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V. A. Mikhailets, V. M. Molyboga (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

SINGULARLY PERTURBED PERIODIC AND SEMIPERIODIC DIFFERENTIAL OPERATORS *

СИНГУЛЯРНО ЗБУРЕНІ ПЕРІОДИЧНІ ТА НАПІВПЕРІОДИЧНІ ДИФЕРЕНЦІАЛЬНІ ОПЕРАТОРИ

Qualitative and spectral properties of the form-sums

$$S_{\pm}(V) := D_{\pm}^{2m} \dot{+} V(x), \quad m \in \mathbb{N},$$

in the Hilbert space $L_2(0,1)$ are studied. Here, (D_+) is the periodic differential operator, (D_-) is the semiperiodic differential operator, $D_{\pm}\colon u \mapsto -iu'$, and $V(x)$ is a 1-periodic complex-valued distribution in the Sobolev spaces $H_{\text{per}}^{-m\alpha}$, $\alpha \in [0,1]$.

Досліджено якісні та спектральні властивості форм-сум

$$S_{\pm}(V) := D_{\pm}^{2m} \dot{+} V(x), \quad m \in \mathbb{N},$$

у гільбертовому просторі $L_2(0,1)$. Тут (D_+) та (D_-) — періодичний та напівперіодичний диференціальні оператори, $D_{\pm}\colon u \mapsto -iu'$, а $V(x)$ — довільна 1-періодична комплекснозначна узагальнена функція з просторів Соболева $H_{\text{per}}^{-m\alpha}$, $\alpha \in [0,1]$.

1. Introduction and statement of results. In this paper, we study the operators $S_+(V)$ and $S_-(V)$ that are not selfadjoint in general and given on the Hilbert space $L_2(0,1)$ by two-terms differential expressions of an even order, with a 1-periodic complex-valued potential $V(x)$, which is a distribution in $\mathcal{D}'_1$, and periodic and semiperiodic boundary conditions,

$$S_{\pm}u \equiv S_{\pm}(V)u := D_{\pm}^{2m}u + V(x)u,$$

$$D_{\pm} := -i\frac{d}{dx}, \qquad \mathrm{Dom}(D_{\pm}) = H_{\pm}^1, \qquad D_{\pm}^{2m} := |D_{\pm}|^{2m},$$

$$\mathrm{Dom}(D_{\pm}^{2m}) = H_{\pm}^{2m}, \quad m \in \mathbb{N},$$

$$V(x) = \sum_{k\in\mathbb{Z}} \widehat{V}(2k)e^{i\,2k\pi x} \in \mathcal{D}'_1,$$

$$u \in \mathrm{Dom}(S_{\pm}).$$

Here by the $H_{\pm}^1 \equiv H_{\pm}^1[0,1]$ and $H_{\pm}^{2m} \equiv H_{\pm}^{2m}[0,1]$ we denote the Sobolev spaces of functions that are 1-periodic and 1-semiperiodic on the interval $[0,1]$, and $\mathcal{D}'_1$ denotes the space of 1-periodic distributions [1, p. 115].

In this paper, we give sufficient conditions for the operators $S_{\pm}(V)$ to exist as form-sums, conduct a detailed study of their *qualitative* properties, prove theorems about their *approximation* and *spectrum decomposition*. The approximation theorem gives another definition of the operators $S_{\pm}(V)$ as a limit, in the generalized convergence sense [2] (Ch. IV, § 2.6), of a sequence of operators with smooth potentials.

Earlier in [3 – 5], the authors have carried out a detailed study of the differential operators $L_{\pm}(V)$ generated on the finite interval by the same differential expressions as the operators $S_{\pm}(V)$ but defined on the *negative* Sobolev spaces $H_{\pm}^{-m}$. The case $m = 1$ for operators $L_{\pm}(V)$ was treated in [6, 7] (see also closely related papers [8 – 11]).

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So, for an arbitrary $s \in \mathbb{R}$, the Sobolev spaces of 1-periodic and 1-semiperiodic functions or distributions are defined in a natural fashion by means of their Fourier coefficients,

$$H_+^s \equiv H_+^s[0,1] := \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(2k) e^{i2k\pi x} \,\Big|\, \| f \|_{H_+^s} < \infty \right\},$$

$$\| f \|_{H_+^s} := \left(\sum_{k\in\mathbb{Z}} \langle 2k \rangle^{2s} \left| \widehat{f}(2k) \right|^2 \right)^{1/2}, \quad \langle k \rangle := 1 + |k|,$$

$$\widehat{f}(2k) := \left\langle f, e^{i2k\pi x} \right\rangle_+, \quad k \in \mathbb{Z},$$

and

$$H_-^s \equiv H_-^s[0,1] := \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(2k+1) e^{i(2k+1)\pi x} \,\Big|\, \| f \|_{H_-^s} < \infty \right\},$$

$$\| f \|_{H_-^s} := \left(\sum_{k\in\mathbb{Z}} \langle 2k+1 \rangle^{2s} \left| \widehat{f}(2k+1) \right|^2 \right)^{1/2}, \quad \langle k \rangle = 1 + |k|,$$

$$\widehat{f}(2k+1) := \left\langle f, e^{i(2k+1)\pi x} \right\rangle_-, \quad k \in \mathbb{Z}.$$

By $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_-$ we denote the sesquilinear forms that define the pairing between the dual spaces $H_\pm^s$ and $H_\pm^{-s}$ with respect to the zero space $L_2(0,1)$; these pairings are obtained by extending the inner product in $L_2(0,1)$ by continuity [12, p. 47],

$$(f, g) = \int_0^1 f(x) \overline{g(x)} \, dx, \quad f, g \in L_2(0,1).$$

It will be useful to notice that the two-sided scales of Sobolev spaces $\{H_\pm^s\}_{s\in\mathbb{R}}$ coincide up to equivalent norms with scales generated by powers of the non-negative selfadjoint operators $|D_\pm|$ [13] (Ch. II, § 2.1).

