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A PROPERTY OF THE β-CAUCHY TYPE INTEGRAL WITH A CONTINUOUS DENSITY*

ПРО ОДНУ ВЛАСТИВІСТЬ ІНТЕГРАЛА ТИПУ β-КОШІ З НЕПЕРЕРВНОЮ ЩІЛЬНІСТЮ

The aim of this paper is to extend a theorem from classical complex analysis proved by Davydov in 1949 to the theory of solutions of a special case of the Beltrami equation in the z-complex plane (i.e., null solutions of the differential operator $\partial_{\bar z} - \beta \frac{z}{\bar z} \partial_z$, $0 \le \beta < 1$).

We prove that if γ is a rectifiable Jordan closed curve and f is a continuous complex-valued function on γ such that the integral

$$\int\limits_{\gamma \setminus \{\zeta \in \gamma : |\zeta - t| \le r\}} \frac{|f(\zeta) - f(t)|}{|\zeta - t|t/\zeta|^{\theta}|} \left| \left(n(\zeta) - \beta \frac{\zeta}{\bar\zeta} \bar n(\zeta) \right) \right| ds, \quad \theta = \frac{2\beta}{1-\beta},$$

converges uniformly on γ as $r \to 0$, where $n(\zeta)$ is the exterior unit normal vector on γ at a point ζ and ds is the arc length differential, then the β-Cauchy type integral

$$\frac{1}{2(1-\beta)\pi} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z|z/\zeta|^{\theta}} \left(n(\zeta) - \beta \frac{\zeta}{\bar\zeta} \bar n(\zeta) \right) ds, \quad z \notin \gamma,$$

admits a continuous extension to γ and a version of the Sokhotski – Plemelj formulae holds.

Метою цієї статті є узагальнення теореми із класичного комплексного аналізу, що була доведена Давидовим у 1949 р., для теорії розв’язків окремого випадку рівняння Бельтрамі у z-комплексній площині (тобто нульових розв’язків диференціального оператора $\partial_{\bar z} - \beta \frac{z}{\bar z} \partial_z$, $0 \le \beta < 1$).

Доведено, що коли γ є спрямлюваною замкненою кривою Жордана і f є неперервною комплекснозначною функцією на γ такою, що інтеграл

$$\int\limits_{\gamma \setminus \{\zeta \in \gamma : |\zeta - t| \le r\}} \frac{|f(\zeta) - f(t)|}{|\zeta - t|t/\zeta|^{\theta}|} \left| \left(n(\zeta) - \beta \frac{\zeta}{\bar\zeta} \bar n(\zeta) \right) \right| ds, \quad \theta = \frac{2\beta}{1-\beta},$$

рівномірно збігається на γ при $r \to 0$, де $n(\zeta)$ — зовнішній одиничний нормальний вектор на γ у точці ζ, а ds — диференціал довжини дуги, тоді інтеграл типу β-Коші

$$\frac{1}{2(1-\beta)\pi} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z|z/\zeta|^{\theta}} \left(n(\zeta) - \beta \frac{\zeta}{\bar\zeta} \bar n(\zeta) \right) ds, \quad z \notin \gamma,$$

дозволяє неперервне розширення на γ і один із варіантів формули Сохоцького – Племея виконується.

1. The β-Cauchy type integral and Davydov’s theorem. *1.1.* The classical Beltrami equation can be considered as a remarkable generalization of the Cauchy – Riemann equation in the complex plane. Its solutions have many properties analogous to those of analytic functions of one complex variable and numerous problems of

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analysis and geometry can be reduced to solving this equation. For a survey of recent research and historical details on the Beltrami equation we refer the reader to [1 – 3].

It is our purpose to study a Cauchy type integral associated to the theory of solutions of a special case of Beltrami equation, which are called β-analytic functions, namely, solutions of the following linear first order partial differential equation:

$$\partial_{\bar z} f = \beta \frac{z}{\bar z} \partial_z f, \quad z = x + iy,$$

where $0 \le \beta < 1$ and as usual $\partial_{\bar z} := \frac{1}{2}(\partial_x + i\partial_y)$, $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$.

