Journal of Mathematical Physics, Analysis, Geometry
2018, Vol. 14, No. 1, pp. 54–66
doi: https://doi.org/10.15407/mag14.01.054

The Interaction of the Maxwell Flows of General Form for the Bryan–Pidduck Model

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The interaction between the two Maxwell flows of general form in a gas of rough spheres is studied. The approximate solution of the Bryan–Pidduck equation describing the interaction is a bimodal distribution with specially selected coefficient functions. It is shown that under certain additional conditions imposed on these functions and hydrodynamic parameters of the flows, the norm of the difference between the parts of the Bryan–Pidduck equation can be arbitrarily small.

Key words: rough spheres, Bryan–Pidduck equation, error, Maxwellian flows, bimodal distribution, hydrodynamic parameters.

Mathematical Subject Classification 2010: 76P05, 45K05, 82C40, 35Q55.

1. Statement of the problem

This article describes a model of rough spheres [4] first introduced by Bryan in 1894. The methods developed by Chapman and Enskog for general non-rotating spherical molecules were extended to Bryan's model by Pidduck in 1922. The advantage of the model over all other variably rotating models is that no variables specifying its orientation in the space are required.

The statement that the molecules are perfectly elastic and perfectly rough is to be interpreted as follows. When two molecules collide, the points which come into contact will not, in general, possess the same velocity. It is supposed that the two spheres grip each other without slipping; first each sphere is strained by the other, and then the strain energy is reconverted into kinetic energy of translation and rotation, no energy being lost; the effect is that the relative velocity of the spheres at their point of contact is reversed by the impact.

The model is applied to monatomic molecules and taking into account its ability to rotate, is considered to be more physical than the model of hard spheres and thus more interesting to explore.

The Boltzmann equation for the model of rough spheres (or the Bryan–Pidduck equation) has the form [3, 4, 6, 7]:

$$D(f) = Q(f, f); \tag{1}$$

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$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right); \tag{2}$$

$$Q(f,f) \equiv \frac{d^2}{2} \int_{R^3} dV_1 \int_{R^3} d\omega_1 \int_{\Sigma} d\alpha B(V - V_1, \alpha)\big[f(t, V_1^*, x, \omega_1^*) f(t, V^*, x, \omega^*) - f(t, V, x, \omega) f(t, V_1, x, \omega_1)\big]. \tag{3}$$

Here d is the diameter of the molecule, which is associated with the moment of inertia I by the relation

$$I = \frac{bd^2}{4}, \tag{4}$$

where b, $b \in \left(0, \frac{2}{3}\right]$, is the parameter characterizing the isotropic distribution of matter inside the gas particles; t is the time; $x = (x^1, x^2, x^3) \in R^3$ is the spatial coordinate; $V = (V^1, V^2, V^3)$ and $w = (w^1, w^2, w^3) \in R^3$ are the linear and angular velocities of the molecule, respectively; $\frac{\partial f}{\partial x}$ is the gradient of the function f of the variable x; Σ is the unit sphere in the space R^3; α is the unit vector of R^3 directed along the line connecting the centers of the colliding molecules;

$$B\left(V - V_1, \alpha\right) = |(V - V_1, \alpha)| - (V - V_1, \alpha) \tag{5}$$

is the collision term.

The linear (V^*, V_1^*) and angular (w^*, w_1^*) molecular velocities after the collision can be expressed by the appropriate values before the collision:

$$V^* = V - \frac{1}{b+1}\left(b(V_1 - V) - \frac{bd}{2}\alpha \times (\omega + \omega_1) + \alpha(\alpha, V_1 - V)\right),$$
$$V_1^* = V_1 + \frac{1}{b+1}\left(b(V_1 - V) - \frac{bd}{2}\alpha \times (\omega + \omega_1) + \alpha(\alpha, V_1 - V)\right),$$
$$\omega^* = \omega + \frac{2}{d(b+1)}\left\{\alpha \times (V - V_1) + \frac{d}{2}\left[\alpha(\omega + \omega_1, \alpha) - \omega - \omega_1\right]\right\},$$
$$\omega_1^* = \omega_1 + \frac{2}{d(b+1)}\left\{\alpha \times (V - V_1) + \frac{d}{2}\left[\alpha(\omega + \omega_1, \alpha) - \omega - \omega_1\right]\right\},$$

where the symbol $\times$ indicates the vector product. These formulas can be obtained using the laws of conservation of momentum, the total energy of translational and rotational motion (for the first time they were given in [1]).

