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Surfaces of Revolution with Vanishing Curvature in Galilean 3-Space

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In the paper, three types of surfaces of revolution in the Galilean 3-space are defined and studied. The construction of the well-known surface of revolution, defined as the trace of a planar curve rotated about an axis in the supporting plane of the curve, is given for the Galilean 3-space. Then we classify the surfaces of revolution with vanishing Gaussian curvature or vanishing mean curvature in the Galilean 3-space.

Key words: surface of revolution, flat surface, minimal surface, Galilean 3-space.

Mathematical Subject Classification 2010: 53A10, 53A35, 53A40.

1. Introduction

Together with the ruled surfaces, the surfaces of revolution were among the first studied subjects in differential geometry, besides the curves. Being the trace of a rotated planar curve, the surfaces of revolution are also to be found widely in the "real world". The catenoid, obtained by rotating a catenary, is one of the elementary minimal surfaces. Although surfaces of revolution or rotational surfaces have been studied thoroughly for centuries, we believe we can make a modest contribution to this research area by defining and studying them in a Galilean 3-space. The geometry of this non-Euclidean space was first studied intensively by Röschel [11]. In the last decade, this space was used by several researchers as an ambient space for the well-known Euclidean concepts (see [2,3, 7,9,10] for more examples on special surfaces). The study of the surfaces in the Galilean 3-space can also be found in [1].

In [7], two types of surfaces of revolution in the Galilean 3-space are used as examples. It is pointed out that they are Weingarten surfaces, that is, there exists a non-trivial functional dependence between the Gaussian curvature and the mean curvature. Further, no systematic studies of the curvatures of these surfaces have been found. The authors of this paper are aimed to study twisted surfaces, which are surfaces that are traced out by a planar curve on which two simultaneous rotations are performed, and hence which are a generalization of surfaces of revolution (see [6] and the references therein). We did it first in [4]. To the best of our knowledge this has not been done before.

In this paper, we first recall the necessary preliminaries on the Galilean space. Then we define three different types of surfaces of revolution in the Galilean space and study the curvature properties of these surfaces. We give the classification theorems of surfaces of revolution with vanishing Gaussian curvature or vanishing mean curvature.

2. Preliminaries

For an in-depth study of the Galilean 3-space, see [11]. Here, we recall the properties that we need from this work.

The Galilean 3-space $\mathbb{G}^3$ arises in a Cayley–Klein way by pointing out an absolute figure $\{\omega, f, I\}$ in the 3-dimensional real projective space. Here ω is the absolute plane, f is the absolute line and I is the fixed elliptic involution of points of f. Then the homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ are introduced such that ω is given by $x_0 = 0$, f is given by $x_0 = x_1 = 0$ and I, by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : -x_2)$. The group of motions of $\mathbb{G}^3$ is a six-parameter group. Regarding this group of motions, except the absolute plane, there exist two classes of planes in $\mathbb{G}^3$: *Euclidean planes* that contain f and in which the induced metric is Euclidean and *isotropic planes* that do not contain f and in which the induced metric is isotropic. Also, there are four types of lines in $\mathbb{G}^3$: *isotropic lines* that intersect f, *non-isotropic lines* that do not intersect f, non-isotropic lines in ω and the absolute line f.

In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x_1 : x_2 : x_3)$, the distance between two points $P_i = (x_i, y_i, z_i)$ with $i \in \{1, 2\}$ is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| & \text{if } x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2} & \text{if } x_1 = x_2. \end{cases}$$

The vector $\vec{a} = (x, y, z)$ is isotropic if $x = 0$ and non-isotropic otherwise. Hence, for standard coordinates (x, y, z), the x-axis is non-isotropic while the y-axis and the z-axis are isotropic. The yz-plane, $x = 0$, is Euclidean and the xy-plane and the xz-plane are isotropic. The Galilean scalar product of two vectors $\vec{a} = (x, y, z)$ and $\vec{b} = (x_1, y_1, z_1)$ is defined by

$$\langle \vec{a}, \vec{b} \rangle = \begin{cases} xx_1 & \text{if } x \neq 0 \text{ or } x_1 \neq 0, \\ yy_1 + zz_1 & \text{if } x = x_1 = 0. \end{cases}$$