Suppose that, in $\mathbb{C}$, a domain Ω with boundary γ is given. We refer to [4] for the theory of β-analytic functions in Ω having proved a new integral representation formula. In particular, the Cauchy integral formula

$$(\mathbf{C}_\gamma^\beta f)(z) = \begin{cases} f(z), & z \in \Omega, \\ 0, & z \in \mathbb{C} \setminus \overline{\Omega}, \end{cases}$$

where

$$(\mathbf{C}_\gamma^\beta f)(z) := \frac{1}{2(1-\beta)\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta} \left(d\zeta + \beta \frac{\zeta}{\bar\zeta} d\bar\zeta \right), \quad z \notin \gamma,$$

and $\theta = \dfrac{2\beta}{1-\beta}$.

In this way $\mathbf{C}_\gamma^\beta f$ plays the role of the Cauchy type integral in the theory of β-analytic functions and we shall call it the β-Cauchy type integral.

Note that the complex element of integration $d\zeta$ may be written as $d\zeta = i n(\zeta) ds$, where $n(\zeta)$ is the exterior unit normal vector on γ at a point ζ, writing it as a complex number and ds is the arc length differential.

We can thus write $(\mathbf{C}_\gamma^\beta f)(z)$ in the form

$$(\mathbf{C}_\gamma^\beta f)(z) = \frac{1}{2(1-\beta)\pi} \int_\gamma \frac{f(\zeta)}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta} \left(n(\zeta) - \beta \frac{\zeta}{\bar\zeta} \bar n(\zeta) \right) ds, \quad z \notin \gamma. \tag{1}$$

Here and subsequently γ denotes a rectifiable positively oriented closed Jordan curve in $\mathbb{C}$. Let Ω^+ and Ω^- be, respectively, the interior and exterior domains bounded by γ. There is no loss of generality in assuming $0 \in \Omega^+$.

An important point to note here is that if $\beta = 0$, i.e., the case of analytic functions, we recovered the standard Cauchy type integral.

In the theory of analytic functions of one complex variable, the Cauchy type integral is proved to be a very deep and crucial object of research. Moreover, its study has led to numerous discoveries both in analytic function theory itself and in many other areas, such as the Hardy space theory, singular integral equations as well as potential and elasticity theories.

Properties of this integral on a bounded domain with quite enough smooth boundary are sufficiently well understood, see [5, 6]. There are numerous investigations on the evolution of the limit boundary values of the Cauchy type integral in such smooth bounded domains. A well-known first result is that it possesses continuous limit boundary values on γ if its density belongs to the Lipschitz class.

By far much more general is the case of Davydov's theorem (see [7]) which is related to the problem of establishing a sufficient condition for the Cauchy type integral to be continuously extended to the closure of a domain bounded by a rectifiable Jordan closed curve.

This result requires the uniform existence of some integral which restricts both the set of curves and classes of densities, it provides a sufficient condition guaranteeing that the Sokhotski – Plemelj formulae hold. As an example of such restriction, Davydov has considered the Lipschitz class and an arbitrary rectifiable closed Jordan curve.

Moreover, the result presented in [8] under the much more stronger condition of being γ a regular curve (i.e., the quotient of the measure of γ inside any circle with center at any point of this curve to the radius of the circle is less than some fixed constant) is a very good piece of work on the subject that attracts considerable attention.

Generalizations of the Davydov's theorem were the subjects of research in a number of papers, see [9 – 13], both in the complex and hypercomplex context. Hence, this can be thought of as a good motivation for the analog of the above for the theory of the β-Cauchy type integral and the question arises about the existence of a reasonable extension of Davydov's theorem. Our paper deals with this situation.

For a deep discussion on the existence of continuous limit values of the integral (1) along a regular curve we refer the reader to [14].

In addition, to illustrate how β-Cauchy type integral works, we refer the reader to [15], where some higher order Cauchy – Pompeiu representation formulas for β-analytic functions are given trying to determine particular solutions to some differential equations.

