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# NEW ASPECTS OF KREIN'S EXTENSION THEORY

# ДЕЯКІ НОВІ АСПЕКТИ ТЕОРІЇ РОЗШИРЕНЬ КРЕЙНА

The extension problem for closed symmetric operators with a gap is studied. A new kind of parametrization of extensions (the so-called Krein model) is developed. The notion of a singular operator plays the key role in our approach. We give the explicit description of extensions and establish the spectral properties of extended operators.

Досліджується проблема розширень замкнених симетричних операторів із щілиною в спектрі. Розвинуто новий спосіб параметризації розширень (так звана модель Крейна). Ключову роль у нашому підході відіграє поняття сингулярного оператора. Дано явний опис розширень і встановлені спектральні властивості розширених операторів.

**1. Introduction.** The phenomenon of the "extension of a closed symmetric operator", which appeared in the works of von Neumann [35] and Friedrichs [19] and later in the famous papers of Krein [30, 31] and was completed by the papers of Birman [8] and Vishik [45], has been transformed into an extensive theory with numerous modifications, applications, and open problems. The above-mentioned works do not only contain fundamental results; they also have generated a series of new questions and problems and have led to a powerful wave of publications [40 – 43, 6, 37, 23, 24, 33, 36, 39, 38, 34, 20, 17, 32, 18, 10 – 15, 27]. Roughly speaking, one can survey all these papers in the following three directions: various kinds of parametrization of the extensions, the properties (in particular, spectral properties) of the extended operators, and applications. Of course, the chosen type of parametrization determines the properties of the extended operators and the range of applications one can envelop. The original von Neumann approach is an abstract one, it has led to general theorems [1]. More suitable for applications is Krein's approach which has generated a lot of imitations with deep results [8, 23, 20, 17, 10]. The most successful approach appeared in [8, 40, 23, 20] and was developed in [17, 18, 5].

In [10], a special kind of parametrization of symmetric and self-adjoint extensions of a given symmetric operator, the so-called Krein model, was presented. In [10] and [11], this model was used when solving various operator theoretical problems. In Section 2, we show how to introduce the Krein model in a simpler way by using the so-called singular operators.

In Section 3, we present the parametrization of the self-adjoint extensions of a symmetric operator via the so-called singular perturbations and singularly perturbed operators. The basic idea is very simple. Every extension of a closed symmetric operator is given by some abstract boundary condition. We consider this condition as the perturbation of a certain suitable extension (the Friedrichs or Krein extension). Note that every boundary condition is zero on a dense set of vectors in the original Hilbert space. This means that the boundary condition can be represented as a singular operator or as a singular bilinear form. The explicit properties of singular objects imply the corresponding properties (for example, the spectral properties) of the extended operator. The efficiency of this approach is verified by applications (see, for example, the remarkable monographs [2, 3]).

In Section 4, we investigate the point spectrum of singularly perturbed operators and show how to construct a singularly perturbed operator with a given point spectrum. We refer to [12, 17], and [32] for the related results.

In Section 5, we examine the problem of determining which types of the spectrum the self-adjoint extensions of $A$ may possess within $(a, b)$. Here, $A$ is a symmetric operator with the gap $(a, b)$. In other words, we present the first steps towards the inverse spectral theory of self-adjoint extensions of a symmetric operator.

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Finally, in Section 6, we present a detailed study of singular perturbations of the operator $-\Delta + 1$ in $L^2(R^3)$. On one hand, our results in this section can be regarded as an illustration of the previous abstract results and, on the other hand, they are of independent interest for applications in mathematical physics.

**2. Models.** Let $A$ be a densely defined closed symmetric operator with equal nonzero deficiency indices in a separable Hilbert space $\mathcal{H}$. In what follows, we use the standard notation for domains, range, etc.

Assume that $\ker A = \{0\}$ and that the inverse operator $L = A^{-1}$ is bounded. Its domain $\mathcal{D}(L) = \mathcal{R}(A)$ is a closed subspace in $\mathcal{H}$, which we denote by $\mathcal{R}$; its range $\mathcal{R}(L) = \mathcal{D}(A)$ is dense in $\mathcal{H}$, and we denote it by $\mathcal{D}$. Evidently, $\mathcal{H} = \mathcal{R} \oplus N$, where $N = \mathcal{R}^{\perp} = \ker A^*$. Let $P_N$ denote the orthogonal projector in $\mathcal{H}$ onto $N$ and let $P_{\mathcal{R}}$ be the projector on $\mathcal{R}$.

**Lemma 1.** *The operator* $A_0 : P_{\mathcal{R}}\varphi \to A\varphi$, $\varphi \in \mathcal{D}$ *is self-adjoint in* $\mathcal{R}$ : $\mathcal{R}(A_0) = \mathcal{R}$, $\ker A_0 = \{0\}$.

***Proof.*** The domain $\mathcal{D}(A_0) = P_{\mathcal{R}}\mathcal{D}$ is dense in $\mathcal{R}$, since $\mathcal{D}$ is dense in $\mathcal{H}$. The operator $A_0$ is correctly defined. Indeed, if $P_{\mathcal{R}}\varphi = 0$, then $\varphi = 0$ because $N \cap \cap \mathcal{D} = \{0\}$. This follows from the facts that $\ker A = \{0\}$ and $N = \ker A^*$. Evidently, $A_0$ is symmetric in $\mathcal{R}$. Actually, it is self-adjoint since $\mathcal{R}(A_0) = \mathcal{R}$. Hence, $A_0 = \{0\}$.

**Lemma 2.** *The restriction* $P_{\mathcal{R}} \restriction \mathcal{D}$ *is an invertible operator in* $\mathcal{H}$.

***Proof.*** We just have seen that $P_{\mathcal{R}}\varphi = 0$, $\varphi \in \mathcal{D}$, implies $\varphi \equiv 0$.

Introduce the operator

$$\Gamma := P_N (P_{\mathcal{R}} \restriction \mathcal{D})^{-1}, \quad \Gamma : P_{\mathcal{R}}\varphi \to P_N\varphi, \quad \varphi \in \mathcal{D}, \tag{1}$$

$$\mathcal{D}(\Gamma) = \mathcal{D}(A_0) = P_{\mathcal{R}}\mathcal{D}, \quad \mathcal{R}(\Gamma) \subset N.$$

Thus, $\Gamma$ maps $\mathcal{R}$ into $N$. It is a densely defined operator in $\mathcal{R}$ and its range is dense in $N$.

**Lemma 3.** *The operator* $\Gamma$ *is singular in the following sense: For every* $\psi \in \in \mathcal{D}(\Gamma) \subset \mathcal{R}$, *there exists a sequence* $\{\psi_n\}$ *in* $\mathcal{D}(\Gamma)$ *such that* $\psi_n \to \psi$ *in* $\mathcal{R}$ *and* $\Gamma\psi_n \to 0$ *in* $N$.

***Proof.*** Let $\varphi_n \in \mathcal{D}$, $\varphi_n \to \psi \in \mathcal{R}$ in $\mathcal{H}$. Then, clearly, $P_N\varphi_n \to 0$ in $N$ and $\psi_n := P_{\mathcal{R}}\varphi_n \to \psi$ in $\mathcal{R}$. By the definition of $\Gamma$, we have $\Gamma\psi_n = P_N\varphi_n$. Hence, $\Gamma\psi_n \to 0$ in $N$.

***Remark* 1.** The operator $\Gamma$ is also singular in another equivalent sense [22, 29]. For every $\psi \in \mathcal{D}(\Gamma)$, there exists a sequence $\{\psi'_n\}$ in $\mathcal{D}(\Gamma)$ such that $\psi'_n \to 0$ and $\Gamma\psi'_n \to \Gamma\psi$.

Indeed, we can set $\psi'_n = \psi - \psi_n$.

**Lemma 4.** *The operator* $\Gamma$ *is closed in the graph norm of* $A_0$.

***Proof.*** Let $\varphi_n \to \varphi \in \mathcal{R}$, $\varphi_n \in \mathcal{D}(\Gamma)$, $A_0\varphi_n \to h \in \mathcal{R}$, and $\Gamma\varphi_n \to \eta \in N$. Since $A_0$ is self-adjoint, the vector $\varphi$ is in $\mathcal{D}(A_0)$ and $A_0\varphi = h$. Further, since $A^{-1}$ is bounded, we have $A^{-1}A_0\varphi_n = \varphi_n \oplus \Gamma\varphi_n \to A^{-1}h = \varphi \oplus \eta$, i.e., $\varphi \in \mathcal{D}(\Gamma)$ and $\Gamma\varphi \to \eta$.