The vector $\vec{a}$ is a unit vector if $\|\vec{a}\| := \sqrt{\langle \vec{a}, \vec{a} \rangle} = 1$. In [7], the Galilean cross product of two vectors $\vec{a} = (x, y, z)$ and $\vec{b} = (x_1, y_1, z_1)$ is defined as

$$\vec{a} \wedge \vec{b} = \begin{vmatrix} 0 & e_2 & e_3 \\ x & y & z \\ x_1 & y_1 & z_1 \end{vmatrix}.$$

In order to define surfaces of revolution, we need the two types of rotations in $\mathbb{G}^3$. A *Euclidean rotation* about the non-isotropic x-axis is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where θ is the Euclidean angle. An *isotropic rotation* is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} c\theta \\ \frac{c}{2}\theta^2 \\ 0 \end{pmatrix},$$

where θ is the isotropic angle and $c \in \mathbb{R}_0$. Here the bundle of fixed planes is given by $z = \text{const}$ [8].

Finally, to calculate the curvatures, we have to be able to perform patch computations for a surface in $\mathbb{G}^3$. If a surface in $\mathbb{G}^3$ is parameterized by

$$\varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)),$$

then we denote the first-order derivatives for $i \in \{1, 2\}$ by $\varphi_{,i} = \frac{\partial\varphi}{\partial v^i}(v^1, v^2)$ [2,11]. Here we will always assume that the surfaces are *admissible*, that is, the tangent plane is nowhere Euclidean.

The unit normal vector N of the surface is defined by

$$N = \frac{\varphi_{,1} \wedge \varphi_{,2}}{w} \qquad \text{where } w = \|\varphi_{,1} \wedge \varphi_{,2}\|.$$

The coefficients of the second fundamental form are given by

$$L_{ij} = \langle \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{x_{,1}}, N \rangle = \langle \frac{\varphi_{,ij} x_{,2} - x_{,ij} \varphi_{,2}}{x_{,2}}, N \rangle. \tag{2.1}$$

The Gaussian curvature K and the mean curvature H of the surface are defined in [11], analogously to the Euclidean space, by

$$K = \frac{L_{11} L_{22} - L_{12}^2}{w^2} \qquad \text{and} \qquad 2H = \sum_{i,j=1}^{2} g^{ij} L_{ij}, \tag{2.2}$$

where

$$g^1 = \frac{x_{,2}}{w}, \quad g^2 = -\frac{x_{,1}}{w}, \quad \text{and} \quad g^{ij} = g^i g^j \qquad \text{for } i, j \in \{1, 2\}.$$

As proved in [11, Satz 19.5, p. 107], the mean curvature at the point p of a surface in $\mathbb{G}^3$ is the curvature of intersection of the surface with the Euclidean plane that contains the point p.

3. Surfaces of revolution in the Galilean 3-space

We construct a surface of revolution in the Galilean 3-space analogously to that constructed in the Euclidean 3-space.

Definition 3.1. A *surface of revolution* in $\mathbb{G}^3$ is a surface that is traced out by a planar curve, the *profile curve*, rotated in $\mathbb{G}^3$. The rotation is either a Euclidean rotation about an axis in the supporting plane of the profile curve, or an isotropic rotation for which a bundle of fixed planes is chosen.

Because of the existence of different kinds of planes in $\mathbb{G}^3$, we consider two possibilities for the supporting plane of the profile curve of a surface of revolution in $\mathbb{G}^3$: the profile curve lies either in a Euclidean plane, or in an isotropic plane.

Since a Euclidean plane contains only isotropic vectors, while an isotropic plane contains both isotropic and non-isotropic vectors, there are three types of surfaces of revolution to be defined in $\mathbb{G}^3$.

3.1. Type I surfaces of revolution in $\mathbb{G}^3$.

Without losing generality, we can assume that the profile curve α lies in the Euclidean yz-plane and it is parameterized by $\alpha(t) = (0, f(t), g(t))$ where f and g are real functions. On this profile curve, we perform an isotropic rotation with $c \in \mathbb{R}_0$, for instance,

$$\begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f(t) \\ g(t) \end{pmatrix} + \begin{pmatrix} cs \\ \frac{c}{2}s^2 \\ 0 \end{pmatrix}.$$

Thus, up to a transformation, a type I surface of revolution in $\mathbb{G}^3$ is parameterized by

$$\varphi(s,t) = \left(cs, f(t) + \frac{c}{2}s^2, g(t)\right).$$

This type I surface of revolution is one of the two kinds of surfaces of revolution that are mentioned as examples in [7].