1.2. For the study of the behavior of the β-Cauchy type integral near the integration curve we also need the singular β-Cauchy type integral given by

$$(\mathbf{S}_\gamma^\beta f)(t) \;:=\; \lim_{r\to 0}\frac{1}{2(1-\beta)\pi}\int\limits_{\gamma\setminus\{\zeta\in\gamma:|\zeta-t|\le r\}}\frac{f(\zeta)-f(t)}{\zeta-t\left|\dfrac{t}{\zeta}\right|^{\theta}}\left(n(\zeta)-\beta\,\frac{\zeta}{\overline{\zeta}}\,\overline{n}(\zeta)\right)ds,\quad t\in\gamma.$$

It is a simple exercise to see that the β-Cauchy type integral is an example of a β-analytic function in $\mathbb{C}\setminus\gamma$ that vanishes at infinity. Another example of a β-analytic function is the function $\zeta(z):=z|z|^{\theta}$.

Let us remark that the transformation $(z,\overline{z})\to(\zeta,\overline{\zeta})$ is continuously differentiable in $\mathbb{C}\setminus\{0\}$ and its Jacobian is

$$J(\zeta) \;=\; \frac{1+\beta}{1-\beta}|\zeta|^{2\theta}.$$

Using the above remark and our assumption on γ $(0\notin\gamma)$, it is easy to see that for z_1, z_2 sufficiently close to γ we have the following inequalities:

$$c^{-1}\;\le\;\frac{\left|z_1|z_1|^{\theta}-z_2|z_1|^{\theta}\right|}{|z_1-z_2|}\;\le\;c,\quad z_1\ne z_2.$$

In several cases we will make use of these inequalities. We will use the symbol c for constants depending on γ which may vary from one occurence to the next.

The generalization of Davydov's theorem for the β-analytic function theory to be proved is formulated as follows:

Theorem 1. *Let γ be a rectifiable Jordan closed curve and let f be a continuous complex-valued function on γ. If the integral*

$$\int\limits_{\gamma\setminus\{\zeta\in\gamma:|\zeta-t|\le r\}}\frac{|f(\zeta)-f(t)|}{\left|\zeta-t\left|\dfrac{t}{\zeta}\right|^{\theta}\right|}\left|\left(n(\zeta)-\beta\,\frac{\zeta}{\overline{\zeta}}\,\overline{n}(\zeta)\right)\right|ds \tag{2}$$

converges uniformly on γ *as* $r \to 0$, *then the singular* β*-Cauchy type integral exists and then the* β*-Cauchy type integral of* f *has continuous limit values on* γ. *Moreover, the Sokhotski – Plemelj formulae hold* :

$$\lim_{\Omega^+ \ni z \to t} (\mathbf{C}_\gamma^\beta f)(z) = (\mathbf{S}_\gamma^\beta f)(t) + f(t), \quad t \in \gamma, \tag{3}$$

$$\lim_{\Omega^- \ni z \to t} (\mathbf{C}_\gamma^\beta f)(z) = (\mathbf{S}_\gamma^\beta f)(t), \quad t \in \gamma. \tag{4}$$

2. Proof of the theorem. We first show that if the integral (2) converges uniformly on γ as $r \to 0$, then the singular β-Cauchy type integral exists and is continuous on γ.

Denote

$$\mathcal{W}_f(r, t) = \int\limits_{\gamma \setminus \{\zeta \in \gamma : |\zeta - t| \le r\}} \frac{|f(\zeta) - f(t)|}{\left|\zeta - t\left|\frac{t}{\zeta}\right|^\theta\right|} \left| n(\zeta) - \beta \frac{\zeta}{\bar{\zeta}} \bar{n}(\zeta) \right| ds,$$

$$\mathcal{V}_f(r, t) = \frac{1}{2(1-\beta)\pi} \int\limits_{\gamma \setminus \{\zeta \in \gamma : |\zeta - t| \le r\}} \frac{f(\zeta) - f(t)}{\zeta - t\left|\frac{t}{\zeta}\right|^\theta} \left(n(\zeta) - \beta \frac{\zeta}{\bar{\zeta}} \bar{n}(\zeta) \right) ds.$$