**Theorem 1** (*Krein model*). *Every densely defined invertible closed symmetric operator* $\boldsymbol{A}$ *in* $\mathcal{H}$ *with a bounded inverse is uniquely determined by the following three objects*:

(i) *the resolution* $\mathcal{H} = \mathcal{R} \oplus N$, $N = \ker \boldsymbol{A}^*$;

(ii) *the self-adjoint operator* $A_0$ *in* $\mathcal{R}$, $\ker A_0 = \{0\}$, $\mathcal{R}(A_0) = \mathcal{R}$;

(iii) *the singular operator* $\Gamma : \mathcal{R} \to N$ *such that* $\mathcal{D}(\Gamma) = \mathcal{D}(A_0)$ *and* $\mathcal{R}(\Gamma)$ *is dense in* $N$, *which is closed in the graph norm of* $A_0$.

***Proof.*** We have already shown that the original operator $\boldsymbol{A}$ defines $\mathcal{R} = \mathcal{R}(\boldsymbol{A})$, $N = \ker \boldsymbol{A}^*$, $A_0$ (see Lemma 1) with $\ker A_0 = \{0\}$, and the operator $\Gamma$ (see (1)) with the above stated properties. Conversely, let three objects (i)–(iii) be given. We must show that

$$\boldsymbol{A} : f = \varphi \oplus \Gamma\varphi \to A_0 \varphi, \quad \varphi \in \mathcal{D}(A_0) \tag{2}$$

is an operator with the required properties. First, we prove that $\mathcal{D}(\boldsymbol{A}) := \{f \in \mathcal{H} \mid f = \varphi \oplus \Gamma\varphi,\ \varphi \in \mathcal{D}(A_0)\}$ is dense in $\mathcal{H}$. Since $\Gamma$ is singular, there exist $f_n = \psi_n \oplus \Gamma\psi_n$ such that $f_n \to \psi \oplus 0$ for any $\psi \in \mathcal{D}(A_0)$. Remark 1 implies the existence of $f_n' = \psi_n' \oplus \Gamma\psi_n'$ such that $f_n' \to 0 \oplus \Gamma\psi$ for any $\psi \in \mathcal{D}(A_0)$. However, $\mathcal{D}(A_0)$ is dense in $\mathcal{R}$ and $\mathcal{R}(\Gamma)$ is dense in $N$. Therefore, $\mathcal{D}(\boldsymbol{A})$ is dense in $\mathcal{H}$. Evidently, $\boldsymbol{A}$ is closed. Indeed, let $f_n = \varphi_n \oplus \Gamma\varphi_n \to f$ in $\mathcal{H}$. This means that $\varphi_n \to \varphi$ in $\mathcal{R}$ for some $\varphi$. If, in addition, $\boldsymbol{A} f_n = A_0 \varphi_n \to h$, then $\varphi \in \mathcal{D}(\boldsymbol{A})$ and $A_0 \varphi = h$, since $A_0$ is a self-adjoint operator. Hence, $g = \varphi \oplus \Gamma\varphi \in \mathcal{D}(\boldsymbol{A})$ and $\boldsymbol{A}\, g = A_0 \varphi$. We assert that $\Gamma\varphi_n \to \Gamma\varphi$ as $\Gamma$ is closed in the graph norm of $A_0$. Thus, $f = g \in \mathcal{D}(\boldsymbol{A})$, $\boldsymbol{A} f = h$ is closed. Further, $\boldsymbol{A}$ is symmetric, since

$$(\boldsymbol{A} f, g) = (A_0 \varphi, \psi)_{\mathcal{R}} = (\varphi, A_0 \psi)_{\mathcal{R}} = (f, \boldsymbol{A}\, g) \tag{3}$$

if $f = \varphi \oplus \Gamma\varphi$ and $g = \psi \oplus \Gamma\psi$, $\varphi, \psi \in \mathcal{D}(A_0)$. Finally, $\ker \boldsymbol{A} = \{0\}$ because $\boldsymbol{A} f = 0$ implies that $A_0 \varphi = 0$ and $\varphi = 0$, since $\ker A_0 = \{0\}$. Hence, $f = 0$ as well. Of course, if we construct $\mathcal{R}$, $N$, $A_0$, and $\Gamma$ starting from the operator $\boldsymbol{A}$ obtained, then we return to objects (i) – (iii) given above, and vice versa.

***Remark* 2.** If, for some $a, b \in R^1$ with $a < 0 < b$, the operator $A_0$ satisfies the condition $a^{-1} \le A_0 \le b^{-1}$, then the interval $(a, b)$ is a gap for $\boldsymbol{A}$.

***Remark* 3.** For the operator $\Gamma$, the following condition holds: $\mathcal{R}(\Gamma^*) \cap \mathcal{D}(A_0) = \{0\}$.

Indeed, let $(\Gamma\varphi, \eta)_N = l_\eta(\varphi)$ be a linear continuous functional on $\varphi \in \mathcal{D}(\Gamma) = \mathcal{D}(A_0)$. In this case, we have $l_\eta(\varphi) = (\varphi, \eta^*)_{\mathcal{R}}$ for some $\eta^* \in \mathcal{R}$. Assume that $\eta^* \in \mathcal{D}(A_0)$. Then we can take $\varphi = \eta^*$. Owing to the singularity of $\Gamma$ in $\mathcal{R}$, there exists a sequence $\psi_n \to \varphi$ such that $\Gamma\psi_n \to 0$. Then $(\Gamma\psi_n, \eta)_N \to 0$ and, therefore, $\eta^* = 0$.

Thus, in the Krein model, each closed symmetric operator $\boldsymbol{A}$, $\ker \boldsymbol{A} = 0$, is given in the Hilbert space $\mathcal{H}$ by the resolution $\mathcal{H} = \mathcal{R} \oplus N$, by the self-adjoint operator $A_0$ in $\mathcal{R}$ and by the singular operator $\Gamma : \mathcal{R} \to N$. In the next section, we show that every self-adjoint extension $\tilde{A}$ of $\boldsymbol{A}$, $\ker \tilde{A} = 0$, is fixed by another singular operator $T$, which acts in the rigged Hilbert space constructed by using the Friedrichs extension of $\boldsymbol{A}$.

**3. Extension as a Singularly Perturbed Operator.** Let $A = A^* \geq 1$ be a self-adjoint unbounded operator in a separable complex Hilbert space $\mathcal{H}$.

**Definition 1.** *A self-adjoint operator $\tilde{A}$ in $\mathcal{H}$ is called singularly perturbed with respect to $A$ if the linear set*

$$\mathcal{D} := \{ f \in \mathcal{D}(A) \cap \mathcal{D}(\tilde{A}) \mid Af = \tilde{A} f \} \tag{4}$$

*is dense in $\mathcal{H}$.*

We write $\tilde{A} \in \mathcal{A}_S(A)$ if, in addition, $0 \in \rho(\tilde{A})$, and $\tilde{A} \in \mathcal{A}_S^+(A)$ if $\tilde{A}$ is positive definite. Here, $\rho(\cdot)$ denotes the set of regular points of the operator.

Thus, the self-adjoint operator $\tilde{A}$ in $\mathcal{H}$ belongs to the class $\mathcal{A}_S(A)$ if $\tilde{A}^{-1}$ exists and is bounded and the pair $A$ and $\tilde{A}$ has a common symmetric part, i.e.,

$$\boldsymbol{A} := A \restriction \mathcal{D} = \tilde{A} \restriction \mathcal{D} \tag{5}$$

is a densely defined symmetric operator in $\mathcal{H}$. It is closed and $\ker \boldsymbol{A} = 0$. Hence, each singularly perturbed operator $\tilde{A}$ is a self-adjoint extension of some symmetric restriction $\boldsymbol{A}$ of $A$.

Below, we show that each restriction $\boldsymbol{A}$ of $A$ and each extension $\tilde{A}$ of $\boldsymbol{A}$ are uniquely determined by a certain singular perturbation of $A$.

We now introduce the precise definition of the concept "singular perturbation". In fact, it arises from the concept of a singular operator or a singular bilinear form [22].

First, let us define by $A$ the rigged Hilbert space (see [7])

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \tag{6}$$

where $\mathcal{H}_+ = \mathcal{D}(A)$ with respect to the inner product $(\cdot, \cdot)_+ = (A \cdot, A \cdot)$ and $\mathcal{H}_-$ is the completion of $\mathcal{H}$ in the norm $\|\cdot\|_- = \|A^{-1} \cdot\|$. We denote the duality between $\mathcal{H}_-$ and $\mathcal{H}_+$ by $\langle \cdot, \cdot \rangle$. Note that

$$(\varphi, \psi)_+ = \langle D\varphi, \psi \rangle = \langle \varphi, D\psi \rangle = \langle D\varphi, D\psi \rangle_-, \quad \varphi, \psi \in \mathcal{H}_+, \tag{7}$$

where $D = A^{cl} A : \mathcal{H}_+ \to \mathcal{H}_-$ is a canonical unitary isomorphism ($A^{cl}$ denotes the closure of $A : \mathcal{H} \to \mathcal{H}_-$).

