UDC 517.9

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ON INVERSE PROBLEM FOR SINGULAR STURM – LIOUVILLE OPERATOR FROM TWO SPECTRA

ПРО ОБЕРНЕНУ ЗАДАЧУ ДЛЯ СИНГУЛЯРНОГО ОПЕРАТОРА ШТУРМА – ЛІУВІЛЛЯ ВІД ДВОХ СПЕКТРІВ

In the paper, an inverse problem with two given spectra for second order differential operator with singularity of type $\dfrac{2}{r}+\dfrac{\ell(\ell+1)}{r^2}$ (here, l is a positive integer or zero) at zero point is studied. It is well known that two spectra $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely determine the potential function $q(r)$ in a singular Sturm – Liouville equation defined on interval $(0,\pi]$.

One of the aims of the paper is to prove the generalized degeneracy of the kernel $K(r,s)$. In particular, we obtain a new proof of Hochstadt's theorem concerning the structure of the difference $\tilde q(r)-q(r)$.

Вивчається обернена задача з використанням двох заданих спектрів для диференціального оператора другого порядку з сингулярністю типу $\dfrac{2}{r}+\dfrac{\ell(\ell+1)}{r^2}$ (l — додатне ціле число або нуль) у нульовій точці. Відомо, що два спектри $\{\lambda_n\}$ та $\{\mu_n\}$ встановлюють єдиним чином функцію потенціалу $q(r)$ у сингулярному рівнянні Штурма – Ліувілля, визначеному на інтервалі $(0,\pi]$.

Однією з цілей роботи є доведення узагальненої виродженості ядра $K(r,s)$. Зокрема, одержано нове доведення теореми Гохштадта щодо структури різниці $\tilde q(r)-q(r)$.

Introduction. We will consider the equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\ell(\ell+1)}{r^2}R+\left(E+\frac{2}{r}\right)R=0,\quad 0<r<\infty. \tag{1}$$

In quantum mechanics, the study of the energy levels of a hydrogen atom leads to this equation [1]. The substitution $R=y/r$ reduces equation (1) to the form

$$\frac{d^2y}{dr^2}+\left\{E+\frac{2}{r}-\frac{\ell(\ell+1)}{r^2}\right\}y=0. \tag{2}$$

Just as in the case of Bessel's equation, one can show that, in a finite interval $[0,b]$, the spectrum is discrete.

As known [2, 3], for a solution of (2) which is bounded at zero, one has the following asymptotic formula for $\lambda\to\infty$ $(E=\lambda)$:

$$\varphi(r,\lambda)=\frac{e^{\frac{\pi}{2\sqrt{\lambda}}}}{\left|\Gamma\left(\ell+1+\dfrac{i}{\sqrt{\lambda}}\right)\right|}\frac{1}{\sqrt{\lambda}}\cos\left[\sqrt{\lambda}r+\frac{1}{\sqrt{\lambda}}\ln\sqrt{\lambda}r-(\ell+1)\frac{\pi}{2}+\alpha\right]+o(1), \tag{3}$$

where $\alpha=\arg\Gamma\left(\ell+1+\dfrac{i}{\sqrt{\lambda}}\right)$.

We consider two singular Sturm – Liouville problems

$$-y''+\left[\frac{\ell(\ell+1)}{r^2}-\frac{2}{r}+q(r)\right]y=\lambda y,\quad 0<r\le\pi, \tag{4}$$

$$y(0)=0, \tag{5}$$

$$y'(\pi)+Hy(\pi)=0, \tag{6}$$

$$-y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + \tilde{q}(r)\right] y = \lambda y, \quad 0 < r \leq \pi, \tag{7}$$

$$y\,(0) = 0,$$

$$y'(\pi) + \tilde{H} y(\pi) = 0, \tag{8}$$

in which the functions $q(r)$ and $\tilde{q}(r)$ are assumed to be real-valued and square integrable. H and $\tilde{H}$ are finite real numbers.

We denote the spectrum of the first problem by $\{\lambda_n\}_0^\infty$ and the spectrum of the second by $\{\tilde{\lambda}_n\}_0^\infty$.

Next, we denote by $\varphi\,(r,\lambda)$ the solution of (4) and we denote by $\tilde{\varphi}\,(r,\lambda)$ the solution of (7) satisfying the initial condition (5).