3.2. Type II surfaces of revolution in $\mathbb{G}^3$.

Now we assume, again without losing generality, that the profile curve α lies in the isotropic xy-plane and it is parameterized by $\alpha(t) = (f(t), g(t), 0)$ where f and g are real functions. Also on this profile curve we perform an isotropic rotation with $c \in \mathbb{R}_0$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{pmatrix} \begin{pmatrix} f(t) \\ g(t) \\ 0 \end{pmatrix} + \begin{pmatrix} cs \\ 0 \\ \frac{c}{2}s^2 \end{pmatrix}.$$

Then, up to a transformation, a type II surface of revolution in $\mathbb{G}^3$ is parameterized by

$$\varphi(s,t) = \left(f(t) + cs, g(t), sf(t) + \frac{c}{2}s^2\right).$$

As far as we know, this type of surfaces of revolution in $\mathbb{G}^3$ has not been defined before.

3.3. Type III surfaces of revolution in $\mathbb{G}^3$. We start again with a profile curve $\alpha(t) = (f(t), g(t), 0)$ in the isotropic xy-plane, where f and g are real functions, but this time we perform a Euclidean rotation about the x-axis on it,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix} \begin{pmatrix} f(t) \\ g(t) \\ 0 \end{pmatrix}.$$

So, up to a transformation, a type III surface of revolution in $\mathbb{G}^3$ is parameterized by

$$\varphi(s,t) = (f(t), g(t)\cos s, -g(t)\sin s).$$

These type III surfaces of revolution are also used as examples in [7].

4. Zero curvature surfaces of revolution in $\mathbb{G}^3$

For the three types of surfaces of revolution in $\mathbb{G}^3$, we calculate the Gaussian curvature and the mean curvature, and we examine when they vanish. As in the Euclidean 3-space, the surfaces with vanishing Gaussian curvature are called *flat* and the surfaces with vanishing mean curvature are called *minimal*. From [11], we recall the following important theorem which classifies all minimal surfaces in $\mathbb{G}^3$.

Theorem 4.1 ([11])**.** *The minimal surfaces in the Galilean 3-space are cones whose vertices are on the absolute line f and the ruled surfaces of type C that are conoidal surfaces having the absolute line f as the directional line at infinity.*

A type C ruled surface can be parameterized as follows:

$$\varphi(s,t) = (s, f(s), 0) + t\,(0, \beta_2(s), \beta_3(s))$$

where f, β_2 and β_3 are three times continuous differentiable real functions such that $\beta_2(s)^2 + \beta_3(s)^2 = 1$.

4.1. Zero curvature type I surfaces of revolution in $\mathbb{G}^3$. For a type I surface of revolution in $\mathbb{G}^3$, parametrized by

$$\varphi(s,t) = \left(cs, f(t) + \frac{c}{2}s^2, g(t)\right) \tag{4.1}$$

with $c \in \mathbb{R}_0$, the partial derivatives of type I surfaces of revolution are obtained as

$$\varphi_s = (c, cs, 0), \varphi_t = (0, f'(t), g'(t)),$$

where primes denote derivative respect to t. It follows that

$$w = \|\varphi_s \wedge \varphi_t\| = |c|\sqrt{f'(t)^2 + g'(t)^2}.$$

By using the partial derivatives of type I surfaces of revolution and (2), one calculates

$$g^{11} = g^{12} = 0, \qquad g^{22} = \frac{1}{f'(t)^2 + g'(t)^2}. \tag{4.2}$$

From (2.1), we have the coefficients of the second fundamental form II as

$$L_{11} = \operatorname{sgn}(c)\frac{-cg'(t)}{\sqrt{f'(t)^2+g'(t)^2}}, \tag{4.3}$$

$$L_{12} = 0, \tag{4.4}$$

$$L_{22} = \operatorname{sgn}(c)\frac{f'(t)g''(t)-f''(t)g'(t)}{\sqrt{f'(t)^2+g'(t)^2}}. \tag{4.5}$$

Here, by sgn we mean the sign function. The substituting of (4.3)–(4.5) and (4.2) into (2.2) gives

$$K = \frac{g'(f''g'-f'g'')}{c(f'^2+g'^2)^2} \quad \text{and} \quad H = \operatorname{sgn}(c)\frac{f'g''-f''g'}{2(f'^2+g'^2)^{3/2}}.$$

Here and in the remainder of the paper we often drop the parameter of the functions f and g for reasons of readability. Since the surface is admissible, then f' and g' can not be both identically zero.