**Definition 2.** *A linear closed operator $T : \mathcal{H}_+ \to \mathcal{H}_-$ is called a singular perturbation of $A$ if the linear set*

$$F_{0,T} := \ker T \tag{8}$$

*is dense in $\mathcal{H}$.*

We write $T \in \mathcal{T}_S(A)$ if $T$ is self-adjoint and its range $\mathcal{R}(T)$ is a closed subspace of $\mathcal{H}_-$ such that

$$\mathcal{R}(T) \cap \mathcal{H} = \{0\}. \tag{9}$$

Recall that $T : \mathcal{H}_+ \to \mathcal{H}_-$ is self-adjoint if $T = T^*$. By definition, a vector $\psi \in \mathcal{H}_+$ belongs to $\mathcal{D}(T^*)$ if $l(\varphi) = \langle T\varphi, \psi \rangle$ is a linear continuous functional on $\mathcal{H}_+$. Then there exists a unique $\psi^* \in \mathcal{H}_-$ such that $l(\varphi) = \langle \varphi, \psi^* \rangle$. We set $T^* \psi = \psi^*$.

Thus, a self-adjoint operator $T : \mathcal{H}_+ \to \mathcal{H}_-$ belongs to the class $\mathcal{T}_S(A)$ if its restriction $\boldsymbol{T} := T \restriction F$, $F = \mathcal{H}_+ \ominus \ker T$ is boundedly invertible and (9) holds.

Note that each $T \in \mathcal{T}_S(A)$ is a singular operator in $\mathcal{H}$ [22] and $\gamma_T(\varphi, \psi) :=$

$:= \langle T\varphi, \psi\rangle$ is a singular bilinear form in $\mathcal{H}$ [28].

The following important result establishes the connection between singularly perturbed operators and singular perturbations for fixed $A$. The connection is based on the Krein extension theory. It gives the parametrization of all self-adjoint extensions $\tilde{A} \in \mathcal{A}_S(A)$ of the symmetric operator $\boldsymbol{A}$ from (5) in terms of the singular perturbation $T \in \mathcal{T}_S(A)$.

**Theorem 2** [28]. *There is a one to one correspondence between* $\tilde{A} \in \mathcal{A}_S(A)$ *and* $T \in \mathcal{T}_S(A)$. *For each* $\tilde{A}$, *the operator* $T$ *is defined as follows*:

$$T = A^{cl}(0 \oplus B)A, \tag{10}$$

*where* $B$ *is fixed by the difference*

$$\tilde{A}^{-1} - A^{-1} = \tilde{B}^{-1} = \begin{cases} B^{-1} & in \quad N, \\ 0 & on \quad \mathcal{R}, \end{cases} \tag{11}$$

*where*

$$\mathcal{H} = \mathcal{R} \oplus N, \quad \mathcal{R} = \mathcal{R}(A), \quad N = \ker \boldsymbol{A}^* \tag{12}$$

*and* $\boldsymbol{A}$ *is the largest common symmetric part of* $A$, $\tilde{A}$ (*see* (5)). *Conversely, each* $T$ *defines* $\tilde{A}$ *by* (11) *if we set*

$$\mathcal{R} = AF_{0,T}, \quad N = AF, \quad \mathcal{H}_+ = F_{0,T} \oplus F, \tag{13}$$

$$B = (A^{cl})^{-1}\mathbf{T}A^{-1}, \quad \mathbf{T} = T \restriction F. \tag{14}$$

***Proof.*** Let $\tilde{A} \in \mathcal{A}_S(A)$. Then both $A^{-1}$ and $\tilde{A}^{-1}$ are bounded and self-adjoint. Hence, their difference $\tilde{B}^{-1}$ is also bounded and self-adjoint in $\mathcal{H}$. Moreover, it follows from $\mathcal{R} = A\mathcal{D} = \tilde{A}\mathcal{D}$ ($\mathcal{D}$ is defined by (4)) that $\tilde{B}^{-1}$ is zero on $\mathcal{R}$ and $\ker \tilde{B}^{-1} = \mathcal{R}$. Clearly, $N$ is reduced by $\tilde{B}^{-1}$. Therefore, the restriction $B^{-1}$ of $\tilde{B}^{-1}$ to $N$ is a bounded self-adjoint invertible operator in $N$. We set $B = (B^{-1})^{-1}$. It is necessary to show that $T$ defined by (10) belongs to the class $\mathcal{T}_S(A)$. It is easy to see that $T$ is self-adjoint and

$$\ker T \equiv F_{0,T} = A^{-1}\mathcal{R}, \tag{15}$$

$$\mathcal{R}(T) \equiv N_- = A^{cl}N \ (N = \mathcal{R}(B)). \tag{16}$$

The subspace $F_{0,T}$ of $\mathcal{H}_+ = \mathcal{D}(A)$ is dense in $\mathcal{H}$ because it coincides with $\mathcal{D}$ from (4) ($\mathcal{D} = A^{-1}\mathcal{R} = \tilde{A}^{-1}\mathcal{R}$). By construction, the range $\mathcal{R}(B)$ is equal to $N$ and, therefore, $N_-$ is a closed subspace in $\mathcal{H}_-$ ($A^{cl} : \mathcal{H} \to \mathcal{H}_-$ is unitary). We obtain condition (9) from (8) by virtue of Lemma 5 (presented below). Conversely, let $T \in \mathcal{T}_S(A)$ be given. We set $\mathcal{D} = F_{0,T}$ and define the restriction

$$\boldsymbol{A} := A \restriction \ker T, \quad \mathcal{D}(\boldsymbol{A}) = \mathcal{D}. \tag{17}$$

This is a closed symmetric operator in $\mathcal{H}$. Evidently, the range $\mathcal{R}(\boldsymbol{A}) = \mathcal{R}$, and $N = \ker \boldsymbol{A}^*$ is the deficiency subspace of $\boldsymbol{A}$ (see (13)). By construction (see (14)), $B$ is a self-adjoint operator in $N$ such that $\mathcal{R}(B) = N$. Hence, $B^{-1}$ exists, is bounded and self-adjoint in $N$. We denote its trivial extension onto $\mathcal{R}$ by $\tilde{B}^{-1}$. We can now define $\tilde{A}^{-1} = A^{-1} + \tilde{B}^{-1}$. This is a self-adjoint bounded operator in $\mathcal{H}$. It coincides

with $A^{-1}$ on $\mathcal{R}$ and, therefore, $\tilde{A}$ is one of the self-adjoint extensions of $\boldsymbol{A}$. Of course, $\tilde{A}^{-1}$ is invertible because $\tilde{A}^{-1}h = 0$ implies $h = 0$, since $A^{-1}h \in \mathcal{D}$, $\tilde{B}^{-1}h \in N$, and $N \cap \mathcal{D} = \{0\}$. Thus, $\tilde{A} \in \mathcal{A}_S(A)$.

In the proof of Theorem 2, we have used the following lemma:

**Lemma 5.** *For the rigged Hilbert space* (6), *let* $\mathcal{H}_+ = F_0 \oplus F$ *and let* $T : \mathcal{H}_+ \to \mathcal{H}_-$ *be a self-adjoint operator such that* $F_0 = \ker T$ *and* $N_- := \mathcal{R}(T)$ *is a closed subspace of* $\mathcal{H}_-$. *Then*

$$\overline{F_0} = \mathcal{H} \Leftrightarrow N_- \cap \mathcal{H} = \{0\}, \tag{18}$$

*where* $\overline{F_0}$ *denotes the closure of* $F_0$ *in* $\mathcal{H}$.

***Proof.*** In $\mathcal{H}_+$, we consider the operator $V := D^{-1}T$, where $D$ is the canonical isomorphism defined by (7). It is clear that the operator $V$ is self-adjoint, $\ker V = F_0$, and the range $\mathcal{R}(V) = D^{-1}N_- = F$, i.e.,

$$N_- = DF = A^{cl}AF. \tag{19}$$

Assume that $N_- \cap \mathcal{H} = \{0\}$. Then, by virtue of (7), we have:

$$(\psi, F_0) = 0 = \langle \psi, F_0 \rangle = (D^{-1}\psi, F_0)_+, \tag{20}$$

for any $\psi \in \mathcal{H}$, $\psi \perp F_0$, i.e., $D^{-1}\psi \in F$ and, again by (7), $\psi \in N_-$. Hence, $\psi = 0$ and $F_0$ is dense in $\mathcal{H}$. Conversely, assume that $\overline{F_0} = \mathcal{H}$. Then, by virtue of (7), for any $\psi \in N_- \cap \mathcal{H}$, we have $D^{-1}\psi \in F$ and

$$(D^{-1}\psi, F_0)_+ = 0 = \langle \psi, F_0 \rangle = (\psi, F_0), \tag{21}$$

i.e., $\psi \perp F_0$ and, therefore, $\psi = 0$.