It is well known that there exists a function $K(r,s)$ such that

$$\tilde{\varphi}\,(r,\lambda) = \varphi\,(r,\lambda) + \int\limits_0^r K(r,s)\varphi\,(s,\lambda)\,ds. \tag{9}$$

The function $K(r,s)$ satisfies the equation

$$\frac{\partial^2 K}{\partial r^2} - \left[\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + \tilde{q}(r)\right] K = \frac{\partial^2 K}{\partial s^2} - \left[\frac{2}{s} - \frac{\ell(\ell+1)}{s^2} + q\,(s)\right] K \tag{10}$$

and the conditions

$$K(r,r) = \frac{1}{2}\int\limits_0^r \left[\tilde{q}\,(t) - q\,(t)\right] dt, \tag{11}$$

$$K\,(r,0) = 0. \tag{12}$$

After the transformations

$$z = \frac{1}{4}(r+s)^2, \quad w = \frac{1}{4}(r-s)^2, \quad K(r,s) = (z-w)^{-\nu+\frac{1}{2}} u(z,w),$$

we obtain the following problem $(-\nu + \dfrac{1}{2} = \beta)$:

$$\frac{\partial^2 u}{\partial z \partial w} - \frac{\beta}{z-w}\frac{\partial u}{\partial z} + \frac{\beta}{z-w}\frac{\partial u}{\partial w} = \frac{(\tilde{q}-q)\,u}{4\sqrt{zw}} - \frac{u}{\sqrt{z}\,(z-w)}$$

$$\frac{\partial u}{\partial z} + \frac{\beta}{z}u = \frac{1}{4}\left[\tilde{q}\left(\sqrt{z}\right) - q\left(\sqrt{z}\right)\right] z^{\nu-1}, \quad u\,(z, z-\delta) = 0.$$

This problem can be solved by using the Riemann method [4 – 6].

We put

$$c_n = \int\limits_0^\pi \varphi^2\,(r,\lambda_n)\,dr, \qquad \tilde{c}_n = \int\limits_0^\pi \tilde{\varphi}^2\left(r,\tilde{\lambda}_n\right) dr,$$

$$\rho(\lambda) = \sum_{\lambda_n<\lambda} \frac{1}{c_n}, \qquad \tilde{\rho}(\lambda) = \sum_{\tilde{\lambda}_n<\lambda} \frac{1}{\tilde{c}_n}.$$

The function $\rho(\lambda)$ $(\tilde{\rho}(\lambda))$ is called the spectral function of problem (4) – (6) ((7), (8)). Problem (4) – (6) will be regarded as an unperturbed problem, while (7), (8) will be considered to be a perturbation of (4) – (6).

It is a known [7] fact that the knowledge of two spectra for a given singular Sturm – Liouville equation makes it possible to recover its spectral function, i.e., to find numbers $\{c_n\}$. More exactly, suppose that, in addition to the spectrum of problem (4) – (6), we also know the spectrum $\{\mu_n\}$ of the problem

$$-y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + q(r)\right] y = \lambda y$$
$$y(0) = 0, y'(\pi) + H_1 y(\pi) = 0, \quad H_1 \neq H. \tag{13}$$

Knowing $\{\lambda_n\}$ and $\{\mu_n\}$, we can calculate the numbers $\{c_n\}$. Similarly, for (7), if besides $\{\tilde{\lambda}_n\}$ we also know the spectrum $\{\tilde{\mu}_n\}$ determined by the boundary conditions

$$y\,(0) = 0, \quad y'(\pi) + \tilde{H}_1 y(\pi) = 0, \quad \tilde{H}_1 \neq \tilde{H}, \tag{14}$$

it then follows that we can determine the numbers $\{\tilde{c}_n\}$.

It is also shown that

$$\sqrt{\lambda_n} = \left[n + \frac{\ell}{2}\right] + \frac{1}{\pi}\frac{\ln(n + \ell/2)}{n + \ell/2} + O\left(\frac{1}{n^2}\right),$$

$$\|\varphi_n\|^2 = \int_0^\pi \varphi_n^2(r)dr = \frac{\pi}{2} + \frac{\pi^2}{2}\frac{1}{n + \ell/2} + O\left(\frac{\ln n}{n^2}\right).$$

Theorem 1. *Consider the operator*

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + q(r)\right] y, \tag{15}$$

subject to boundary conditions

$$y\,(0) = 0, \tag{16}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{17}$$

where q is square integrable on $(0, \pi]$. Let $\{\lambda_n\}$ be the spectrum of L subject to (16) *and* (17).

If (17) is replaced by a new boundary condition

$$y'(\pi) + H_1 y(\pi) = 0, \tag{18}$$

a new operator and a new spectrum, say $\{\mu_n\}$, result.