As is known, the general form of the Maxwellian solution of the Boltzmann equation for the model of hard spheres was obtained in [5,8,11], and its description and study can also be found in [2,9,12]. A similar problem for the Bryan–Pidduck model was finally solved in [9].

In [9], it is shown that the most general form of local Maxwellians, which is feasible for the Bryan–Pidduck model, has the form

$$M_i = \rho_i I^{3/2} \left(\frac{\beta_i}{\pi}\right)^3 e^{-\beta_i\left(\left(V - \overline{V}_i\right)^2 + I\omega^2\right)}, \tag{6}$$

where ρ_i is the gas density (here and throughout what follows, the index i takes values 1 and 2) which has the following analytical representation:

$$\rho_i = \rho_{0i} e^{\beta_i \left(\overline{\omega}_i^2 r_i^2 - 2\overline{w}_i x\right)}, \tag{7}$$

ρ_{0i} is the positive constant, $\beta_i = \frac{1}{2T_i}$ is the value inverse to the temperature T_i, $\overline{\omega}_i$ is the angular velocity of the gas flow; r_i^2 denotes the scalar expression

$$r_i^2 = \frac{1}{\overline{\omega}_i^2} \left[\overline{\omega}_i \times (x - \overline{x}_{0i} - \overline{u}_{0i} t)\right]^2 ; \tag{8}$$

the mass velocity of molecules $\overline{V}_i$ has the form

$$\overline{V}_i = \widehat{V}_i + \overline{w}_i t + \left[\overline{\omega}_i \times (x - x_{0i} - \overline{u}_{0i} t)\right], \tag{9}$$

the vector $\overline{u}_{0i} \perp \overline{\omega}_i$, the axis of speed x_{0i} and the density $\overline{x}_{0i}$ at the moment of time $t = 0$ have the form

$$x_{0i} = \frac{1}{\overline{\omega}_i^2} \left[\overline{\omega}_i \times \widetilde{V}_i\right], \quad \overline{x}_{0i} = \frac{1}{\overline{\omega}_i^2} \left[\overline{\omega}_i \times \left(\widetilde{V}_i - \overline{u}_{0i}\right)\right], \tag{10}$$

$\widetilde{V}_i$ are the arbitrary constant vectors of the space R^3, but arbitrary vectors $\widehat{V}_i, \overline{w}_i$ are parallel to the angular velocity $\overline{\omega}_i$.

We consider the problem of constructing the approximate solution of the Bryan–Pidduck equations (1)–(3) in the form of a bimodal distribution

$$f = \varphi_1 M_1 + \varphi_2 M_2, \tag{11}$$

where Maxwellians M_i are described by (6), and the desired coefficient functions $\varphi_i(t, x)$ are chosen to be such that the deviation between the parts of equation (1) is arbitrarily small due to the conditions imposed on the hydrodynamic parameters included in distribution (6). In this work, as a deflection between the parts of equation (1), we use the uniform-integral error from [10]:

$$\Delta = \sup_{(t,x)\in R^4} \int_{R^3} dV \int_{R^3} d\omega \left|D(f) - Q(f, f)\right|. \tag{12}$$