The Sobolev spaces

$$H_{\mathrm{per}}^s \equiv H_{\mathrm{per}}^s[-1,1] := \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik\pi x} \,\Big|\, \| f \|_{H_{\mathrm{per}}^s} < \infty \right\}, \quad s \in \mathbb{R},$$

of 2-periodic elements (functions or distributions) are defined in a similar way.

Now, we are ready to formulate the main results obtained in the paper. But first recall that an operator A on a Hilbert space is said to be *m-sectorial* if its numerical range $\Theta(A)$, i.e., the set

$$\Theta(A) := (Au, u), \quad u \in \mathrm{Dom}(A), \quad \|u\| = 1,$$

is contained in a sector of the complex plane,

$$\Theta(A) \subseteq \mathrm{Sect}(\gamma, \theta),$$

$$\mathrm{Sect}(\gamma, \theta) := \left\{ \lambda \in \mathbb{C} \mid \left| \arg(\lambda - \gamma) \right| \leq \theta \right\}, \quad 0 \leq \theta < \frac{\pi}{2},$$

and the exterior of the sector $\mathrm{Sect}(\gamma, \theta)$ belongs to the resolvent set $\mathrm{Resol}(A)$ of the operator A [2] (Ch. V, § 3.10).

Theorem 1. *Let a* 1*-periodic complex-valued distribution V(x) be in the space* H_+^{-m}*. Then the operators* $S_\pm(V)$ *are well defined on the Hilbert space* $L_2(0,1)$ *as* m*-sectorial operators — form-sums,*

$$S_\pm(V) = D_\pm^{2m} \dotplus V(x),$$

associated with densely defined, closed, sectorial sesquilinear forms defined on $L_2(0,1)$ *by*

$$t_{S_\pm}[u,v] \equiv t_\pm[u,v] := \left\langle D_\pm^{2m}u, v\right\rangle_\pm + \left\langle V(x)u, v\right\rangle_\pm, \quad \mathrm{Dom}(t_{S_\pm}) = H_\pm^m,$$

and act on the dense domains

$$\mathrm{Dom}(S_\pm) = \left\{u \in H_\pm^m \;\middle|\; D_\pm^{2m}u + V(x)u \in L_2(0,1)\right\}$$

as

$$S_\pm(V)u = D_\pm^{2m}u + V(x)u, \quad u \in \mathrm{Dom}(S_\pm).$$

Let us remark that, in virtue of the convolution lemma (see Lemma 1 below), a 1-periodic complex-valued distribution $V(x) \in H_+^{-m}$ defines, on the Hilbert space $L_2(0,1)$, *two* sesquilinear forms,

$$t_V^+[u,v] := \left\langle V(x)\cdot u, v\right\rangle_+, \quad u,v \in H_+^m,$$

$$t_V^-[u,v] := \left\langle V(x)\cdot u, v\right\rangle_-, \quad u,v \in H_-^m,$$

where $V(x)\cdot u$ denotes the formal product, which converges in the Sobolev spaces $H_\pm^{-m}$, of the Fourier series of the distribution $V(x) \in H_+^{-m}$ and the function $u \in H_\pm^m$.

If the distribution $V(x)$ has additional smoothness in the scale $\{H_\pm^s\}_{s\in\mathbb{R}}$ of the Hilbert spaces, then functions in the domains of the operators $S_\pm(V)$ have an additional regularity.

Theorem 2. *Let* $V(x) \in H_+^{-m\alpha}$, $\alpha \in [0,1]$. *Then the inclusion*

$$\mathrm{Dom}(S_\pm) \subseteq H_\pm^{m(2-\alpha)}$$

holds.

In the case $\alpha \neq 0$, i.e., for

$$V(x) \in H_+^{-m\alpha}, \quad \alpha \in (0,1],$$

the question about locality of the operators $S_\pm(V)$ is meaningful. Let us recall that an operator A on a function space is called *local* if

$$\mathrm{supp}(Au) \subseteq \mathrm{supp}(u), \quad u \in \mathrm{Dom}(A).$$

For the Hilbert space $L_2(0,1)$, this is equivalent to the following:

$$u\,\big|_{(\alpha,\beta)} = 0 \Rightarrow Au\,\big|_{(\alpha,\beta)} = 0, \qquad u \in \mathrm{Dom}(A), \quad (\alpha,\beta) \subset [0,1].$$

Theorem 3. *If* $V(x) \in H_+^{-m}$, *the operators* $S_+(V)$ *and* $S_-(V)$ *are local.*

The following theorem describes qualitative properties of the operators $S_\pm(V)$.

Theorem 4. *Let a* 1*-periodic complex-valued distribution* $V(x)$ *be in the space* H_{+}^{-m}.

a) *The operators* $S_{\pm}(V)$ *are* m*-sectorial with respect to an arbitrary angle containing the positive half-axis.*

b) *The operators* $S_{\pm}(V)$ *are selfadjoint if and only if the distribution* $V(x)$ *is real-valued, i.e., if*

$$\widehat{V}(2k) = \overline{\widehat{V}(-2k)}, \quad k \in \mathbb{Z}.$$

c) *The operators* $S_{\pm}(V)$ *have discrete spectra.*

The following theorem allows to give another alternative definition of the operators $S_{\pm}(V)$ described in Theorem 1.