***Remark* 4.** In Theorem 2, we have used the operators $B$ acting in the deficiency subspace $N = \ker \boldsymbol{A}^*$, as an intermediate step in the parametrization of self-adjoint extensions of the symmetric operator $\boldsymbol{A}$. This type of parametrization represents the essential point of the so-called Birman–Krein–Vishik theory [5]. We work with a singular operator $T$ (see (14)) instead of $B$. This is more suitable for applications.

***Remark* 5.** Definitions 1 and 2 and Theorem 2 have a natural generalization to the case where the original operator $A$ is bounded from below or has a gap $(a, b)$ in its spectrum. Moreover, in the general case, all formulas (10) – (14) may be rewritten in terms of resolvents.

**Theorem 3** [28]. *The domain* $\mathcal{D}(\tilde{A})$ *of the operator* $\tilde{A} \in \mathcal{A}_S(A)$ *and its action possess the following description:*

$$\mathcal{D}(\tilde{A}) = \{g \in \mathcal{H} \mid g = f + B^{-1}P_N Af, \quad f \in \mathcal{D}(A)\}, \tag{22}$$

$$\tilde{A}g = Af. \tag{23}$$

***Proof.*** For each $\tilde{A} \in \mathcal{A}_S(A)$, we have $\mathcal{R}(\tilde{A}) = \mathcal{H} = \mathcal{R}(A)$. Hence, for any $h \in \mathcal{H}$, there exist $g \in \mathcal{D}(\tilde{A})$ and $f \in \mathcal{D}(A)$ such that $h = \tilde{A}g = Af$. Thus, by using (11), we obtain (22) and (23).

**Theorem 4** [28]. *The resolvent* $\tilde{R}_z$ *of the operator* $\tilde{A} \in \mathcal{A}_S(A)$ *is expressed by the following Krein's formula:*

$$\tilde{R}_z = R_z + \tilde{B}_z^{-1}, \quad z \in \rho(\tilde{A}) \cap \rho(A), \tag{24}$$

*where*

$$\tilde{B}_z^{-1} = \begin{cases} B_z^{-1} & in \;\; N_z, N_z = \ker(A^* - z), \\ 0 & on \;\; M_z, M_z = N_z^{\perp} \end{cases} \tag{25}$$

*and*

$$B_z^{-1} = U_{z,0}(B - z G_{x,0})^{-1} P_N, \tag{26}$$

*where* $U_{z,0} = A R_z$, $R_z = (A - z)^{-1}$, $G_{z,0} = P_z(1 - z A^{-1})$, *and* $P_z$ *is the orthogonal projector onto* $N_z$ *in* $\mathcal{H}$.

**4. The Additional Point Spectrum of** $\tilde{A} \in \mathcal{A}_S(A)$. First, we consider the case where the original operator $A$ is perturbed by $T \in \mathcal{T}_S^n(A)$, where $n := \operatorname{rank} T = 1$. Then $T$ is fixed by a vector $\omega \in \mathcal{H}_- \setminus \mathcal{H}$, $\|\omega\|_- = 1$, and a number $\lambda \in R^1$:

$$T\omega \equiv T_{\lambda,\omega}\varphi := \lambda \langle \varphi, \omega \rangle \omega, \quad \varphi \in \mathcal{H}_+ = \mathcal{D}(T). \tag{27}$$

For a singularly perturbed operator, we write $\tilde{A} \in \mathcal{A}_S^n(A)$ if $\dim N = n$. Theorems 2 and 3 now have the following consequences:

**Theorem 5.** *There is a one to one correspondence between the sets* $\mathcal{T}_S^1(A)$ *and* $\mathcal{A}_S^1(A)$. *Namely, for each* $T_{\lambda,\omega}$, *the operator* $\tilde{A} \equiv A_{\lambda,\omega}$ *is given by*

$$A_{\lambda,\omega} g := A f, \tag{28}$$

$$\mathcal{D}(A_{\lambda,\omega}) := \{ g \in \mathcal{H} \mid g = f + \lambda^{-1} \langle f, \omega \rangle \eta, \; f \in \mathcal{D}(A) \}. \tag{29}$$

*Here,* $\eta := (A^{cl})^{-1}\omega$. *Conversely, each* $\tilde{A} \in \mathcal{A}_S^1(A)$ *defines* $T_{\lambda,\omega}$ *by the vector*

$$\omega := A^{cl}\eta, \quad where \quad \eta \in N, \quad \|\eta\| = 1, \quad N = (A\mathcal{D})^{\perp}, \tag{30}$$

*and by the number*

$$\lambda^{-1} := ((\tilde{A}^{-1} - A^{-1})\eta, \eta). \tag{31}$$

***Proof.*** It follows from Definitions (28) and (29) that $A_{\lambda,\omega}$ is a densely defined symmetric operator. Actually, it is self-adjoint because its range is the entire space $\mathcal{H}$. In Lemma 5, we now take the subspace $F_0 = \mathcal{D}_\omega$, where

$$\mathcal{D}_\omega = \{ f \in \mathcal{D}(A) \mid \langle f, \omega \rangle = (A f, \eta) = 0 \}.$$

Then $\omega \in \mathcal{H}_- \setminus \mathcal{H}$ implies that $F_0$ is dense in $\mathcal{H}$. Therefore, $A_{\lambda,\omega} \in \mathcal{A}_S^1(A)$. Conversely, starting from $\tilde{A} \in \mathcal{A}_S^1(A)$, we can introduce $\omega$ and $\lambda$ as in (30) and (31) and define the operator $T_{\lambda,\omega}$ by (27). The set $\mathcal{D}$ (see (4)) is dense in $\mathcal{H}$ (clearly, it coincides with $\mathcal{D}_\omega$) and (27) now implies that $\mathcal{D} = \mathcal{D}_\omega = \ker T_{\lambda,\omega}$. Moreover, due to Lemma 5, the vector $\omega$ lies in $\mathcal{H}_- \setminus \mathcal{H}$. This means that $T_{\lambda,\omega} \in \mathcal{T}_S^1(A)$.

A more detailed version of the proof of this theorem is given in [28] (Chapter 3, Theorem 1.1). In what follows, we use Krein's resolvent formula for $\tilde{A} \in \mathcal{A}_S^1(A)$.

In the case of $\tilde{A} = A_{\lambda,\omega}$, Krein's formula for resolvents (24) becomes simpler. It follows from (28) and (29) that, for each $g \in \mathcal{D}(A_{\lambda,\omega})$, by using $\langle f, \omega \rangle \equiv (A f, \eta)$, we can write

$$g = f + \lambda^{-1} P_0 A f, \quad f \in \mathcal{D}(A), \tag{32}$$

where $P_0$ is the projector onto $N$ in $\mathcal{H}$. By virtue of (28), this yields

$$g = A^{-1} h + \lambda^{-1} P_0 h,$$

where $h = Af = A_{\lambda,\omega} g$. Hence, we obtain

$$A_{\lambda,\omega}^{-1} = A^{-1} + \lambda^{-1} P_0. \tag{33}$$

A similar formula is true for any common real regular point $\alpha \in \rho := \rho(\tilde{A}) \cap \rho(A)$.

**Theorem 6.** *For any real $0 \neq \alpha \in \rho$, the resolvents of the operators $\tilde{A} = T_{\lambda,\omega}$ and $A$ are connected by*

$$\tilde{R}_\alpha = R_\alpha + \lambda_\alpha^{-1} P_\alpha, \tag{34}$$

*where $\lambda_\alpha = \lambda - \alpha(\eta_0, \eta_\alpha)$, $\eta_0 \equiv \eta = (A^{cl})^{-1}\omega$, $\eta_\alpha := R_\alpha^{cl}\omega$, and $P_\alpha$ denotes the orthogonal projector in $\mathcal{H}$ onto the deficiency subspace $N_\alpha$.*

***Proof.*** First, we note that the subspaces $N_\alpha$, $\alpha \in (-\infty, 1)$, for the operator $A$, admit the following representation:

$$N_\alpha = \{c(A^{cl} - \alpha)^{-1}\omega : c \in \mathbb{C}\}. \tag{35}$$

Indeed,

$$((A - \alpha)\mathcal{D}, N_\alpha) = 0 = (A\mathcal{D}, N_\alpha) - \alpha(\mathcal{D}, N_\alpha) = \langle \mathcal{D}, A^{cl} N_\alpha\rangle - \alpha\langle \mathcal{D}, N_\alpha\rangle =$$

$$= \langle \mathcal{D}, (A^{cl} - \alpha) N_\alpha\rangle = (D\mathcal{D}, (A^{cl} - \alpha) N_\alpha)_-.$$

This means that the one-dimensional subspace $N_{-,\alpha} = (A^{cl} - \alpha) N_\alpha$ in $\mathcal{H}_-$ coincides with $\{c\omega : c \in \mathbb{C}\}$. Therefore, (35) is true. Clearly, the operators $(A^{cl} - \alpha)^{-1}$ and $R_\alpha$ coincide on $\mathcal{H} \subset \mathcal{H}_-$. A simple calculation shows that, for any $\omega \in \mathcal{H}_-$, there exists a vector $g = (1 - \alpha A^{-1})^{-1}(A^{cl})^{-1}\omega$ such that $(A^{cl} - \alpha) g = \omega$. Thus, $\mathcal{H}_- = = \mathcal{D}(R_\alpha^{cl})$, $(A^{cl} - \alpha)^{-1} = R_\alpha^{cl}$, and $\mathcal{R}(R_\alpha^{cl}) = \mathcal{H}$. Now (34) follows from (33) by virtue of the resolvent identity.