2. The main results

Theorem 2.1. *Let the coefficient functions $\varphi_i(t, x)$ in distribution (11) have the form*

$$\varphi_i(t, x) = \psi_i(t, x) e^{-\beta_i \left(\overline{\omega}_i^2 r_i^2 - 2\overline{w}_i x\right)}, \tag{13}$$

where $\psi_i(t, x)$ are smooth, nonnegative and bounded on R^4 functions. Assume that the expressions

$$t\psi_i, \quad (x, \overline{u}_{0i})\psi_i, \quad \frac{\partial \psi_i}{\partial t}, \quad \left|\frac{\partial \psi_i}{\partial x}\right|, \quad \left|\frac{\partial \psi_i}{\partial x}\right| t, \quad \left(x, \frac{\partial \psi_i}{\partial x}\right) \tag{14}$$

are also bounded. In addition, consider the representations:

$$\overline{\omega}_i = \frac{\overline{\omega}_{0i}}{\beta_i^n}, \quad \overline{w}_i = \frac{\overline{w}_{0i}}{\beta_i^k}, \qquad n, k > 1. \tag{15}$$

Then there exists a value Δ' such that $\Delta \leqslant \Delta'$, and we have the equality

$$\begin{aligned}\lim_{\beta_i \to +\infty} \Delta' = \sum_{i=1}^{2} \rho_{0i} \sup_{(t,x)\in R^4} & \left| \frac{\partial \psi_i}{\partial t} + \left(\frac{\partial \psi_i}{\partial x}, \widehat{V}_i + \widetilde{V}_i - \frac{1}{\overline{\omega}_{0i}^2} \overline{\omega}_{0i} \left(\overline{\omega}_{0i}, \widetilde{V}_i \right) \right) \right| \\ & + 4\pi d^2 \rho_{01}\rho_{02} \left(\sup_{(t,x)\in R^4} (\psi_1\psi_2) \right) \left| \widehat{V}_1 - \widehat{V}_2 + \widetilde{V}_1 - \widetilde{V}_2 \right. \\ & \left. - \frac{1}{\overline{\omega}_{01}^2} \overline{\omega}_{01} \left(\overline{\omega}_{01}, \widetilde{V}_1 \right) + \frac{1}{\overline{\omega}_{02}^2} \overline{\omega}_{02} \left(\overline{\omega}_{02}, \widetilde{V}_2 \right) \right|. \end{aligned} \tag{16}$$

Proof. Substitute bimodal distribution (11) to the differential operator $D(f)$:

$$\begin{aligned} D(f) &= M_1 D(\varphi_1) + M_2 D(\varphi_2) \\ &= M_1 \left(\frac{\partial \varphi_1}{\partial t} + V \frac{\partial \varphi_1}{\partial x} \right) + M_2 \left(\frac{\partial \varphi_2}{\partial t} + V \frac{\partial \varphi_2}{\partial x} \right). \end{aligned}$$

After elementary transformations, the collision integral takes the form

$$Q(f,f) = \varphi_1\varphi_2 \left[Q(M_1, M_2) + Q(M_2, M_1) \right].$$

Further we will use the well-known decomposition of the collision integral

$$Q(f,g) = G(f,g) - fL(g), \tag{17}$$

where the gain and the loss terms of the collision integral have the form (see [2,4]):

$$G(f,g) = \frac{d^2}{2} \int_{R^3} dV_1 \int_{R^3} d\omega_1 \int_{\sum} d\alpha B(V - V_1, \alpha) f(t,x,V_1^*, \omega_1^*) g(t,x,V^*,\omega^*),$$

and

$$L(g) = \frac{d^2}{2} \int_{R^3} dV_1 \int_{R^3} d\omega_1 \int_{\sum} d\alpha B(V - V_1, \alpha) g(t,x,V_1,\omega_1).$$

As it was shown in [10],

$$\int_{R^3} dV \int_{R^3} d\omega Q(M_i, M_j) = 0, \quad j = 1, 2.$$

By the above equality and (17), we get the equality

$$\int_{R^3} dV \int_{R^3} d\omega G(M_i, M_j) = \int_{R^3} dV \int_{R^3} d\omega M_i L(M_j). \tag{18}$$