Theorem 5. *Let* $V_n(x)$, $n \in \mathbb{N}$, *and* $V(x)$ *be defined on the space* H_{+}^{-m}, *and suppose that*

$$V_n(x) \xrightarrow{H_{+}^{-m}} V(x), \quad n \to \infty.$$

Then the operators $S_{\pm}^{(n)} \equiv S_{\pm}(V_n)$ *converge to the operators* $S_{\pm} \equiv S_{\pm}(V)$ *in the uniform resolvent convergent sense,*

$$\left\| R(\lambda, S_{\pm}^{(n)}) - R(\lambda, S_{\pm}) \right\| \to 0, \quad n \to \infty.$$

So, by virtue of Theorem 5, the operators $S_{\pm}(V)$ can be defined as a limit of a sequence of the operators $S_{\pm}^{(n)}$ with smooth potentials $V_n(x)$ in the generalized convergence sense [2] (Ch. IV, § 2.6).

As an example, consider

$$V(x) = \sum_{k \in \mathbb{Z}} \widehat{V}(2k) e^{i\,2k\pi x} \in H_{+}^{-m},$$

the trigonometric polynomials

$$V_n(x) = \sum_{|k| \leq n} \widehat{V}(2k) e^{i\,2k\pi x} \in H_{+}^{\infty},$$

form the necessary sequence,

$$V_n(x) \xrightarrow{H_{+}^{-m}} V(x), \quad n \to \infty,$$

which yields the convergence

$$\left\| R(\lambda, S_{\pm}^{(n)}) - R(\lambda, S_{\pm}) \right\| \to 0, \quad n \to \infty.$$

Due to Theorem 5 we also have that

$$\sigma(S_{\pm}^{(n)}) \to \sigma(S_{\pm}), \quad n \to \infty,$$

where the convergence of spectra is upper semicontinuous in general [2] (Ch. IV, § 3.1) and, for real-valued potentials, it is continuous [14] (Theorems VIII.23 and VIII.24); by $\sigma(S_{\pm}^{(n)})$ and $\sigma(S_{\pm})$ we denote unordered spectra of the corresponding operators.

Now, let us consider, on the Hilbert space $L_2(-1,1)$, the m-sectorial operators — form-sums $S(V)$ with 1-periodic complex-valued potentials that are distributions $V(x) \in \in H_{\text{per}}^{-m}$, i.e., $\widehat{V}(2k+1) = 0 \ \forall k \in \mathbb{Z}$,

$$S \equiv S(V) := D^{2m} \dotplus V(x),$$

$$D := -i\frac{d}{dx}, \qquad \mathrm{Dom}(D) = H^1_{\mathrm{per}}, \qquad D^{2m} := |D|^{2m}, \qquad \mathrm{Dom}(D^{2m}) = H^{2m}_{\mathrm{per}},$$

$$V(x) = \sum_{k\in\mathbb{Z}} \widehat{V}(2k)e^{i\,2k\pi x} \in H^{-m}_{\mathrm{per}},$$

$$\mathrm{Dom}(S) = \left\{ u \in H^m_{\mathrm{per}} \;\middle|\; D^{2m}u + V(x)u \in L_2(-1,1) \right\}, \quad m \in \mathbb{N}.$$

Analogs of Theorems 2–4 and 5 hold for the operators $S(V)$. In particular, they have discrete spectra.

Let us study the structure of spectra of the operators $S(V)$, $S_+(V)$, and $S_-(V)$ in more details.

Denote by $\mathrm{spec}(A)$ the discrete spectrum of the operator A, taking into account the algebraic multiplicity of the eigenvalues that ordered lexicographically. Namely, we will say that an eigenvalue λ_k precedes an eigenvalue λ_{k+1} for $k \in \mathbb{Z}_+$ if

$$\mathrm{Re}\,\lambda_k < \mathrm{Re}\,\lambda_{k+1}, \quad \text{or} \quad \mathrm{Re}\,\lambda_k = \mathrm{Re}\,\lambda_{k+1} \quad \text{and} \quad \mathrm{Im}\,\lambda_k \le \mathrm{Im}\,\lambda_{k+1}, \quad k \in \mathbb{Z}_+.$$

It is easy to see that

$$\mathrm{spec}(D^{2m}) =$$
$$\left\{0; 1, 1; 2^{2m}, 2^{2m}; \ldots; (2k-1)^{2m}, (2k-1)^{2m}; (2k)^{2m}, (2k)^{2m}; \ldots \right\} \cdot \pi^{2m},$$

$$\mathrm{spec}(D^{2m}_+) = \left\{0; 2^{2m}, 2^{2m}; \ldots; (2k)^{2m}, (2k)^{2m}; \ldots \right\} \cdot \pi^{2m},$$

$$\mathrm{spec}(D^{2m}_-) = \left\{1, 1; 3^{2m}, 3^{2m}; \ldots; (2k-1)^{2m}, (2k-1)^{2m}; \ldots \right\} \cdot \pi^{2m}.$$

And thus we get

$$\mathrm{spec}(D^{2m}) = \mathrm{spec}(D^{2m}_+) \sqcup \mathrm{spec}(D^{2m}_-) \quad \text{(the disjoint sum)}.$$

The following theorem about spectra decomposition is a non-trivial generalization of the last equality for the perturbed m-sectorial operators — form-sums $S(V)$, $S_+(V)$, and $S_-(V)$.

Theorem 6. *Let $S(V)$, $S_+(V)$ and $S_-(V)$ be the m-sectorial operators, where the potential $V(x)$ is a 1-periodic complex-valued distribution from the Sobolev spaces H^{-m}_{per} and H^{-m}_+ for the first and the second two operators, respectively. Then*

$$S(V) = S_+(V) \oplus S_-(V),$$

and we have the decomposition

$$\mathrm{spec}(S) = \mathrm{spec}(S_+) \cup \mathrm{spec}(S_-).$$

A part of results are announced in [15] and contained in [16].