Of course, the analogous relation is true for any $z \in \rho = \rho(A) \cap \rho(\tilde{A})$.

Thus, the difference $\tilde{R}_\alpha - R_\alpha = \tilde{B}_\alpha^{-1}$ is a rank-one operator. Therefore, the essential spectrum of $\tilde{A}$ is the same as for $A$.

Let us now investigate the form of the additional point spectrum of $\tilde{A} = A_{\lambda,\omega}$. Assume that $\omega \in \mathcal{H}_- \setminus \mathcal{H}$, $\|\omega\|_- = 1$, is fixed and $0 \neq \alpha \in \rho(A) = (-\infty, 1)$. We want to solve the following eigenvalue problem:

$$A_{\lambda,\omega}\psi_\alpha = \alpha\psi_\alpha \tag{36}$$

for some $\lambda = \lambda(\alpha) \in R^1$ and $\psi_\alpha \in \mathcal{D}(A_{\lambda,\omega})$. By using representation (32), we can write

$$\psi_\alpha = \varphi_\alpha + \lambda^{-1}\langle \varphi_\alpha, \omega\rangle \eta, \quad \varphi_\alpha \in \mathcal{D}(A).$$

It follows from (36) that

$$\alpha\,\psi_\alpha = \alpha\,\varphi_\alpha + \alpha\,\lambda^{-1}\langle\varphi_\alpha, \omega\rangle\,\eta = A\,\varphi_\alpha.$$

By changing the length of the vector $\varphi_\alpha$, we can satisfy the condition

$$\langle\varphi_\alpha, \omega\rangle = 1. \tag{37}$$

Then $A\,\varphi_\alpha = \alpha\,\varphi_\alpha + \alpha\,\lambda^{-1}\,\eta$, and we obtain

$$\varphi_\alpha = \alpha\,\lambda^{-1} R_\alpha\,\eta$$

and

$$\psi_\alpha = \varphi_\alpha + \lambda^{-1}\,\eta = \lambda^{-1}(\alpha\,R_\alpha + 1)\,\eta.$$

It follows from the resolvent identity that

$$\alpha\,R_\alpha + 1 = A\,R_\alpha.$$

Hence,

$$\psi_\alpha = \lambda^{-1} A\,R_\alpha\eta = \lambda^{-1} R_\alpha^{cl} A\,\eta = \lambda^{-1} R_\alpha^{cl}\,\omega = \lambda^{-1}\,\eta_\alpha.$$

In order to find $\lambda = \lambda(\alpha)$, we return to (37):

$$1 = \langle\varphi_\alpha, \omega\rangle = \alpha\,\lambda^{-1}\langle R_\alpha\eta, \omega\rangle = \alpha\,\lambda^{-1}(\eta_\alpha, \eta).$$

Thus, $\lambda(\alpha) = \alpha\,(\eta_\alpha, \eta)$, and the following theorem is proved:

**Theorem 7.** *For every fixed* $\omega \in \mathcal{H}_-\setminus\mathcal{H}$, $\|\omega\|_- = 1$, *a point* $\alpha \in (-\infty, 1)$ *belongs to the set* $\sigma_p(A_{\lambda,\omega})$ *of the eigenvalues of the operator* $A_{\lambda,\omega}$ *if*

$$\lambda = \lambda(\alpha) = \alpha\,(\eta_\alpha, \eta), \quad \eta = (A^{cl})^{-1}\,\omega, \quad \eta_\alpha = R_\alpha^{cl}\,\omega. \tag{38}$$

*The corresponding eigenvector has the form:*

$$\psi_\alpha = \lambda^{-1}\,\eta_\alpha \in N_\alpha. \tag{39}$$

Theorem 7 admits an immediate generalization to the case where $T : \mathcal{H}_+ \to \mathcal{H}_-$ is a singular finite rank operator, i.e., $\mathcal{D}(T) = \mathcal{H}_+$ and

$$T\,\varphi = \sum_{i=1}^{n} \lambda_i\langle\varphi, \omega_i\rangle\omega_i, \quad \varphi \in \mathcal{H}_+, \tag{40}$$

where $n < \infty$, all $\omega_i$ belong to $\mathcal{H}_-\setminus\mathcal{H}$ and $(\omega_i, \omega_j)_- = \delta_{ij}$.

**Theorem 8.** *For any set of vectors* $\omega_i \in \mathcal{H}_-\setminus\mathcal{H}$, $(\omega_i, \omega_j)_- = \delta_{ij}$, $i, j \le n < \infty$, *which is linearly independent with respect to* $\mathcal{H}$, *and for any set of nonzero numbers* $\alpha_1, \ldots, \alpha_N \in (-\infty, 1)$, *there exists an operator* $\tilde{A} \in \mathcal{A}_S^n(A)$ *such that* $\alpha_1, \ldots, \alpha_n \in \sigma_p(\tilde{A})$. *This operator* $\tilde{A} \equiv A_T$ *is constructed in accordance with Theorem 2 by the singular perturbation* $T$ *of form* (40), *where*

$$\lambda_i \equiv \lambda(\alpha_i) = \alpha_i(\eta_i, \eta_{\alpha_i}) \tag{41}$$

*and*

$$\eta_i = (A^{cl})^{-1}\,\omega_i, \quad \eta_{\alpha_i} = R_{\alpha_i}^{cl}\,\omega_i, \quad i = 1, \ldots, n. \tag{42}$$

*The corresponding eigenvectors have the form:*

$$\psi_{\alpha_i} = \lambda_i^{-1}\eta_{\alpha_i}. \tag{43}$$

*The multiplicities of the eigenvalues* $\alpha_i$, $1 \le i \le n$, *are equal to the number of* $\alpha_j$, $1 \le j \le n$, *with* $\alpha_i = \alpha_j$.

***Remark* 6.** In Theorems 7 and 8, all numbers $\lambda(\alpha)$ and $\lambda(\alpha_i)$ are nonzero. Indeed, for any $\omega \in \mathcal{H}_- \setminus \mathcal{H}$ and $0 \ne \alpha \in (-\infty, 1)$, we have $(\eta, \eta_\alpha) \ne 0$. This can be shown as follows: We set $\omega = (A^{cl} - \alpha)g$, $g \in \mathcal{H}$. Then $(\eta, \eta_\alpha) = ((1 - \alpha A^{-1})g, g) = \|g\|^2 - \alpha(g, A^{-1}g) \ne 0$, since $(g, A^{-1}g) \le 1$.

Now consider the case where $n = \operatorname{rank} T = \dim F = \infty$; here, $F = \mathcal{H}_+ \ominus \ker T$. Then the operator $A$ (see (40)) has infinite deficiency indices. Consider the singularly perturbed operator $\tilde{A}$, which corresponds to $T$ due to Theorem 2. Let us study its additional point spectrum. From the previous construction, one can easily conclude that any sequence of nonzero numbers $\alpha_1, \ldots, \alpha_i, \ldots \in (-\infty, 1)$ with arbitrary multiplicities may form the additional point spectrum of $\tilde{A}$. To demonstrate this, consider the subspace $N_- := A^{cl} A F_{0,T} \subset \mathcal{H}_- \setminus \mathcal{H}$, $\dim N_- = \infty$. We choose in $N_-$ an orthogonal basis $\{\omega_i\}_{i=1}^{\infty}$ and replace $T$ by $T' = \sum_{i=1}^{\infty} \lambda_i < \varphi, \omega_i > \omega_i$, where the coefficients $\lambda_i = \lambda(\alpha_i)$ are determined by (41) and (42). Then the vectors $\psi_{\alpha_i}$ of form (43) are the eigenvectors of $\tilde{A} = A_{T'} \in \mathcal{A}_S(A)$. Thus, we have the following theorem:

**Theorem 9.** *For each closed subspace* $N_- \subset (\mathcal{H}_- \setminus \mathcal{H}) \cup 0$, $\dim N_- = \infty$, *and any sequence of nonzero numbers* $\alpha_1, \ldots, \alpha_i, \ldots \in (-\infty, 1)$, *there exists a singularly perturbed operator* $\tilde{A} = A_{T'} \in \mathcal{A}_S(A)$ *such that* $\alpha_1, \ldots, \alpha_i, \ldots \in \sigma_p(\tilde{A})$. *The corresponding eigenvectors* $\psi_{\alpha_i} \in N_{\alpha_i} = R_{\alpha_i}^{cl} N_-$ *admit representation* (43).