Consider now a second operator

$$\tilde{L}y = -y'' + \left[\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + \tilde{q}(r)\right] y, \tag{19}$$

where $\tilde{q}$ is square integrable on $(0, \pi]$. Suppose that $\tilde{L}$ has the spectrum $\left\{\tilde{\lambda}_n\right\}$ with $\tilde{\lambda}_n = \lambda_n$ for all n under the boundary conditions (16) and

$$y'(\pi) + \tilde{H}y(\pi) = 0, \tag{20}$$

$\tilde{L}$ with the boundary conditions (16) and

$$y'(\pi) + \tilde{H}_1 y(\pi) = 0 \tag{21}$$

is assumed to have the spectrum $\{\tilde{\mu}_n\}$. We assume that H, $H_1 \neq H$, $\tilde{H}$ and $\tilde{H}_1 \neq \tilde{H}$ are real numbers which are not infinite.

We shall denote by Λ_0 the finite index set for which $\tilde{\mu}_n \neq \mu_n$ and by Λ the infinite index set for which $\tilde{\mu}_n = \mu_n$. Under the above assumptions, it follows that the kernel $K(r,s)$ is degenerate in the extended sense:

$$K(r,s) = \sum_{\Lambda_0} c_n \tilde{\phi}_n(r)\varphi_n(s), \tag{22}$$

where φ_n, $\tilde{\phi}_n$ are suitable solutions of (4) and (7).

Proof. It follows from (9) that

$$\tilde{\varphi}'(r,\lambda) = \varphi'(r,\lambda) + K(r,r)\varphi(r,\lambda) + \int_0^r \frac{\partial K}{\partial r}\varphi(s,\lambda)\,ds \tag{23}$$

and

$$\tilde{\varphi}'(r,\lambda) + \tilde{H}\tilde{\varphi}(r,\lambda) =$$
$$= \varphi'(r,\lambda) + \tilde{H}\varphi(r,\lambda) + K(r,r)\varphi(r,\lambda) + \int_0^r \left(\frac{\partial K}{\partial r} + \tilde{H}K\right)\varphi(s,\lambda)\,ds.$$

Substituting $r = \pi$, $\lambda = \lambda_n$ into the last equation and using boundary conditions (17), (20), we obtain

$$\left(\tilde{H} - H\right)\varphi(\pi,\lambda_n) + K(\pi,\pi)\varphi(\pi,\lambda_n) +$$
$$+ \int_0^\pi \left(\frac{\partial K}{\partial r} + \tilde{H}K\right)_{r=\pi} \varphi(s,\lambda_n)\,ds = 0. \tag{24}$$

As $n \to \infty$ and $\varphi(\pi,\lambda_n) \to o(1)$, the integral on the right-hand side tends to zero. Therefore, from (24) we get

$$K(\pi,\pi) = H - \tilde{H}, \tag{25}$$

$$\int_0^\pi \left(\frac{\partial K}{\partial r} + \tilde{H}K\right)_{r=\pi} \varphi(s,\lambda_n)\,ds = 0, \quad n = 0,1,\ldots. \tag{26}$$

Since the system of functions $\varphi(s,\lambda_n)$ is complete, it follows from the last equation that

$$\left(\frac{\partial K}{\partial r} + \tilde{H}K\right)_{r=\pi} = 0, \quad 0 < s \leq \pi. \tag{27}$$

We now use the condition imposed on the second-mentioned spectrum. Using (9) again, we obtain

$$\tilde{\varphi}'(r,\lambda)+\tilde{H}_1\tilde{\varphi}(r,\lambda)=\varphi'(r,\lambda)+\tilde{H}_1\varphi(r,\lambda)+K(r,r)\varphi(r,\lambda)+ \\ +\int_0^r\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)\varphi(s,\lambda)\,ds. \tag{28}$$

Putting $r=\pi$ and $\lambda=\mu_n$ $(n\in\Lambda)$ and using (18), (21), we obtain

$$\int_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)_{r=\pi}\varphi(s,\mu_n)\,ds+\left(\tilde{H}_1-H_1\right)\varphi(\pi,\mu_n)+ \\ +K(\pi,\pi)\varphi(\pi,\mu_n)=0.$$

In the last equation, as $n\to\infty$, the left-hand side tends to zero and $\varphi(\pi,\mu_n)\to o(1)$. Therefore,

$$K(\pi,\pi)=H_1-\tilde{H}_1, \tag{29}$$

$$\int_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)_{r=\pi}\varphi(s,\mu_n)\,ds=0,\quad n\in\Lambda. \tag{30}$$

Comparing (25) and (29), we obtain $H-\tilde{H}=H_1-\tilde{H}_1$. For $n\in\Lambda_0$, we obtain from (28) (for $r=\pi$ and $\lambda=\mu_n$)

$$\int_0^\pi\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)_{r=\pi}\varphi(s,\mu_n)\,ds=\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n). \tag{31}$$