Further, it is immediate that the Gaussian curvature is identically zero if and only if the mean curvature is identically zero. Hence, a type I surface of revolution in $\mathbb{G}^3$ is flat if and only if it is minimal. Then the following classification theorem is valid.

Theorem 4.2. *A type I surface of revolution in the Galilean 3-space is flat or, equivalently, minimal, if and only if it is either*

1) *a parabolic cylinder parameterized by*

$$\varphi(s,t) = \left(cs, a+\frac{c}{2}s^2, g(t)\right), \tag{4.6}$$

2) *a part of an isotropic plane, consisting of a family of parabolas, parameterized by*

$$\varphi(s,t) = \left(cs, f(t)+\frac{c}{2}s^2, a\right), \tag{4.7}$$

3) *or a parabolic cylinder parameterized by*

$$\varphi(s,t) = \left(cs, f(t)+\frac{c}{2}s^2, af(t)+b\right). \tag{4.8}$$

Here $a, b, c \in \mathbb{R}$ with $c \neq 0$ and $a \neq 0$ in parameterization (4.8).

Proof. From the expressions for the Gaussian curvature and the mean curvature, it is clear that it is sufficient that $f'g''-f''g'=0$.

If $f'=0$ or $g'=0$, then we obtain parameterization (4.6) or (4.7), respectively.

If $f' \neq 0$ and $g' \neq 0$, one has $\frac{f''}{f'} = \frac{g''}{g'}$. Integrating this equation, we obtain $g(t) = af(t)+b$, where $a \neq 0$ and b are real integration constants. Therefore, the profile curve is an isotropic straight line and this leads to parameterization (4.8).

Conversely, it is calculated immediately that the surfaces given by the parameterizations in the statement are flat and minimal. □

The following corollary is immediate.

Corollary 4.3. *A type I surface of revolution in the Galilean 3-space is flat, or equivalently, minimal, if and only if its profile curve is an isotropic straight line.*

Remark 4.4. It is easy to see that in parameterization (4.6) the components satisfy $y = a + \frac{x^2}{2c}$. Similarly, in parameterization (4.8) a straightforward calculation shows that $y = \frac{z-b}{a} + \frac{x^2}{2c}$.

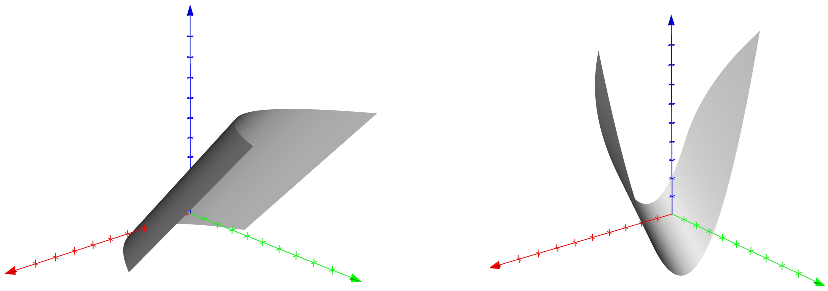
Moreover, reparameterize (4.6) such that $g(t) = v$. Then the resulting parameterization describes a type C ruled surface

$$\varphi(s, v) = \left(cs, a + \frac{c}{2}s^2, 0\right) + v(0, 0, 1).$$

One can proceed similarly for parameterization (4.7). Also, reparameterize (4.8) such that $f(t) = v$, then it becomes

$$\varphi(s, v) = \left(cs, \frac{c}{2}s^2, b\right) + v(0, 1, a).$$

Thus, this is a conoidal surface having the absolute line f as the directional line at infinity. Hence, it is a type C ruled surface. For a drawing of a surface parameterized by parameterization (4.8), see Figure 4.1a.



(a) A flat and minimal type I surface of revolution, parameterized by (4.8) with $a = 1$, $b = c = 2$ and f any function.