***Remark* 7.** We emphasize that the eigenvectors $\psi_{\alpha_i}$ also satisfy the equations $B_{\alpha_i}\psi_{\alpha_i} = 0$, where the operators $B_{\alpha_i}$ are constructed by using $T$ with the help of (14) and (24).

All our previous results can be generalized to the case where the original operator $A$ does not (necessarily) satisfy the condition $A \ge 1$ but has a gap $(a, b)$, $-\infty < a < b < \infty$.

**5. Inverse Spectral Theory for Self-Adjoint Extensions.** In this section, $A$ denotes a closed symmetric operator in a separable Hilbert space $\mathcal{H}$ with a gap $(a, b)$. We continue our discussion of the following **problem**: *What types of spectrum can the self-adjoint extensions of* $A$ *have within the gap* $(a, b)$?

Without loss of generality, we assume that $0 \in (a, b)$. As in the previous sections, we denote the range of $A$ by $\mathcal{R}$, the boundedly invertible self-adjoint operator in $\mathcal{R}$ introduced in Lemma 1 is denoted by $A_0$, and the orthogonal complement of $\mathcal{R}$ in $\mathcal{H}$ is denoted by $N$. We set

$$K := P_N A^{-1} = P_N P_{\mathcal{R}}^{-1} A_0^{-1} = \Gamma A_0^{-1}.$$

and regard $K$ as an operator from the Hilbert space $\mathcal{R}$ to the Hilbert space $N$. Note that $P_{\mathcal{R}}$ is invertible on $D(A_0)$.

First, let $P$ be an arbitrary orthogonal projector in $N$. We have

$$A^{-1} = \begin{pmatrix} A_0^{-1} \\ (1-P)K \\ PK \end{pmatrix} : \mathcal{R} \to \begin{matrix} \mathcal{R} \\ \oplus \\ \mathcal{L} \\ \oplus \\ \mathcal{P} \end{matrix},$$

where $\mathcal{L}$ and $\mathcal{P}$ denote the ranges of operators $1-P$ and $P$. We set

$$\tilde{L} := \begin{pmatrix} A_0^{-1} & ((1-P)K)^* & (PK)^* \\ (1-P)K & C & 0 \\ PK & 0 & Q \end{pmatrix} : \begin{matrix} \mathcal{R} \\ \oplus \\ \mathcal{L} \\ \oplus \\ \mathcal{P} \end{matrix} \to \begin{matrix} \mathcal{R} \\ \oplus \\ \mathcal{L} \\ \oplus \\ \mathcal{P} \end{matrix},$$

where $C$ and $Q$ are arbitrary self-adjoint operators in the corresponding Hilbert space ($P$, $C$, and $Q$ are free parameters) and $\tilde{L}$ is a self-adjoint operator in $\mathcal{H}$. We have $\tilde{L} \supset \boldsymbol{A}^{-1}$ and, therefore, $\mathcal{R}(\tilde{L}) \supset \mathcal{R}(\boldsymbol{A}^{-1}) = \mathcal{D}(\boldsymbol{A})$. Thus,

$$\ker(\tilde{L}) = \ker(\tilde{L}^*) = \mathcal{R}(\tilde{L})^{\perp} = \{0\},$$

i.e., $\tilde{L}$ is invertible. $\tilde{A} := \tilde{L}^{-1}$ is a self-adjoint extension of $\boldsymbol{A}$.

For brevity, we introduce a self-adjoint operator

$$\hat{A}_0^{-1} := \begin{pmatrix} A_0^{-1} & ((1-P)K)^* \\ (1-P)K & C \end{pmatrix}$$

in the Hilbert space $\mathcal{R} \oplus \mathcal{L}$ and set $\hat{K} := (PK \;\; 0)$. Then

$$\tilde{L} = \begin{pmatrix} \hat{A}_0^{-1} & \hat{K}^* \\ \hat{K} & Q \end{pmatrix}.$$

Up to now, we did not impose any restrictions on the parameters $P$, $C$, and $Q$. In what follows, **we only assume that $C$ is such that $\hat{A}_0^{-1}$ is invertible and $(a, b)$ is a gap of $\hat{A}_0 := (\hat{A}_0^{-1})^{-1}$ and that $Q$ is invertible.** It is known that the operators $C$, which satisfy these conditions, exist (cf., e.g., [10], Theorem 4.7 and Corollary 4.8). We set

$$\hat{L} := \begin{pmatrix} \hat{A}_0^{-1} & 0 \\ 0 & Q \end{pmatrix}.$$

Then $\hat{A} := \hat{L}^{-1} = \hat{A}_0 \oplus Q^{-1}$ is self-adjoint and

$$\sigma_{ac}(\hat{A}) \cap (a, b) = (\sigma_{ac}(\hat{A}_0) \cup \sigma_{ac}(Q^{-1})) \cap (a, b) = \sigma_{ac}(Q^{-1}) \cap (a, b),$$

$$\sigma_{\mathrm{ess}}(\hat{A}) \cap (a, b) = \sigma_{\mathrm{ess}}(Q^{-1})) \cap (a, b).$$

Consider the parameter $P$. In what follows, **we only assume that $P$ is such that $PK$ is nuclear**. Then

$$\tilde{A}^{-1} - \hat{A}^{-1} = \tilde{L} - \hat{L} = \begin{pmatrix} 0 & \hat{K}^* \\ \hat{K} & 0 \end{pmatrix}$$

is also nuclear and, therefore, we have $\sigma_{ac}(\tilde{A}) = \sigma_{ac}(\hat{A})$ and $\sigma_{ess}(\tilde{A}) = \sigma_{ess}(\hat{A})$.

Thus,

$$\sigma_{ac}(\tilde{A}) \cap (a, b) = \sigma_{ac}(Q^{-1}) \cap (a, b), \tag{44}$$

$$\sigma_{\mathrm{ess}}(\tilde{A}) \cap (a, b) = \sigma_{\mathrm{ess}}(Q^{-1})) \cap (a, b). \tag{45}$$

Clearly, these relations are useful only if $\mathcal{P}$ is infinite dimensional (recall that $Q$ is an operator in $\mathcal{P}$; thus, $\sigma_{\mathrm{ess}}(Q^{-1}) = 0$ if of $\mathcal{P}$ is finite dimensional). Thus, we arrive at the problem of examining the conditions under which $P$ can be chosen so that both $PK$ is a nuclear operator and $\dim \mathcal{P} = \infty$. This problem is solved completely answer by the following lemma:

**Lemma 6.** *In the notation introduced at the beginning of this section, the following assertion holds: An orthogonal projector $P$ in the Hilbert space $N$, such that the operator $PK$ is nuclear and* $\dim \mathcal{P} = \infty$, *exists if and only if the operator $\boldsymbol{A}$ is significantly deficient in the following sense:*

**Definition 3.** *A closed symmetric operator $\boldsymbol{A}$ is significantly deficient if and only if*

$$P_z \mathcal{D}(\boldsymbol{A}) \neq N_z \tag{46}$$

*for all regular points $z$ of $\boldsymbol{A}$.*

For the proof of Lemma 6, see [14], Lemmas 3.1 and 3.2. A closed symmetric boundedly invertible operator $\boldsymbol{A}$ is significantly deficient if it has infinite deficiency indices and its inverse $\boldsymbol{A}^{-1}$ is compact. Moreover, if (46) holds for one real regular point, then it holds for all regular points and $\boldsymbol{A}$ is significantly deficient (cf. [13], Section 4; in [13], the notion "significantly symmetric" is used).

**Proposition 1.** *Let $\boldsymbol{A}$ be a significantly deficient operator with a gap $(a, b)$. Let $S_{ac} \subset S_c \subset R^1$ be such that $S_{ac}$ is the support of an absolutely continuous measure and $S_c$ is closed. Then $\boldsymbol{A}$ has a self-adjoint extension $\tilde{A}$ with the following properties:*

(i) $\sigma_{\mathrm{ess}}(\tilde{A}) \cap (a, b) = S_c \cap (a, b)$;

(ii) $\sigma_{ac}(\tilde{A}) \cap (a, b) = S_{ac} \cap (a, b)$.

***Proof.*** Since $\boldsymbol{A}$ is significantly deficient, there exists an orthogonal projector $P$ in $N$ such that $PK$ is nuclear and the range of $P$ is infinite-dimensional. We choose an invertible self-adjoint operator $Q$ in the Hilbert space $\mathcal{P}$ such that

$$\sigma_{\mathrm{ess}}(Q^{-1}) = S_c \quad \text{and} \quad \sigma_{ac}(Q^{-1}) = S_{ac}.$$

By using the argument preceding the statement of Proposition 1, we conclude that $\boldsymbol{A}$ has a self-adjoint extension $\tilde{A}$ such that (44) and (45) hold. The proposition is proved.

