frac{\varphi(t_0)}{\varphi(t)} \| x_0 \|, \tag{4}$$

where $x(t; t_0, x_0)$ *is a solution of* (3), $\varphi(t)$ *is a function defined as in the Theorem* 3 *and* $\varphi'(t) > 0$ *on* I.

Then there exists a function $V(t, x)$ *defined on* $I \times \mathbb{R}^n$ *which satisfies the following conditions:*

(i) $\| x \| \le V(t, x) \le K \| x \|$;

(ii) $| V(t, x) - V(t, x_*) | \le K \| x - x_* \|$;

(iii) $V'_{(3)}(t, x) \le -\lambda(t)V(t, x)$, $\lambda(t) = \frac{\varphi'(t)}{\varphi(t)}$ *for all* $t \in I$;

(iv) $V'_{(3)}(t, \varphi(t)x) \le 0$.

Proof. Put $V(t, x) = \sup_{\tau \geq 0} \| x(t+\tau; t, x) \| \dfrac{\varphi(t+\tau)}{\varphi(t)}$.

Due to (4), we can see that condition (i) will be held. In fact,

$$\| x \| \leq \sup_{\tau \geq 0} \| x(t+\tau; t, x) \| \frac{\varphi(t+\tau)}{\varphi(t)} \leq \sup_{\tau \geq 0} \left[K \frac{\varphi(t)}{\varphi(t+\tau)} \| x \| \frac{\varphi(t+\tau)}{\varphi(t)} \right] = K \| x \|.$$

Moreover,

$$| V(t, x) - V(t, x_*) | = \left| \sup_{\tau \geq 0} \| x(t+\tau; t, x) \| \frac{\varphi(t+\tau)}{\varphi(t)} - \sup_{\tau \geq 0} \| x(t+\tau; t, x_*) \| \frac{\varphi(t+\tau)}{\varphi(t)} \right| \leq$$

$$\leq \sup_{\tau \geq 0} \| x(t+\tau; t, x) - x(t+\tau; t, x_*) \| \frac{\varphi(t+\tau)}{\varphi(t)} = \sup_{\tau \geq 0} \| x(t+\tau; t, x - x_*) \| \frac{\varphi(t+\tau)}{\varphi(t)} \leq$$

$$\leq \sup_{\tau \geq 0} \left[K \frac{\varphi(t)}{\varphi(t+\tau)} \| x - x_* \| \frac{\varphi(t+\tau)}{\varphi(t)} \right] = K \| x - x_* \|.$$

Thus, condition (i) is satisfied.

The proof of the continuity of $V(t, x)$ can be performed by the same method used in the proof of Theorem19.1 in [9].

Now, we shall prove (iii). Let $x_* = x(t+h; t, x)$, $h > 0$. Then we have

$$V(t+h, x_*) = \sup_{\tau \geq 0} \| x(t+h+\tau; t+h, x_*) \| \frac{\varphi(t+h+\tau)}{\varphi(t+h)} =$$

$$= \sup_{\tau \geq 0} \| x(t+h+\tau; t, x) \| \frac{\varphi(t+h+\tau)}{\varphi(t)} \frac{\varphi(t)}{\varphi(t+h)} =$$

$$= \sup_{\tau \geq h} \| x(t+\tau; t, x) \| \frac{\varphi(t+\tau)}{\varphi(t)} \frac{\varphi(t)}{\varphi(t+h)} \leq$$

$$\leq \sup_{\tau \geq 0} \| x(t+\tau; t, x) \| \frac{\varphi(t+\tau)}{\varphi(t)} \frac{\varphi(t)}{\varphi(t+h)} = \frac{\varphi(t)}{\varphi(t+h)} V(t, x),$$

which implies

$$\frac{1}{h} [V(t+h, x_*) - V(t, x)] \leq \frac{1}{h} \left[\frac{\varphi(t)}{\varphi(t+h)} - 1 \right] V(t, x).$$

Since the function $\varphi(t)$ is differentiable, the above inequality implies

$$V'_{(3)}(t, x) \leq -\lambda(t) V(t, x),$$

where $\lambda(t) = \dfrac{\varphi'(t)}{\varphi(t)}$, $t \in I$. Condition (iii) is proved.

Finally, we shall establish (iv). Since system (3) is linear, we have the relation $x\big(t; t_0, \varphi(t_0) x_0\big) = \varphi(t_0) x\big(t; t_0, x_0\big)$, whence we obtain

$$V(t+h, \varphi(t+h) x_*) = \sup_{\tau \geq 0} \| x(t+h+\tau; t+h, \varphi(t+h) x_*) \| \frac{\varphi(t+h+\tau)}{\varphi(t+h)} =$$

$$= \sup_{\tau \geq 0} \| x(t+h+\tau; t+h, x(t+h, t, x)) \| \varphi(t+h) \frac{\varphi(t+h+\tau)}{\varphi(t+h)} =$$

$$= \sup_{\tau \geq 0} \| x(t+h+\tau; t, x) \| \varphi(t+h+\tau) = \sup_{\tau \geq h} \| x(t+\tau; t, x) \| \varphi(t+\tau) =$$

$$= \sup_{\tau \geq h} \| x(t+\tau; t, \varphi(t)x) \| \frac{\varphi(t+\tau)}{\varphi(t)} \leq$$

$$\leq \sup_{\tau \geq 0} \| x(t+\tau; t, \varphi(t)x) \| \frac{\varphi(t+\tau)}{\varphi(t)} = V(t, \varphi(t)x),$$

which implies $V(t+h, \varphi(t+h)x_*) - V(t, \varphi(t)x) \leq 0$ and then $V'_{(7)}(t, \varphi(t)x) \leq 0$.

Theorem 7. *Suppose that there exists* $K \geq 1$ *such that* $\| x(t; t_0, x_0) \| \leq \leq K \frac{\varphi(t_0)}{\varphi(t)} \| x_0 \|$, *where* $x(t; t_0, x_0)$ *is a solution of* (3), $\varphi(t)$ *is a function defined as in the Theorem* 4.

Then there exists a function $V(t, x)$ *defined on* $I \times \mathbb{R}^n$ *which satisfies the following conditions:*

$$V(t, 0) \equiv 0, \quad \varphi(t)\| x \| \leq V(t, x), \quad | V(t, x) - V(t, x_*) | \leq K\varphi(t) \| x - x_* \|$$

and

$$V'_{(7)}(t, x) \leq 0.$$

Proof. By the same idea used in the proof of Theorem 6, this theorem can be proved by choosing $V(t, x) = \sup_{\tau \geq 0} \| x(t+\tau; t, x) \| \varphi(t+\tau)$.

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