2. The proofs. At first, we will recall some known facts and results that will be necessary.

Consider the Hilbert spaces of two-sided weighted sequences,

$$h^s \equiv h^s(\mathbb{Z}; \mathbb{C}), \quad s \in \mathbb{R},$$

$$h^s := \left\{ a = (a(k))_{k\in\mathbb{Z}} \mid \|a\|_{h^s} < \infty \right\},$$

$$(a,b)_{h^s} := \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} a(k)\,\overline{b(k)}, \quad \langle k\rangle = 1 + |k|,$$

$$\|a\|_{h^s} := \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |a(k)|^2 \right)^{1/2}.$$

The Fourier transform establishes an isometric isomorphisms between the Sobolev spaces H^s_{per}, $H^s_{\pm}$ and the Hilbert spaces h^s of two-sided weighted sequences,

$$\mathcal{F}\colon H^s_{\text{per}} \ni f \mapsto (\widehat{f}) = \left(\widehat{f}(k)\right)_{k\in\mathbb{Z}} \in h^s,$$

$$\mathcal{F}_+\colon H^s_+ \ni f \mapsto (\widehat{f}) = \left(\widehat{f}(2k)\right)_{k\in\mathbb{Z}} \in h^s,$$

$$\mathcal{F}_-\colon H^s_- \ni f \mapsto (\widehat{f}) = \left(\widehat{f}(2k+1)\right)_{k\in\mathbb{Z}} \in h^s.$$

This, together with the convolution lemma (see bellow), allows to give sufficient conditions of existence of the formal product

$$V(x)\cdot u(x) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widehat{V}(k-j)\widehat{u}(j)\, e^{i\,k\pi x}.$$

To this end, introduce in the scale of the Hilbert spaces of two-sided weighted sequences $\{h^s\}_{s\in\mathbb{R}}$ a commutative convolution operation. For arbitrary sequences

$$a = (a(k))_{k\in\mathbb{Z}} \quad \text{and} \quad b = (b(k))_{k\in\mathbb{Z}},$$

it is defined in a natural fashion,

$$(a,b) \mapsto a * b,$$

$$(a*b)(k) := \sum_{j\in\mathbb{Z}} a(k-j)b(j).$$

The following known lemma (see, for example, [7], Lemma 1.5.4) holds.

Lemma 1 (Convolution lemma). *Let* $s, r \geq 0$, *and* $t \in \mathbb{R}$ *with* $t \leq \min(s,r)$.

(I) *If* $s + r - t > 1/2$, *then the convolution map*

$$(a,b) \mapsto a * b$$

is continuous when viewed as the maps

(a) $h^r \times h^s \to h^t$,

(b) $h^{-t} \times h^s \to h^{-r}$.

(II) *If* $s + r - t < 1/2$, *then this statement fails to hold.*

***2.1. Proof of Theorem* 1.** Due to the convolution lemma for

$$V(x) \in H^{-m}_+ \quad \text{and} \quad u(x) \in H^m_{\pm},$$

the products $V(x)\cdot u(x)$ are well defined in the Sobolev spaces $H^{-m}_{\pm}$. Therefore, the sesquilinear forms

$$t^+_V[u,v] = \langle V(x)u, v\rangle_+, \qquad \text{Dom}(t^+_V) = H^m_+,$$

$$t_V^-[u,v] = \langle V(x)u, v\rangle_-, \qquad \mathrm{Dom}(t_V^-) = H_-^m,$$

are well defined in the Hilbert space $L_2(0,1)$.

Further, set

$$\tau_+[u,v] := \langle D_+^{2m}u, v\rangle_+, \qquad \mathrm{Dom}(\tau_+) = H_+^m,$$

$$\tau_-[u,v] := \langle D_-^{2m}u, v\rangle_-, \qquad \mathrm{Dom}(\tau_-) = H_-^m.$$

The sesquilinear forms $\tau_\pm[u,v]$ are well defined in the Hilbert space $L_2(0,1)$, they are densely defined, closed, and nonnegative.

The following assertion is true.

Proposition 1. *The sesquilinear forms $t_V^\pm[u,v]$ are $\tau_\pm$-bounded with $\tau_\pm$-boundary that equals zero, i.e., we have $V(x) \prec\prec D_\pm^{2m}$.*

Proof. Represent the 1-periodic distribution

$$V(x) = \sum_{k\in\mathbb{Z}} \widehat{V}(2k)e^{i\,2k\pi x} \in H_+^{-m}$$

as the sum

$$V(x) = V_0(x) + V_\delta(x), \tag{1}$$

where $V_0(x)$ is a smooth function and $V_\delta(x)$ is a distribution with an arbitrarily small norm,

$$V_0(x) \in H_+^m,$$

and

$$V_\delta(x) \in H_+^{-m} \quad \text{with} \quad \|V_\delta\|_{H_+^{-m}} \le \frac{\delta}{C_m}.$$

The constant C_m is defined from the convolution lemma and is fixed. The decomposition (1) is possible, since H_+^m is densely embedded into the space H_+^{-m}.

So, for

$$u \in \mathrm{Dom}(\tau_\pm) \subset \mathrm{Dom}(t_V^\pm),$$

we have

$$\left|t_V^\pm[u]\right| = |\langle V(x)u,u\rangle_\pm| \le |\langle V_0(x)u,u\rangle_\pm| + |\langle V_\delta(x)u,u\rangle_\pm| \le$$

$$\le \|V_0(x)u\|_{L_2(0,1)}\|u\|_{L_2(0,1)} + \|V_\delta(x)u\|_{H_\pm^{-m}}\|u\|_{H_\pm^m} \le$$

$$\le C_m\|V_0(x)\|_{H_+^m}\|u\|^2_{L_2(0,1)} + \delta\|u\|^2_{H_\pm^m}.$$

Taking into account that

$$\|u\|^2_{H_\pm^m} \le \|u\|^2_{L_2(0,1)} + \|u^{(m)}\|^2_{L_2(0,1)} = \|u\|^2_{L_2(0,1)} + \langle D_\pm^{2m}u,u\rangle_\pm$$

for an arbitrary $\delta > 0$ we obtain the necessary estimate,

$$\left|t_V^\pm[u]\right| \le \delta\tau_\pm[u] + \left(C_m\|V_0\|_{H_+^m[0,1]} + \delta\right)\|u\|^2_{L_2(0,1)}. \tag{2}$$

The proof is complete.