**Example.** Let $\Omega$ be a nonempty bounded domain in $R^d$, $d \geq 2$, and

$$\mathcal{D}(A') := C_0^\infty(\Omega), \quad A'f := -\Delta f, \quad f \in C_0^\infty(\Omega),$$

where $\Delta := \sum_{j=1}^d D_j^2$ and $D_j := \partial / \partial x_j$. The closure $\boldsymbol{A}$ of $A'$ is a closed symmetric operator and the real line is covered by the gaps of $\boldsymbol{A}$. Since $d \geq 2$, the operator $\boldsymbol{A}$ has infinite deficiency indices. Since $\Omega$ is bounded, the inverse $\boldsymbol{A}^{-1}$ is compact. Thus, $\boldsymbol{A}$ is significantly deficient and, by Proposition 1, $\boldsymbol{A}$ has self-adjoint extensions with nonempty absolutely continuous spectrum despite the fact that the domain $\Omega$ is bounded!

Proposition 1 can be considerably extended:

**Theorem 10.** *Let $\boldsymbol{A}$ be a significantly deficient symmetric operator with a gap $(a, b)$ in the separable complex Hilbert space $\mathcal{H}$. Let $F_{ac}$ be the support of an ab-*

*solutely continuous positive Radon measure on* $R^1$ *and let* $G_{sc} \subset R^1$ *be an open set such that* $F_{ac} \cap G_{sc} = \varnothing$. *Assume that* $\varepsilon$ *is a countable subset of* $(a, b)$ *and* $p : \varepsilon \to N \cup \{\infty\}$ *is an arbitrary mapping. Then* $\mathbf{A}$ *has a self-adjoint extension* $\tilde{A}$ *with the following properties:*

(i) *let* $\lambda \in (a, b)$. *Then* $\lambda \in \sigma_{ess}(\tilde{A})$ *if and only if* $\lambda \in F_{ac} \cup \overline{G}_{sc}$ *or* $\lambda \in \varepsilon$ *and* $p(\lambda) = \infty$ *or* $\lambda = \lim_{n\to\infty}\lambda_n$ *for some sequence* $\{\lambda_n\}_{n \in N}$ *in* $\varepsilon \setminus \{\lambda\}$;

(ii) $\sigma_{ac}(\tilde{A}) \cap (a, b) = F_{ac} \cap (a, b)$;

(iii) $G_{sc} \subset \sigma_{sc}(\tilde{A}) \cap (a, b) \subset \overline{G}_{sc} \cup \partial F_{ac}$. *In particular,* $\sigma_{sc}(\tilde{A}) \cap (a, b) = \overline{G}_{sc} \cap (a, b)$ *if* $\partial F_{ac}$ *is countable.*

(iv) *every* $\lambda \in \varepsilon$ *is an eigenvalue of* $\tilde{A}$ *whose multiplicity is at least* $p(\lambda)$. *The multiplicity is finite if* $p(\lambda)$ *is finite.*

(v) $\sigma_p(\tilde{A}) \cap F \cap (a, b) = \varepsilon \cap F \cap (a, b)$ *and each* $\lambda \in \varepsilon \cap F \cap (a, b)$ *has the multiplicity* $p(\lambda)$. *Here,* $F := F_{ac} \cup \overline{G}_{sc}$.

We postpone the (very long) proof of this theorem to the forthcoming paper [15].

**6. Singular Perturbations of the Laplacian.** In the applications to mathematical physics, it proves to be quite interesting to construct the self-adjoint extensions of the operator $-\Delta \restriction C_0^\infty(R^d \setminus S)$ in the Hilbert space $\mathcal{H} := L^2(R^d)$ and to study their properties. Here, $S$ is a closed subset of $R^d$ with Lebesgue measure zero and the operator $-\Delta$ in $\mathcal{H}$ is given by

$$\mathcal{D}(-\Delta) := W_2^2(R^d),$$

$$-\Delta f := -\sum_{j=1}^{d} D_j^2 f, \quad f \in \mathcal{D}(-\Delta),$$

where

$$W_2^\alpha(R^d) := \{\mathcal{F}^{-1} g \mid g \in L^2(R^d, (1+p^2)^\alpha)\, dp\}$$

for each $\alpha \in R^1$. ($\hat{g}$ and $\mathcal{F}g$ denote the Fourier transform of $g$; $\check{g}$ and $\mathcal{F}^{-1}g$ stand for its inverse Fourier transform.)

In this section, we define the deficiency subspace of the operator $A \restriction C_0^\infty(R^d \setminus S)$, where $A := -\Delta + 1$. Then the general results in Section 3 can be used when constructing the self-adjoint extensions of the operator $-\Delta \restriction C_0^\infty(R^d \setminus S)$. Furthermore, we discuss the spectral properties of certain singular perturbations of the operator $A$. In this way, we get, in particular, the results concerning the spectral properties of the certain self-adjoint extensions of the operator $-\Delta \restriction C_0^\infty(R^d \setminus S)$. First, we give an explicit description of the objects introduced in Section 3 for the case where $A = -\Delta + 1$.

Since the space $L^2(R^d, (1+p^2)^{-2}\, dp)$ is the completion of the space $\mathcal{H}$ in norm

$$f \mapsto \left(\int |(1+p^2)^{-1} f(p)|^2 dp\right)^{1/2},$$

we get

$$\mathcal{H}_- = W_2^{-2}(R^d),$$

$$(f, g)_- = (\hat{f}, \hat{g})_{L^2(R^d,\, (1+p^2)^{-2} dp)}, \quad f, g \in \mathcal{H}_-.$$

Note that, in general, the elements of $\mathcal{H}_-$ are distributions but not functions. For instance, the $\delta$-distribution $\delta$ belongs to $\mathcal{H}_-$ if the dimensionality $d$ of $R^d$ is at most 3 and even its derivative $\delta'$ is in $\mathcal{H}_-$ if $d = 1$.

For $z \in C\setminus(1, \infty)$, let $G_z$ be the convolution kernel of the operator $(A - z)^{-1}$ in $\mathcal{H}$ and $G := G_0$. Let $\mu$ be a positive Radon measure on $R^d$. If $\mu \in \mathcal{H}_-$, then

$$(A^{cl} - z)^{-1}\mu = G_z * \mu \quad \text{and} \quad A(A - z)^{-1} G * \mu = G_z * \mu \tag{47}$$

If $\mu \in W_2^{-1}(R^d)$, then, for $1 \le j \le d$, we have $D_j\mu \in \mathcal{H}_-$, $G_z * \mu \in W_2^1(R^d)$,

$$(A^{cl} - z)^{-1}(D_j\mu) = D_j(G_z * \mu), \tag{48}$$

and

$$A(A - z)^{-1} D_j(G * \mu) = D_j(G_z * \mu). \tag{49}$$

These assertions can be easily established by passing to the Fourier transforms.

Let $T \in \mathcal{T}_S(A)$, let $\tilde{A} \in \mathcal{A}_S(A)$ be the operator which corresponds to $T$ in the sense of Theorem 3, and let $A$ be the common part of $A$ and $\tilde{A}$. By Theorem 2, $N = A(\mathcal{H}_+ \ominus \ker T)$. Here, $N := N_0$ and $N_z := \ker(A^* - z)$ for each $z \in C\setminus(1, \infty)$.

Suppose that $N$ is spanned by

$$\{G * \mu_n \mid n \in J\} \cup \{D_{j_n}(G * \mu_n) \mid n \in J'\},$$

where $\mu_n$, $n \in J \cup J'$, are positive Radon measures on $R^d$, $\mu_n \in \mathcal{H}_-$ for each $n \in J$, $\mu_n \in W_2^{-1}(R^d)$, and $1 \le j_n \le d$ for each $n \in J'$ ($J$ and $J'$ are disjoint index sets). Then, by relations (47), (48), and (49), the space $N_z$ is spanned by

$$\{G_z * \mu_n \mid n \in J\} \cup \{D_{j_n}(G_z * \mu_n) \mid n \in J'\}$$

for each $z \in \mathbb{C}$, since the operator $A(A - z)^{-1}$ is a bicontinuous bijection from $N$ onto $N_z$.