Then it is possible to obtain the inequality

$$|D(f) - Q(f,f)| \leqslant M_1 \left(|D(\varphi_1)| + \varphi_1\varphi_2 L(M_2) \right) + M_2 \left(|D(\varphi_2)| + \varphi_1\varphi_2 L(M_1) \right)$$

$$+\varphi_1\varphi_2\left(G(M_1,M_2)+G(M_2,M_1)\right).$$

After integrating the last inequality over the space of linear and angular velocities and taking into account (18), we get the estimation

$$\int_{R^3} dV \int_{R^3} d\omega\, |D(f)-Q(f,f)| \leqslant \sum_{\substack{i,j=1\\ i\neq j}}^{2} \int_{R^3} dV \int_{R^3} d\omega \left(|D(\varphi_i)| + \varphi_i\varphi_j L(M_j)\right) M_i + 2\varphi_1\varphi_2 \int_{R^3} dV \int_{R^3} d\omega G(M_1,M_2)$$
$$\leqslant \sum_{i=1}^{2} \int_{R^3} dV \int_{R^3} d\omega |D(\varphi_i)| M_i + 4\varphi_1\varphi_2 \int_{R^3} dV \int_{R^3} d\omega G(M_1,M_2).$$

From [7], we use the relation

$$\int_{R^3} dV \int_{R^3} d\omega G(M_1,M_2) = \frac{d^2\rho_1\rho_2}{\pi^2} \int_{R^3} dq \int_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \overline{V}_1 - \overline{V}_2 \right| \tag{19}$$

to continue the estimation by using (19) and the form of Maxwellians (6):

$$\int_{R^3} dV \int_{R^3} d\omega\, |D(f)-Q(f,f)| \leqslant \sum_{i=1}^{2} \int_{R^3} dV \int_{R^3} d\omega \left| \frac{\partial\varphi_i}{\partial t} + \left(V, \frac{\partial\varphi_i}{\partial x}\right) \right| \rho_i I^{3/2} \left(\frac{\beta_i}{\pi}\right)^3 e^{-\beta_i\left(\left(V-\overline{V}_i\right)^2 + I\omega^2\right)}$$
$$+ \frac{4d^2\rho_1\rho_2}{\pi^2}\varphi_1\varphi_2 \int_{R^3} dq \int_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \overline{V}_1 - \overline{V}_2 \right|.$$

Calculating the integral of the angular velocity $\int_{R^3} d\omega e^{-\beta_i I\omega^2} = \left(\frac{\pi}{\beta_i I}\right)^{3/2}$, we will have

$$\int_{R^3} dV \int_{R^3} d\omega\, |D(f)-Q(f,f)| \leqslant \sum_{i=1}^{2} \rho_i \left(\frac{\beta_i}{\pi}\right)^{3/2} \int_{R^3} dV \left| \frac{\partial\varphi_i}{\partial t} + \left(V, \frac{\partial\varphi_i}{\partial x}\right) \right| e^{-\beta_i\left(V-\overline{V}_i\right)^2}$$
$$+ \frac{4d^2\rho_1\rho_2}{\pi^2}\varphi_1\varphi_2 \int_{R^3} dq \int_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \overline{V}_1 - \overline{V}_2 \right|.$$

Next, let us change the variables in the integral under the sum $V = \frac{p}{\sqrt{\beta_i}} + \overline{V}_i$, whose Jacobian is $J = \frac{1}{\beta_i^{3/2}}$, to get the estimation

$$\int_{R^3} dV \int_{R^3} d\omega\, |D(f)-Q(f,f)|$$

$$\leqslant \frac{1}{\pi^{3/2}}\sum_{i=1}^{2}\rho_i\int_{R^3}\left|\frac{\partial\varphi_i}{\partial t}+\left(\frac{p}{\sqrt{\beta_i}}+\overline{V}_i,\frac{\partial\varphi_i}{\partial x}\right)\right|e^{-p^2}dp$$
$$+\frac{4d^2\rho_1\rho_2}{\pi^2}\varphi_1\varphi_2\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\overline{V}_1-\overline{V}_2\right|. \quad (20)$$