Proposition 1, together with [2] (Theorem IV.1.33) yields the following corollary.

Corollary. *The sesquilinear forms*

$$t_{\pm}[u,v] := \langle D_{\pm}^{2m}u, v\rangle_{\pm} + \langle V(x)u, v\rangle_{\pm}, \quad \mathrm{Dom}(t_{\pm}) = H_{\pm}^{m},$$

are densely defined, closed, and sectorial in the Hilbert space $L_2(0,1)$.

According to the first representation theorem [2] (Theorem VI.2.1), there exist m-sectorial operators $S_{\pm}(V)$ associated with the forms $t_{\pm}[u,v]$ such that

i) $\mathrm{Dom}(S_{\pm}) \subseteq \mathrm{Dom}(t_{\pm})$ and

$$t_{\pm}[u,v] = (S_{\pm}u, v)$$

for every $u \in \mathrm{Dom}(S_{\pm})$ and $v \in \mathrm{Dom}(t_{\pm})$;

ii) $\mathrm{Dom}(S_{\pm})$ are cores of $t_{\pm}[u,v]$;

iii) if $u \in \mathrm{Dom}(t_{\pm})$, $w \in L_2(0,1)$, and

$$t_{\pm}[u,v] = (w, v)$$

holds for every v belonging to the cores of $t_{\pm}[u,v]$, then $u \in \mathrm{Dom}(S_{\pm})$ and $S_{\pm}(V)u = w$.

The m-sectorial operators $S_{\pm}(V)$ are uniquely defined by condition i).

Now, investigate the operators $S_{\pm}(V)$ associated with the forms $t_{\pm}[u,v]$ in more details.

Let

$$u \in \mathrm{Dom}(S_{\pm}) \quad \text{and} \quad v \in \mathrm{Dom}(t_{\pm}).$$

Then we have

$$t_{\pm}[u,v] = \langle D_{\pm}^{2m}u, v\rangle_{\pm} + \langle V(x)u, v\rangle_{\pm} = \langle D_{\pm}^{2m}u + V(x)u, v\rangle_{\pm} =$$
$$= (S_{\pm}u, v) = \langle S_{\pm}u, v\rangle_{\pm}.$$

This shows that we have the equality

$$\langle D_{\pm}^{2m}u + V(x)u, v\rangle_{\pm} = \langle S_{\pm}u, v\rangle_{\pm}, \quad v \in H_{\pm}^{m},$$

of linear forms. So, we can conclude that

$$S_{\pm}(V)u = D_{\pm}^{2m}u + V(x)u \in L_2(0,1), \quad u \in \mathrm{Dom}(S_{\pm}),$$

and that the inclusions

$$\mathrm{Dom}(S_{\pm}) \subseteq \left\{u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V(x)u \in L_2(0,1)\right\}$$

hold. It remains to verify that the inverse inclusions hold.

Let

$$u \in \left\{u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V(x)u \in L_2(0,1)\right\} \quad \text{and} \quad v \in \mathrm{Dom}(t_{\pm}).$$

Then

$$t_{\pm}[u,v] = \langle D_{\pm}^{2m}u, v\rangle_{\pm} + \langle V(x)u, v\rangle_{\pm} =$$
$$= \langle D_{\pm}^{2m}u + V(x)u, v\rangle_{\pm} = \left(D_{\pm}^{2m}u + V(x)u, v\right),$$

and using the first representation theorem iii) (see above) we get the necessary estimate,

$$u \in \mathrm{Dom}(S_{\pm}),$$

which implies that

$$\left\{ u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V(x)u \in L_2(0,1) \right\} \subseteq \mathrm{Dom}(S_{\pm})$$

and

$$S_{\pm}(V)u = D_{\pm}^{2m}u + V(x)u \in L_2(0,1).$$

So,

$$\mathrm{Dom}(S_{\pm}) = \left\{ u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V(x)u \in L_2(0,1) \right\}$$

and

$$S_{\pm}(V)u = D_{\pm}^{2m}u + V(x)u \in L_2(0,1), \qquad u \in \mathrm{Dom}(S_{\pm}).$$

Theorem 1 is proved completely.

***Remark* 1.** Throughout the rest of the paper we will often use the notations

$$t_{S_{\pm}}[u,v] \equiv t_{\pm}[u,v]$$

to underline the dual relations between the sesquilinear forms $t_{\pm}[u,v]$ and the associated with them operators $S_{\pm}(V)$, see [2] ([Theorem VI.2.7).

***2.2. Proof of Theorem* 2.** Let the 1-periodic distribution $V(x)$ belong to the space $H_{+}^{-m\alpha}$, $\alpha \in [0,1]$. Then for any $u \in \mathrm{Dom}(S_{\pm})$, due to the convolution lemma, we have

$$V(x)u \in H_{+}^{-m\alpha},$$

and therefore

$$D_{\pm}^{2m}u \in H_{\pm}^{-m\alpha}.$$

From this we conclude that

$$u \in H_{\pm}^{m(2-\alpha)}.$$

***2.3. Proof of Theorem* 3.** Let

$$u \in \mathrm{Dom}(S_{\pm})$$

and

$$u\,\big|_{(\alpha,\beta)} = 0 \quad \text{with} \quad (\alpha,\beta) \subset [0,1],$$

and let

$$\varphi(x) \in C_0^{\infty}[0,1] \quad \text{with} \quad \mathrm{supp}(\varphi) \Subset (\alpha,\beta).$$