By assumption, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all t in γ and all $0 < r_1 < r_2 < \delta(\varepsilon)$, we have

$$\mathcal{W}_f(r_1, t) - \mathcal{W}_f(r_2, t) =$$

$$= \int\limits_{\{\zeta \in \gamma : |\zeta - t| \le r_2\} \setminus \{\zeta \in \gamma : |\zeta - t| \le r_1\}} \frac{|f(\zeta) - f(t)|}{\left|\zeta - t\left|\frac{t}{\zeta}\right|^\theta\right|} \left| n(\zeta) - \beta \frac{\zeta}{\bar{\zeta}} \bar{n}(\zeta) \right| ds < 2(1-\beta)\pi\varepsilon.$$

From the above it follows that for all t in γ and all $0 < r_1 < r_2 < \delta(\varepsilon)$ we obtain

$$\left| \mathcal{V}_f(r_1, t) - \mathcal{V}_f(r_2, t) \right| =$$

$$= \left| \frac{1}{2(1-\beta)\pi} \int\limits_{\{\zeta \in \gamma : |\zeta - t| \le r_2\} \setminus \{\zeta \in \gamma : |\zeta - t| \le r_1\}} \frac{f(\zeta) - f(t)}{\zeta - t\left|\frac{t}{\zeta}\right|^\theta} \left(n(\zeta) - \beta \frac{\zeta}{\bar{\zeta}} \bar{n}(\zeta) \right) ds \right| \le$$

$$\le \frac{1}{2(1-\beta)\pi} \int\limits_{\{\zeta \in \gamma : |\zeta - t| \le r_2\} \setminus \{\zeta \in \gamma : |\zeta - t| \le r_1\}} \frac{|f(\zeta) - f(t)|}{\left|\zeta - t\left|\frac{t}{\zeta}\right|^\theta\right|} \left| n(\zeta) - \beta \frac{\zeta}{\bar{\zeta}} \bar{n}(\zeta) \right| ds < \varepsilon.$$

Hence, $\mathcal{V}_f(r, t)$ converges uniformly on γ to $(\mathbf{S}_\gamma^\beta f)(t)$ as $r \to 0$. Therefore, the singular β-Cauchy type integral $(\mathbf{S}_\gamma^\beta f)$ is continuous on γ.

Now, let us prove (3). The relation (4) can be proved similarly.

Let t be a fixed point of γ and let $z \in \Omega^+$. If $t_z \in \{\zeta \in \gamma : |z - \zeta| = \operatorname{dist}(z, \gamma)\}$ we have that

$$\left| (\mathbf{C}_\gamma^\beta f)(z) - (\mathbf{S}_\gamma^\beta f)(t) - f(t) \right| \le$$

$$\le \left| (\mathbf{C}_\gamma^\beta f)(z) - f(t_z) - (\mathbf{S}_\gamma^\beta f)(t_z) \right| + \left| (\mathbf{S}_\gamma^\beta f)(t_z) - (\mathbf{S}_\gamma^\beta f)(t) \right| + \left| f(t_z) - f(t) \right|.$$

By continuity, the last two summands on the right-hand side of the previous inequality tend to zero as $z \to t$.

We now turn to the first summand. Note that for any $r > 0$ we have

$$2(1-\beta)\pi\left|(\mathbf{C}_\gamma^\beta f)(z) - f(t_z) - (\mathbf{S}_\gamma^\beta f)(t_z)\right| =$$

$$= \left|\int_\gamma \left[\frac{1}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta} - \frac{1}{\zeta - t_z\left|\frac{t_z}{\zeta}\right|^\theta}\right](f(\zeta) - f(t_z))\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)ds\right| \le$$

$$\le \left|\int_{\{\zeta\in\gamma:|\zeta - t_z|\le r\}} \frac{f(\zeta) - f(t_z)}{\zeta - t_z\left|\frac{t_z}{\zeta}\right|^\theta}\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)ds\right| +$$