(b) A flat and minimal type II surface of revolution, parameterized by (4.10) with $a = c = 1$ and $b = -2$.

Fig. 4.1: Flat and minimal type I and type II surfaces of revolution.

Remark 4.5. As mentioned in [7], since the Gaussian curvature and the mean curvature of type I surfaces of revolution are the functions of one variable only, these type I surfaces of revolution are Weingarten surfaces. Indeed, $\operatorname{sgn}(c)c\left(f'^2 + g'^2\right)^{1/2} K = 2g'H$.

4.2. Zero curvature type II surfaces of revolution in $\mathbb{G}^3$. A type II surface of revolution in $\mathbb{G}^3$ parametrized by

$$\varphi(s, t) = \left(f(t) + cs, g(t), sf(t) + \frac{c}{2}s^2\right)$$

with $c \in \mathbb{R}_0$, using (2.2), is calculated to have the Gaussian curvature and the mean curvature

$$K = \frac{c^2 g'}{w^4}\left[f(f'g'' - f''g') - f'^2 g'\right]$$

and

$$H = \frac{c^2}{2w^3}\left[f(f'g'' - f''g') - f'^2 g'\right].$$

Here $w^2 = f^2 f'^2 + c^2 g'^2$. Again, since the surface is admissible, then f' and g' can not be both identically zero.

Also now, the Gaussian curvature is identically zero if and only if the mean curvature is identically zero. Thus, a type II surface of revolution in $\mathbb{G}^3$ is flat if and only if it is minimal. The following classification theorem can be proved similarly to that for the type I surfaces of revolution.

Theorem 4.6. *A type II surface of revolution in the Galilean 3-space is flat or, equivalently, minimal, if and only if it is either*

1) *a part of an isotropic plane, consisting of a family of parabolas, parameterized by*

$$\varphi(s,t) = \left(f(t) + cs, a, sf(t) + \frac{c}{2}s^2\right),$$

2) *a parabolic cylinder parameterized by*

$$\varphi(s,t) = \left(a + cs, g(t), as + \frac{c}{2}s^2\right), \tag{4.9}$$

3) *or a cyclic surface (parabolic sphere) parameterized by*

$$\varphi(s,t) = \left(t + cs, at^2 + b, st + \frac{c}{2}s^2\right), \tag{4.10}$$

where $a, b, c \in \mathbb{R}$ and $c \neq 0$.

In order to find parameterization (4.10), in Theorem 4.6 we assume that the profile curve is parameterized by arclength, then $f(t) = t$.

Remark 4.7. A straightforward calculation shows that the components of parameterization (4.9) satisfy

$$z = \frac{x^2}{2c} - \frac{a^2}{2c}.$$

Similarly, for the components of parameterization (4.10) one sees that

$$x^2 - \frac{y}{a} - 2cz + \frac{b}{a} = 0.$$

Again, reparameterize (4.9) using $g(t) = v$. Then we get a conoidal surface that has the absolute line f as a directional line at infinity. Hence, it is a type C ruled surface

$$\varphi(s,v) = \left(a + cs, 0, as + \frac{c}{2}s^2\right) + v(0,1,0).$$

Similarly, parameterize (4.10), setting $u = t + cs$ and $v = -s\left(t + \frac{c}{2}s\right)$. Then we have

$$\varphi(u, v) = \left(u, au^2 + b, 0\right) + v(0, ac, -1).$$

Remark that in parameterization (4.10) the profile curve is an isotropic circle, see [11]. In a Galilean space this parameterization represents a cyclic surface constructed analogously to a Euclidean sphere, see [12]. For a drawing of the surface, see Figure 4.1b.

Remark 4.8. Also for type II surfaces of revolution, the Gaussian curvature and the mean curvature are the functions of one variable only, therefore, type II surfaces of revolution are Weingarten surfaces. Indeed, $wK = 2g'H$.

4.3. Zero curvature type III surfaces of revolution in $\mathbb{G}^3$. For a type III surface of revolution in $\mathbb{G}^3$ given by the parametrization

$$\varphi(s, t) = (f(t), g(t)\cos s, -g(t)\sin s)\,,$$

using (2.2), it is calculated that

$$K = \frac{f''g' - f'g''}{f'^3 g} \quad \text{and} \quad H = \frac{\operatorname{sgn}(f'g)}{2g}.$$

Here f' should be non-zero in order to have an admissible surface. Also the function g must be non-zero of course.