Then we have

$$(S_{\pm}u,\varphi) = \langle S_{\pm}u,\varphi\rangle_{\pm} = \langle D_{\pm}^{2m}u + V(x)u,\varphi\rangle_{\pm} = \langle D_{\pm}^{2m}u,\varphi\rangle_{\pm} + \langle V(x)u,\varphi\rangle_{\pm} =$$

$$= \langle u, D_{\pm}^{2m}\varphi\rangle_{\pm} + \langle V(x),\overline{u}\varphi\rangle_{\pm} = \langle V(x),0\rangle_{\pm} = 0,$$

which yields the necessary statement,

$$(S_{\pm}u)\,\big|_{(\alpha,\beta)} = 0.$$

***2.4. Proof of Theorem* 4.** (a) The m-sectoriality of the operators $S_{\pm}(V)$ have been proved in Theorem 1. Let us prove the second part of the assertion, i.e., we need to show that for any $\varepsilon > 0$ and some constant $c_\varepsilon \geq 0$ the following estimates hold:

$$|\arg\left((S_{\pm} + c_\varepsilon Id)u, u\right)| \leq \varepsilon, \quad u \in \mathrm{Dom}(S_{\pm}).$$

For this we have to make sure that

$$|\mathrm{Im}(S_{\pm}u, u)| \leq \varepsilon \mathrm{Re}(S_{\pm}u, u) + c_\varepsilon \|u\|^2_{L_2(0,1)}, \quad u \in \mathrm{Dom}(S_{\pm}),$$

for any $\varepsilon > 0$ and some constant $c_\varepsilon \geq 0$.

So, take $0 < \varepsilon < 1/2$. From Proposition 1 (see (2)) we get

$$\left|t_V^{\pm}[u]\right| \leq \frac{\varepsilon}{2}\tau_{\pm}[u] + \left(C_m \|V_0(x)\|_{H_+^m} + \frac{\varepsilon}{2}\right)\|u\|^2_{L_2(0,1)}, \quad u \in \mathrm{Dom}(\tau_{\pm}),$$

and, hence,

$$-\varepsilon \mathrm{Re}\, t_V^{\pm}[u] \leq \frac{\varepsilon}{2}\tau_{\pm}[u] + \left(C_m \|V_0(x)\|_{H_+^m} + \frac{\varepsilon}{2}\right)\|u\|^2_{L_2(0,1)}, \quad u \in \mathrm{Dom}(\tau_{\pm}).$$

Further, taking into account that

$$\mathrm{Re}(S_{\pm}u, u) = \langle D_{\pm}^{2m}u, u\rangle_{\pm} + \mathrm{Re}\langle V(x)u, u\rangle_{\pm},$$

$$\mathrm{Im}(S_{\pm}u, u) = \mathrm{Im}\langle V(x)u, u\rangle_{\pm}, \quad u \in \mathrm{Dom}(S_{\pm}),$$

we obtain the necessary estimates,

$$|\mathrm{Im}(S_{\pm}u, u)| \leq |\langle V(x)u, u\rangle_{\pm}| \leq \frac{\varepsilon}{2}\tau_{\pm}[u] + \left(C_m \|V_0(x)\|_{H_+^m} + \frac{\varepsilon}{2}\right)\|u\|^2_{L_2(0,1)} \leq$$

$$\leq \varepsilon\left(\tau_{\pm}[u] + \mathrm{Re}\, t_V^{\pm}[u]\right) + \left(2C_m \|V_0(x)\|_{H_+^m} + \varepsilon\right)\|u\|^2_{L_2(0,1)} =$$

$$= \varepsilon \mathrm{Re}(S_{\pm}u, u) + c_\varepsilon \|u\|^2_{L_2(0,1)}, \quad u \in \mathrm{Dom}(S_{\pm}).$$

(b) Let the 1-periodic distribution $V(x)$ be real-valued. Then the sesquilinear forms $t_{S_{\pm}}[u, v]$ are symmetric and, consequently, in virtue of [2] (Theorem VI.2.7) (also see the KLMN theorem [17] (Theorem X.17)), the operators are self-adjoint.

Conversely, let the operators $S_{\pm}(V)$ be selfadjoint. In the case of a non-real-valued distribution $V(x)$, the operators $S_{\pm}(V)$ are not symmetric either. This contradiction allows to make conclusion that the distribution $V(x)$ is real-valued.

(c) From [2] (Theorem VI.3.4) and Proposition 1 we immediately obtain the following proposition.

Proposition 2. *The resolvent sets of the operators $S_{\pm}(V)$ are non-empty. Moreover, the resolvents $R(\lambda, S_{\pm}(V))$ of the operators $S_{\pm}(V)$ are compact.*

Proposition 2 implies that the operators $S_{\pm}(V)$ have discrete spectra.

***2.5. Proof of Theorem* 5.** The proof is based on the following proposition.

Proposition 3. *Let the 1-periodic distributions $V_n(x)$, $n \in \mathbb{N}$, and $V(x)$ be in the Sobolev space H_+^{-m}. For*

$$V_n(x) \xrightarrow{H_+^{-m}} V(x), \quad n \to \infty,$$

the operators

$$S_{\pm}^{(n)} \equiv S_{\pm}(V_n) := D_{\pm}^{2m} \dotplus V_n(x),$$

$$\operatorname{Dom}(S_{\pm}^{(n)}) = \left\{u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V_n(x)u \in L_2(0,1)\right\},$$

converge to the operators

$$S_{\pm} \equiv S_{\pm}(V) = D_{\pm}^{2m} \dotplus V(x),$$

$$\operatorname{Dom}(S_{\pm}) = \left\{u \in H_{\pm}^{m} \;\middle|\; D_{\pm}^{2m}u + V(x)u \in L_2(0,1)\right\},$$

in the generalized convergence sense [2] (*Ch.* IV, § 2.6).