Then we have to find the derivatives of the functions $\varphi_i(t,x)$ by the variable t basing on its representation (13):

$$\frac{\partial\varphi_i}{\partial t}=e^{-\beta_i(\overline{\omega}_i^2r_i^2-2\overline{w}_ix)}\left(\frac{\partial\psi_i}{\partial t}+2\beta_i\psi_i\left[\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right]\right).$$

Thus the gradient over the spatial coordinate x has the form

$$\frac{\partial\varphi_i}{\partial x}=e^{-\beta_i(\overline{\omega}_i^2r_i^2-2\overline{w}_ix)}\left(\frac{\partial\psi_i}{\partial x}+2\beta_i\psi_i\left[\overline{w}_i+\overline{\omega}_i\left(\overline{\omega}_i,x\right)-\overline{\omega}_i^2(x-\overline{x}_{0i}-\overline{u}_{0i}t)\right]\right).$$

Continue to evaluate (20) by using the derivatives of the coefficient functions $\varphi_i(t,x)$ and the expression for the density (7):

$$\int_{R^3}dV\int_{R^3}\omega\left|D(f)-Q(f,f)\right|$$
$$\leqslant\frac{1}{\pi^{3/2}}\sum_{i=1}^{2}\rho_{0i}e^{\beta_i(\overline{\omega}_i^2r_i^2-2\overline{w}_ix)}\int_{R^3}dpe^{-p^2}\left|e^{-\beta_i(\overline{\omega}_i^2r_i^2-2\overline{w}_ix)}\left\{\frac{\partial\psi_i}{\partial t}\right.\right.$$
$$\left.+2\beta_i\psi_i\left[\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right]\right\}$$
$$+\left(\frac{p}{\sqrt{\beta_i}}+\overline{V}_i,e^{-\beta_i(\overline{\omega}_i^2r_i^2-2\overline{w}_ix)}\left\{\frac{\partial\psi_i}{\partial x}\right.\right.$$
$$\left.\left.\left.+2\beta_i\psi_i\left[\overline{w}_i+\overline{\omega}_i(\overline{\omega}_i,x)-\overline{\omega}_i^2(x-\overline{x}_{0i}-\overline{u}_{0i}t)\right]\right\}\right)\right|$$
$$+\frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2}\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\overline{V}_1-\overline{V}_2\right|.$$

After elementary transformations and substitution of the expression for mass velocity, $\overline{V}_i$ (9), into the above estimation, we have

$$\int_{R^3}dV\int_{R^3}d\omega\left|D(f)-Q(f,f)\right|$$
$$\leq\frac{1}{\sqrt{\pi^3}}\sum_{i=1}^{2}\rho_{0i}\int_{R^3}dpe^{-p^2}\left|\frac{\partial\psi_i}{\partial t}+2\beta_i\psi_i\left(\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right)\right.$$
$$+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\overline{w}_it+\left[\overline{\omega}_i\times(x-x_{0i}-\overline{u}_{0i}t)\right],\right.$$
$$\left.\left.\frac{\partial\psi_i}{\partial x}+2\beta_i\psi_i\left[\overline{w}_i+\overline{\omega}_i(\overline{\omega}_i,x)-\overline{\omega}_i^2(x-\overline{x}_{0i}-\overline{u}_{0i}t)\right]\right)\right|$$
$$+\frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2}\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+(\overline{w}_1-\overline{w}_2)t\right.$$

$$+ [\overline{\omega}_1 \times (x - x_{01} - \overline{u}_{01}t)] - \Big[\overline{\omega}_2 \times (x - x_{02} - \overline{u}_{02}t)\Big]\Big|. \tag{21}$$