$$+ \left|\int_{\{\zeta\in\gamma:|\zeta - t_z|\le r\}} \frac{f(\zeta) - f(t_z)}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta}\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)ds\right| +$$

$$+ \left|\int_{\gamma\setminus\{\zeta\in\gamma:|\zeta - t|\le r\}} \left[\frac{1}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta} - \frac{1}{\zeta - t_z\left|\frac{t_z}{\zeta}\right|^\theta}\right](f(\zeta) - f(t_z))\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)ds\right| =$$

$$= I_1 + I_2 + I_3.$$

Fix $\varepsilon > 0$. By the uniform convergence of $\mathcal{V}_f(r, t)$, there exists $\delta_1(\varepsilon) > 0$ such that for all $r < \delta_1(\varepsilon)$ we have

$$I_1 < \frac{2(1-\beta)\pi\varepsilon}{3}$$

for all $z \in \Omega^+$.

Since $|\zeta - t_z| \le 2|\zeta - z|$ for all $\zeta \in \gamma$, we get that

$$I_2 \le \int_{\{\zeta\in\gamma:|\zeta - t_z|\le r\}} \frac{|f(\zeta) - f(t_z)|}{\left|\zeta - z\left|\frac{z}{\zeta}\right|^\theta\right|}\left|\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)\right|ds \le$$

$$\le c\int_{\{\zeta\in\gamma:|\zeta - t_z|\le r\}} \frac{|f(\zeta) - f(t_z)|}{\left|\zeta - t_z\left|\frac{t_z}{\zeta}\right|^\theta\right|}\left|\left(n(\zeta) - \beta\frac{\zeta}{\overline{\zeta}}\overline{n}(\zeta)\right)\right|ds.$$

The uniform convergence of the integral (2) now assures the existence of a positive number $\delta_2(\varepsilon)$ such that for all $r < \delta_2(\varepsilon)$ and all $z \in \Omega^+$ we have

$$I_2 < \frac{2(1-\beta)\pi\varepsilon}{3}.$$

Now fix any $r > 0$ strictly less than $\min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. Our next concern is to estimate I_3. First, note that for $\zeta \in \gamma\setminus\{\zeta \in \gamma: |\zeta - t_z| \le r\}$ we have $r < |\zeta - t_z| \le$ $\le 2|\zeta - z|$.

In this way

$$\left|\frac{1}{\zeta - z\left|\frac{z}{\zeta}\right|^\theta} - \frac{1}{\zeta - t_z\left|\frac{t_z}{\zeta}\right|^\theta}\right| \le \frac{|\zeta|^\theta\left|z|z|^\theta - t_z|t_z|^\theta\right|}{\left|\zeta|\zeta|^\theta - z|z|^\theta\right|\left|\zeta|\zeta|^\theta - t_z|t_z|^\theta\right|} \le \frac{c|z - t_z|}{r^2}$$

for $\zeta \in \gamma\setminus\{\zeta \in \gamma: |\zeta - t_z| \le r\}$.

Let us take

$$\delta(\varepsilon) = \frac{2(1-\beta)\pi r^2 \varepsilon}{c \max\limits_{\zeta\in\gamma} |f(\zeta)| s(\gamma)}.$$

Then for all $z \in \Omega^+$ with $|z - t| < \delta(\varepsilon)$, we have

$$I_3 \le \int\limits_{\gamma\setminus\{\zeta\in\gamma:|\zeta - t_z|\le r\}} \left| \frac{1}{\zeta - z\left|\frac{z}{\zeta}\right|^{\theta}} - \frac{1}{\zeta - t_z\left|\frac{t_z}{\zeta}\right|^{\theta}} \right| |f(\zeta) - f(t_z)| ds \le$$

$$\le \frac{c|z - t_z|}{r^2} \max_{\zeta\in\gamma} |f(\zeta)| s(\gamma) < \frac{2(1-\beta)\pi\varepsilon}{3},$$

which completes the proof.

If $\beta = 0$, then an immediate consequence of Theorem 1 is the Davydov theorem [7] mentioned in Section 1.

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