Proof. Set

$$t_{S_{\pm}^{(n)}}[u,v] \equiv t_{\pm}^{(n)}[u,v] := (S_{\pm}^{(n)}u, v), \qquad u \in \operatorname{Dom}(S_{\pm}^{(n)}), v \in \operatorname{Dom}(t_{\pm}^{(n)}) = H_{\pm}^{m},$$

and recall that

$$t_{S_{\pm}}[u,v] \equiv t_{\pm}[u,v] = (S_{\pm}u, v), \qquad u \in \operatorname{Dom}(S_{\pm}), v \in \operatorname{Dom}(t_{\pm}) = H_{\pm}^{m}.$$

Then, for every $u \in \operatorname{Dom}(t_{\pm}) = \operatorname{Dom}(t_{\pm}^{(n)}) = H_{\pm}^{m}$,

$$\left|t_{\pm}^{(n)}[u] - t_{\pm}[u]\right| = \left|\langle (V_n(x) - V(x))u, u\rangle_{\pm}\right| \le \|(V_n(x) - V(x))u\|_{H_{\pm}^{-m}} \|u\|_{H_{\pm}^{m}} \le$$

$$\le C_m \|V_n(x) - V(x)\|_{H_{+}^{-m}} \left(\|u\|_{H_{\pm}^{m}}^2 + \tau_{\pm}[u]\right),$$

where the constant C_m is defined due to the convolution lemma, and $\tau_{\pm}[u,v]$, as above,

$$\tau_{\pm}[u,v] = \langle D_{\pm}^{2m}u, v\rangle_{\pm}, \qquad \operatorname{Dom}(\tau_{\pm}) = H_{\pm}^{m},$$

are sesquilinear, densely defined, closed and nonnegative forms. Since the forms

$$t_V^{\pm}[u,v] = \langle V(x)u, v\rangle_{\pm}, \qquad \operatorname{Dom}(t_V^{\pm}) = H_{\pm}^{m},$$

are $\tau_{\pm}$-bonded with zero $\tau_{\pm}$-boundary for an arbitrary $0 < \varepsilon \le 1/2$, the following estimates hold:

$$2\left|\operatorname{Re} t_V^{\pm}[u]\right| \le \tau_{\pm}[u] + 2\left(C_m \|V_0(x)\|_{H_{+}^{m}} + \varepsilon\right) \|u\|_{L_2(0,1)}^2,$$

and thus

$$2\operatorname{Re} t_V^{\pm}[u] + \tau_{\pm}[u] + 2\left(C_m \|V_0(x)\|_{H_{+}^{m}} + \varepsilon\right) \|u\|_{L_2(0,1)}^2 \ge 0.$$

Taking to account that

$$\operatorname{Re} t_{\pm}[u] = \tau_{\pm}[u] + \operatorname{Re} t_V^{\pm}[u]$$

we get the needed estimates,

$$\left|t_{\pm}^{(n)}[u] - t_{\pm}[u]\right| \le C_m \|V_n(x) - V(x)\|_{H_{+}^{-m}} \left(\|u\|_{H_{\pm}^{m}}^2 + \tau_{\pm}[u]\right) \le$$

$$\le C_m \|V_n(x) - V(x)\|_{H_{+}^{-m}} \times$$

$$\times \left(2\operatorname{Re} t_V^{\pm}[u] + 2\tau_{\pm}[u] + 2\left(C_m \|V_0(x)\|_{H_{+}^{m}} + \varepsilon + 1/2\right) \|u\|_{L_2(0,1)}^2\right) =$$

$$= a_n \|u\|^2_{L_2(0,1)} + b_n \mathrm{Re}\, t_\pm[u],$$

where

$$a_n = 2\left(C_m \|V_0(x)\|_{H_+^m} + 1\right) \|V_n(x) - V(x)\|_{H_+^{-m}} \geq 0$$

and

$$b_n = 2C_m \|V_n(x) - V(x)\|_{H_+^{-m}} \geq 0$$

tend to zero as $n \to \infty$.

To complete the proof it suffices to apply [2] (Theorem VI.3.6).

Proposition 3 and Theorem IV.2.25 [2] together with Proposition 2, give Theorem 5.

***2.6. Proof of Theorem* 6.** Let the operators — form-sums $S(V)$, $S_+(V)$, and $S_-(V)$ be given with $V(x)$ a 1-periodic complex-valued distribution from the Sobolev spaces H^{-m}_{per} and $H^{-m}_{\pm}$, correspondingly.

For an arbitrary $s \in \mathbb{R}$ let us consider the Sobolev spaces

$$H^s_{\mathrm{per}} = \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik\pi x} \;\Big|\; \| f \|_{H^s_{\mathrm{per}}} < \infty \right\},$$

$$H^s_+ = \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(2k) e^{i2k\pi x} \;\Big|\; \| f \|_{H^s_+} < \infty \right\},$$

$$H^s_- = \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(2k+1) e^{i(2k+1)\pi x} \;\Big|\; \| f \|_{H^s_-} < \infty \right\}.$$

It should be remarked that

$$H^0_{\mathrm{per}} \equiv L_2(-1,1) \quad \text{and} \quad H^0_+ \equiv H^0_- \equiv L_2(0,1).$$

Set

$$H^s_{\mathrm{per},+} := \left\{ f \in H^s_{\mathrm{per}} \;\Big|\; \widehat{f}(2k+1) = 0 \quad \forall k \in \mathbb{Z} \right\},$$

$$H^s_{\mathrm{per},-} := \left\{ f \in H^s_{\mathrm{per}} \;\Big|\; \widehat{f}(2k) = 0 \quad \forall k \in \mathbb{Z} \right\},$$

and thus

$$H^s_{\mathrm{per}} = H^s_{\mathrm{per},+} \oplus H^s_{\mathrm{per},-}, \quad s \in \mathbb{R}.$$

Let

$$I_\pm : H^s_\pm \ni f(x) \mapsto f(x) \in H^s_{\mathrm{per},\pm}, \quad s \in \mathbb{R},$$

be extension operators that extend the elements $f(x) \in H^s_\pm$ defined on the interval $[0,1]$ to the elements $f(x) \in H^s_{\mathrm{per},\pm}$ defined on the interval $[-1,1]$. The operators $I_\pm$ establish isometric isomorphisms between the spaces $H^s_\pm$ and $H^s_{\mathrm{per},\pm}$ for $s \in \mathbb{R}$.