It follows from (30) and (31) that

$$\left(\frac{\partial K}{\partial r}+\tilde{H}_1K\right)_{r=\pi}=\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}\varphi(s,\mu_n),\quad 0<s\le\pi. \tag{32}$$

We derive from (27) and (32) the following equations:

$$K(\pi,s)=\frac{1}{\tilde{H}_1-\tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}\varphi(s,\mu_n), \tag{33}$$

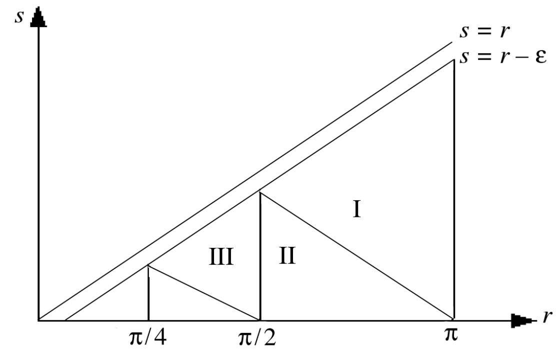
$$\left.\frac{\partial K(r,s)}{\partial r}\right|_{r=\pi}=-\frac{\tilde{H}}{\tilde{H}_1-\tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n)+\tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}\varphi(s,\mu_n), \tag{34}$$
$$0<s\le\pi.$$

The function $K(r,s)$ satisfies (10). Therefore, it follows from the initial conditions (33) and (34) that, in the triangle I (see Figure), we have

$$K(r,s)=\frac{1}{\tilde H_1-\tilde H}\sum_{\Lambda_0}\frac{\tilde\varphi'(\pi,\mu_n)+\tilde H_1\tilde\varphi(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}\times$$
$$\times\left[\tilde c(r,\mu_n)-\tilde H\tilde s(r,\mu_n)\right]\varphi(s,\mu_n), \tag{35}$$

where $\tilde c(r,\lambda)$ and $\tilde s(r,\lambda)$ are solutions of (7) satisfying the initial conditions

$$\tilde c(\pi,\lambda)=\tilde s'(\pi,\lambda)=1,\qquad \tilde c'(\pi,\lambda)=\tilde s(\pi,\lambda)=0.$$



The function $K(r,s)$ and the sum (35) satisfy (12); therefore, they coincide in the triangle II; consequently, they coincide in the triangle III as solutions of (10) satisfy the same initial conditions on the line $r=\pi/2$, etc., i.e., $K(r,s)$ is expressed by (35) throughout the triangle $0<s\le r\le\pi$ (see [8 – 10]).

Hence, we obtain Hochstadt's result in a somewhat more general formulation.

Theorem 2. *If the spectra* $\{\lambda_n\}$ *and* $\{\tilde\lambda_n\}$ *coincide and* $\{\mu_n\}$ *and* $\{\tilde\mu_n\}$ *differ in a finite number of their terms, i.e.,* $\tilde\mu_n=\mu_n$ *for* $n\in\Lambda$, *then*

$$\tilde q(r)-q(r)=\sum_{\Lambda_0}\tilde c_n\frac{d}{dr}\big(\tilde\phi_n,\varphi_n\big),$$

where φ_n , $\tilde\phi_n$ *are suitable solutions of (4) and* (7).

Proof. We obtain from (11) the equation

$$\tilde q(r)-q(r)=2\frac{dK(r,r)}{dr}.$$

Differentiating (35) and putting $s=r$, we obtain

$$\tilde q(r)-q(r)=\frac{2}{\tilde H_1-\tilde H}\sum_{\Lambda_0}\frac{\tilde\varphi'(\pi,\mu_n)+\tilde H_1\tilde\varphi(\pi,\mu_n)}{\|\varphi(s,\mu_n)\|^2}\times$$
$$\times\frac{d}{dr}\left\{\left[\tilde c(r,\mu_n)-\tilde H\tilde s(r,\mu_n)\right]\varphi(\pi,\mu_n)\right\}.$$

Consequently,

$$\tilde q(r)-q(r)=\sum_{\Lambda_0}\tilde c_n\frac{d}{dr}\left(\tilde\phi_n\varphi_n\right),$$

where $\tilde c(r,\mu_n)-\tilde H\tilde s(r,\mu_n)=\tilde\phi_n$, $\varphi(r,\mu_n)=\varphi_n(r,\mu_n)$, and

$$\hat{c}_n = \frac{2\left[\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)\right]}{\left(\tilde{H}_1 - \tilde{H}\right)\|\varphi(s,\mu_n)\|^2}.$$

This completes the proof of Theorem 2. We note that similar problems are investigated in [11 – 14].

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Received 16.05.2005