For these type III surfaces of revolution in $\mathbb{G}^3$, the flat and minimal conditions are not equivalent. Moreover, it is immediate that there do not exist minimal type III surfaces of revolution in $\mathbb{G}^3$.

This time the proof similar to that of Theorem 4.2 leads to the classification of flat type III surfaces of revolution in $\mathbb{G}^3$.

Theorem 4.9. *A type III surface of revolution in the Galilean 3-space is flat if and only if it is either*

1) *a cylinder over a Euclidean circle parameterized by*

$$\varphi(s, t) = (f(t), a\cos s, -a\sin s)\,, \tag{4.11}$$

2) *or a circular cone with vertex $(b, 0, 0)$ parameterized by*

$$\varphi(s, t) = (ag(t) + b, g(t)\cos s, -g(t)\sin s)\,,$$

where $a, b \in \mathbb{R}$ with $a \neq 0$.

Remark 4.10. We can reparameterize (4.11) by setting $f(t) = v$, to obtain

$$\varphi(s, v) = (0, a\cos s, -a\sin s) + v(1, 0, 0).$$

In [11], a surface of this kind is called a type B ruled surface.

Remark 4.11. As mentioned in [7], since the Gaussian curvature and the mean curvature of type III surfaces of revolution are the functions of one variable only, type III surfaces of revolution are Weingarten surfaces. Indeed, $\operatorname{sgn}(f'g)f'^2K = 2(f''g' - f'g'')H$.

Only for type III surfaces of revolution it is possible to determine which surfaces are of constant mean curvature. In this case, the function g is a non-zero constant function. Hence the surfaces considered are flat. Moreover, they are cylinders over Euclidean circles as in parameterization (4.11) of Theorem 4.9.

5. Conclusion and further research

In this work, we defined three types of surfaces of revolution in the Galilean 3-space and studied when they are flat or minimal. For type I and type II surfaces, the flat and minimal conditions are equivalent. There do not exist minimal type III surfaces of revolution in $\mathbb{G}^3$.

Analogously to how a Minkowski 3-space relates to a Euclidean 3-space, one has the notion of pseudo-Galilean 3-space $\mathbb{G}_1^3$. Without going into detail here, $\mathbb{G}_1^3$ is similar to $\mathbb{G}^3$, but the pseudo-Galilean scalar product of two vectors $\vec{a} = (x, y, z)$ and $\vec{b} = (x_1, y_1, z_1)$ is defined by

$$\langle \vec{a}, \vec{b} \rangle = \begin{cases} xx_1 & \text{if } x \neq 0 \text{ or } x_1 \neq 0, \\ yy_1 - zz_1 & \text{if } x = x_1 = 0. \end{cases}$$

Therefore, there exist four types of isotropic vectors $\vec{a} = (0, y, z)$ in $\mathbb{G}_1^3$: spacelike vectors (if $y^2 - z^2 > 0$), timelike vectors (if $y^2 - z^2 < 0$) and two types of lightlike vectors (if $y = \pm z$). In the same way as in a Minkowski 3-space, hyperbolic functions have to be used instead of trigonometric functions to describe rotations. One can also define different types of surfaces of revolution in $\mathbb{G}_1^3$. It was done in [8] but one type seems to be lost there. Thus, this could be the subject of further research, as well as a more elaborate study of the constancy of the curvatures of surfaces of revolution in the Galilean and pseudo-Galilean 3-spaces. See also [5] and [13], where the Gauss map of two types of surfaces of revolution in $\mathbb{G}_1^3$ is studied.

The figures of the paper are made by using VisuMath. For more information see www.visumath.be.

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Поверхні обертання з нульовою кривиною у тривимірному просторі Галілея

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У статті визначено та досліджено три типи поверхонь обертання у тривимірному просторі Галілея. Запропоновано конструкцію поверхні обертання у тривимірному просторі Галілея, визначеної обертанням пласкої кривої навколо осі, що лежить у площині кривої. Класифіковано поверхні обертання у тривимірному просторі Галілея з нульовою гауссовою кривиною та з нульовою середньою кривиною.

Ключові слова: поверхня обертання, пласка поверхня, мінімальна поверхня, тривимірний простір Галілея.