Further, let us consider the operators $S(V)$. Since the potentials $V(x)$ are 1-periodic distributions from the space H^{-m}_{per}, i.e., $V(x) \in H^{-m}_{\mathrm{per},+}$, the operators $S(V)$ are reduced by the space $H^{-m}_{\mathrm{per},+}$ [18] (Ch. IV, § 40). So, we have

$$S(V) = S_{\mathrm{per},+}(V) \oplus S_{\mathrm{per},-}(V), \tag{3}$$

where the operators $S_{\mathrm{per},\pm}(V)$ are defined on the Hilbert spaces $H^0_{\mathrm{per},\pm}$. Taking into account that

$$H^s_+ \overset{I_+}{\simeq} H^s_{\mathrm{per},+} \quad \text{and} \quad H^s_- \overset{I_-}{\simeq} H^s_{\mathrm{per},-}$$

for an arbitrary $s \in \mathbb{R}$ we conclude that the operators $S_{\mathrm{per},\pm}(V)$ and $S_\pm(V)$ are unitary equivalent,

$$S_+(V) \overset{I_+}{\simeq} S_{\mathrm{per},+}(V) \quad \text{and} \quad S_-(V) \overset{I_-}{\simeq} S_{\mathrm{per},-}(V).$$

From the latter relations and decomposition (3), we obtain the need statement,

$$S(V) = S_+(V) \oplus S_-(V),$$

which implies

$$\mathrm{spec}(S) = \mathrm{spec}(S_+) \cup \mathrm{spec}(S_-).$$

The proof of Theorem 6 is completed.

1. *Vladimirov V. S.* Generalized functions in mathematical physics (in Russian). – Moscow: Nauka, 1976. – 280 p.
2. *Kato T.* Perturbation theory for linear operators. – Berlin etc.: Springer, 1966. – 592 p. (Russian edition: Moscow: Mir, 1972. – 740 p.).
3. *Molyboga V.* Estimates for periodic eigenvalues of the differential operator $(-1)^m d^{2m}/dx^{2m} + V$ with V-distribution // Meth. Funct. Anal. and Top. – 2003. – **9**, № 2. – P. 163 – 178.
4. *Mikhailets V. A., Molyboga V. M.* Singular eigenvalues problems on the circle // Ibid. – 2004. – **10**, № 3. – P. 44 – 53.
5. *Mikhailets V. A., Molyboga V. M.* Uniform estimates for the semiperiodic eigenvalues of the singular differential operators // Ibid. – № 4. – P. 30 – 57.
6. *Kappeler T., Möhr C.* Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials // J. Func. Anal. – 2001. – **186**. – P. 62 – 91.
7. *Möhr C.* Schrödinger operators with singular potentials on the circle: spectral analysis and applications // Thes. Univ. Zürich. – 2001. – 134 p.
8. *Mikhailets V. A., Sobolev A. V.* Common eigenvalue problem and periodic Schrödinger operators // J. Funct. Anal. – 1999. – **165**, № 1. – P. 150 – 172.
9. *Hryniv R. O., Mykytyuk Ya. V.* 1-D Schrödinger operators with periodic singular potentials // Meth. Funct. Anal. and Top. – 2001. – **7**, № 4. – P. 31 – 42.
10. *Korotyaev E. L.* Characterization of the spectrum of Schrödinger operators with periodic distributions // Int. Math. Res. Not. – 2003. – **37**. – P. 2019 – 2031.
11. *Djakov P., Mityagin B. S.* Instability zones of one-dimensional periodic Schrödinger and Dirac operators (in Russian) // Uspekhi Mat. Nauk. – 2006. – **61**, № 4. – P. 77 – 182.
12. *Berezansky Yu. M.* Expansions in eigenfunctions of self-adjoint operators. – Providence, RI: Amer. Math. Soc. Transl., 1968. – **17**. – 809 p. (Russian edition: Kiev: Naukova Dumka, 1965. – 800 p.).
13. *Gorbachuk V. I., Gorbachuk M. L.* Boundary value problems for operator differential equations. – Dordrecht etc.: Kluwer, 1991. – 347 p. (Russian edition: Kiev: Naukova Dumka, 1984. – 283 p.).
14. *Reed M., Simon B.* Methods of modern mathematical physics: Vols 1 – 4. – V. 1. Functional analysis. – New York etc.: Acad. Press, 1972. – 400 p. (Russian edition: Moscow: Mir, 1977. – 359 p.).
15. *Mikhailets V. A., Molyboga V. M.* The perturbation of periodic and semiperiodic operators by Schwartz distributions (in Russian) // Repts NAS Ukraine. – 2006. – **7**. – P. 26 – 31.
16. *Mikhailets V. A., Molyboga V. M.* The spectral problems over the periodic classes of distributions (in Ukrainian). – Kyiv, 2004. – 46 p. – (Preprint / NAS Ukraine. Inst. Math.; 2004.10).
17. *Reed M., Simon B.* Methods of modern mathematical physics: Vols 1 – 4. – V. 2. Fourier analysis, self-adjointness. – New York etc.: Acad. Press, 1975. – 361 p. (Russian edition: Moscow: Mir, 1978. – 396 p.).
18. *Akhiezer N. I., Glazman I. M.* Theory of linear operators in Hilbert space. – New York: Dover Publ., Inc., 1993. – 218 p. (Russian edition: Moscow: Nauka, 1966. – 544 p).

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