Consider the special case where $\tilde{A} = A_{\lambda, \omega}$ for some real number $\lambda$ and some $\omega \in \mathcal{H}_-\setminus\mathcal{H}$ with $\|\omega\|_- = 1$, where $\omega$ is a positive Radon measure on $R^d$ or $\omega = D_j\mu$ for some $1 \le j \le d$ and a positive Radon measure $\mu \in W_2^{-1}(R^d)$. By Theorem 6 and relations (47), (48), and (49), we have

$$\tilde{R}_z = R_z + \lambda_z^{-1} P_z, \quad z \in \rho := \rho(A) \cap \rho(\tilde{A}), \tag{50}$$

where

$$\lambda_z = \lambda - z(G * \omega, G_z * \omega), \quad z \in \rho,$$

if $\omega$ is a positive Radon measure and

$$\lambda_z = \lambda - z(D_j(G * \omega), D_j(G_z * \omega)), \quad z \in \rho, \tag{51}$$

otherwise.

It is well known that

$$\sigma_{\text{ess}}(A) = [1, \infty) = \sigma_{ac}(A).$$

Moreover, by Krein's formula, $A^{-1} - \tilde{A}^{-1}$ has a finite rank if the operator $\tilde{A} \in \mathcal{A}_S(A)$ corresponds to a singular perturbation $T$ with finite rank. Thus, in this case, we have

$$\sigma_{\text{ess}}(\tilde{A}) = [1, \infty) = \sigma_{ac}(\tilde{A}).$$

Under weak additional assumptions, one can show that the singular continuous spectrum of $\tilde{A}$ is empty.

**Theorem 11.**

(i) *Assume that* $\mu \in \mathcal{H}_- \setminus \mathcal{H}$ *is a positive Radon measure with compact support and* $\|\mu\|_- = 1$. *Let* $\lambda \in R^1$. *Then the singular continuous spectrum of* $A_{\lambda, \omega}$ *is empty.*

(ii) *Suppose, in addition, that* $\mu \in W_2^{-1}(R^d)$. *Let* $1 \le j \le d$ *and* $\omega := D_j \mu$. *Then the singular continuous spectrum of* $A_{\lambda, \omega}$ *is empty.*

***Proof.*** It suffices to show that there exists an open domain $U$ in $\mathbb{C}$ containing the set $(1, \infty)$ and an analytic function $s : U \to C$ such that $s(z) = \lambda_z$ for all $z \in U$ with $\operatorname{Im}(z) > 0$. In fact, the set $S$ of zeros of $s$ is discrete and $s(\cdot)^{-1}$ is analytic on $U \setminus S$, since the analytic function $s$ is not identically equal to zero. Let $1 < a < b < \infty$ be such that $[a, b] \cap S = \varnothing$. Then we can choose an $\varepsilon > 0$ such that $V \cap S = \varnothing$ and $\overline{V} \subset V$, where

$$V := \{x + iy \mid a \le x \le b,\ 0 < y \le \varepsilon\}$$

and $\overline{V}$ denotes the closure of $V$. Since $\overline{V}$ is a compact subset of $U \setminus S$, we have

$$\sup_{z \in V} |s(z)^{-1}| = \sup_{z \in V} |\lambda_z^{-1}| < \infty, \tag{52}$$

$$\sup_{z \in V} \|P_z\| \le 1 \tag{53}$$

because the operators $P_z$ are orthogonal projectors. Moreover, it is well known (and can be easily shown) that

$$\sup_{z \in V} |(R_z f, f)| < \infty, \quad f \in C_0^\infty(R^d). \tag{54}$$

By Krein's resolvent formula (50) and relations (52), (53), and (54), we obtain

$$\sup_{z \in V} |(\tilde{R}_z f, f)| < \infty, \quad f \in C_0^\infty(R^d).$$

Here, $\tilde{R}_z := (\tilde{A} - z)^{-1}$ and $\tilde{A} := A_{\lambda, \mu}$ (respectively, $\tilde{A} := A_{\lambda, \omega}$). Since $C_0^\infty(R^d)$ is dense in $L^2(R^d)$, the limiting absorption principle gives that.

$$\sigma_{sc}(\tilde{A}) \cap (a, b) = \varnothing.$$

Since $\sigma_{sc}(\tilde{A}) \cap \sigma_{\text{ess}}(\tilde{A}) = [1, \infty)$, this implies that $\sigma_{sc}(\tilde{A}) \subset S$. Since $S$ is countable, we get $\sigma_{sc}(\tilde{A}) = \varnothing$. It remains to prove the existence of an analytic function with the required properties. We give the proof of existence under the hypothesis of assertion (ii). The proof in the other case is virtually the same and even somewhat shorter. We set

$$C^{++} := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1,\ \operatorname{Im}(z) > 0\},$$

$$U := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1,\ \operatorname{Im}(z^{1/2}) < 1/4\},$$

where

$$(re^{i\varphi})^{1/2} := -r^{1/2} e^{i\varphi/2}, \quad r > 0, \quad -\pi < \varphi < \pi.$$

We set

$$\tilde{G}_z(x) := (2\pi)^{-d/2}(|x|/(i(z-1)^{1/2}))^{1-d/2} K_{d/2-1}(i(z-1)^{1/2}|x|),$$

$$x \in R^d, \quad x \neq 0, \quad z \in U,$$

where $K_\alpha$ denotes the modified Bessel function of the second kind. Then $\tilde{G}_z = G_z$ for $z \in C^{++}$ (but $\tilde{G}_z \notin L^2(R^d)$, in particular, $\tilde{G}_z \neq G_z$ if $z \in U \setminus C^{++}$). We set

$$D(x) := D_j G(x) \quad \text{and} \quad D_z(x) := D_j \tilde{G}_z(x), \quad x \in R^d, \quad x \neq 0, \quad z \in U.$$

The function $z \mapsto D_z(x)$ is analytic on $U$ for each $x \in R^d$, $x \neq 0$. Let $V$ be a non-empty set such that its closure $\overline{V}$ is a compact subset of $U$ and $B = \{x \in R^d \mid |x| \leq \leq a\}$ for some $0 < a < \infty$. There exist strictly positive finite constants $c$ and $C$ such that

$$\sup_{z \in V} \left( |D_z(x)| + \left| \frac{\partial}{\partial z} D_z(x) \right| + |D(x)| \right) \leq C|x|^{1-d} \quad \text{on } B \tag{55}$$

and

$$\sup_{z \in V} \left( |D_z(x)D(x)| + \left| \frac{\partial}{\partial z} D_z(x)D(x) \right| \right) \leq C|x|^{-c|z|} \quad \text{on } R^d \setminus B. \tag{56}$$

Let $J$ be the function on $L^1(R^d)$ satisfying

$$\hat{J}(p) = (1 + p^2)^{-1/2} \quad \text{a.e.}$$

Since $\mu \in W_2^{-1}(R^d)$, we have

$$J * \mu \in L^2(R^d). \tag{57}$$

It is known (cf. [44], Chapter V.3) that

$$J(x) \geq 0 \quad \text{a.e.} \quad \text{and} \quad J(x) \geq C'|x|^{1-d} \quad \text{a.e. on } B \tag{58}$$

for some strictly positive constant $C'$. It easily follows from (55) – (57) and (58) that the functions

$$(x, y, y') \mapsto |D_z(x-y)D(x-y')| + \left| \frac{\partial}{\partial z} D_z(x-y)D(x-y') \right|,$$

$$x, y, y' \in R^d, \quad z \in V,$$

admit a $\lambda^d \otimes \mu \otimes \mu$-integrable majorant ($\lambda^d$ denotes the Lebesgue measure) and

$$D * \mu = D_j(G * \mu) \quad \text{and} \quad D_z * \mu = \overline{D_j(G_z * \mu)} \quad \lambda^d\text{-a.e.} \tag{59}$$

for each $z \in C^{++}$ (in the last equality, we have used the fact that $\overline{G}_z = G_z \lambda^d$-a.e.). Thus, the function $s : U \to C$,

$$s(z) := \lambda - z \int_{R^d} \int_{R^d} D_z(x-y)\mu(dy) \int_{R^d} D(z-y')\mu(dy')\,dx,$$

is defined and analytic and, by (51) and (59), $s(z) = \lambda_z$ for each $z \in C^{++}$. Thus, the theorem is proved.

**Remark 8.** Theorem 11 can be easily extended to the case where the singular perturbation $T$ has a finite rank and the underlying Radon measures are not necessarily positive.

The deficiency subspace of the operator $A \upharpoonright C_0^\infty(R^d \backslash S)$ is given by the following theorem (cf. [9]).

**Theorem 12.** *Let $S$ be a closed subset of $R^d$. Then the deficiency subspace of $A \upharpoonright C_0^\infty(R^d \backslash S)$ is the closed span of*

$$\{G * \mu \mid \mu \in M_2(S)\} \cup \bigcup_{j=1}^{d} \{D_j(G * \mu) \mid \mu \in M_1(S)\}.$$

*Here, $M_j(S)$ denotes the set of positive Radon measures $\mu \in W_2^{-j}(R^d)$ with compact support in $S$, $j = 1, 2$.*

We refer to [2], Chapter 6, [24], and [9] for the related results in the extension theory in $L^2$-spaces and its generalizations.

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