Let us regroup the terms in the right-hand side of the last inequality in the following way:

$$\begin{aligned}
&\int_{R^3} dV \int_{R^3} d\omega\, |D(f) - Q(f,f)| \\
&\quad \leqslant \frac{1}{\sqrt{\pi^3}} \sum_{i=1}^{2} \rho_{0i} \int_{R^3} dp e^{-p^2} \left| \frac{\partial \psi_i}{\partial t} \right. \\
&\quad + \left(\frac{\partial \psi_i}{\partial x}, \frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \overline{w}_i t + [\overline{\omega}_i \times (x - x_{0i} - \overline{u}_{0i}t)] \right) \\
&\quad + 2\beta_i \psi_i \left(\overline{\omega}_i^2 (x, \overline{u}_{0i}) - \overline{u}_{0i}^2 \overline{\omega}_i^2 t - \left(\overline{\omega}_i \times \widetilde{V}_i, \overline{u}_{0i} \right) \right) + 2\psi_i \sqrt{\beta}_i (p, \overline{\omega}_i)(\overline{\omega}_i, x) \\
&\quad + 2\beta_i \psi_i \left. \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \overline{w}_i t + [\overline{\omega}_i \times (x - x_{0i} - \overline{u}_{0i}t)], \overline{w}_i - \overline{\omega}_i^2 (x - \overline{x}_{0i} - \overline{u}_{0i}t) \right) \right| \\
&\quad + \frac{4d^2 \rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2} \int_{R^3} dq \int_{R^3} dq_1 e^{-q^2 - q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + (\overline{w}_1 - \overline{w}_2)t \right. \\
&\quad + [\overline{\omega}_1 \times (x - x_{01} - \overline{u}_{01}t)] - [\overline{\omega}_2 \times (x - x_{02} - \overline{u}_{02}t)] \Big| .
\end{aligned}$$

As we know from vector algebra, for arbitrary three vectors $\overline{a}, \overline{b}, \overline{c}$, the equality

$$\left[\overline{a} \times \left[\overline{b} \times \overline{c}\right]\right] = \overline{b}(\overline{a}, \overline{c}) - \overline{c}(\overline{a}, \overline{b})$$

is true. Then, taking into account (10), we arrive at

$$[\overline{\omega}_i \times x_{0i}] = \frac{1}{\overline{\omega}_i^2} \overline{\omega}_i (\overline{\omega}_i, \widetilde{V}_i) - \widetilde{V}_i, \tag{22}$$

and due to some elementary transformations, we have

$$\begin{aligned}
&\left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \overline{w}_i t + [\overline{\omega}_i \times (x - x_{0i} - \overline{u}_{0i}t)], \overline{w}_i - \overline{\omega}_i^2 (x - \overline{x}_{0i} - \overline{u}_{0i}t) \right) \\
&\qquad = \overline{\omega}_i^2 \left(x - \overline{u}_{0i}t, \widetilde{V}_i - \overline{u}_{0i} \right) - \frac{1}{\overline{\omega}_i^2} (\overline{w}_i, \overline{\omega}_i)(\overline{\omega}_i, \widetilde{V}_i) \\
&\qquad + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \widetilde{V}_i + \overline{w}_i t, \overline{w}_i - \overline{\omega}_i^2 (x - \overline{u}_{0i}t) \right. \\
&\qquad + \left. \left[\overline{\omega}_i \times \left(\widetilde{V}_i - \overline{u}_{0i} \right) \right] + (\overline{w}_i, [\overline{\omega}_i \times (x - \overline{u}_{0i}t)]) \right).
\end{aligned}$$

Thus, we have the following estimation:

$$\begin{aligned}
&\int_{R^3} dV \int_{R^3} d\omega\, |D(f) - Q(f,f)| \leqslant \frac{1}{\sqrt{\pi^3}} \sum_{i=1}^{2} \rho_{0i} \int_{R^3} dp e^{-p^2} \left| \frac{\partial \psi_i}{\partial t} \right. \\
&\quad + \left(\frac{\partial \psi_i}{\partial x}, \frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \widetilde{V}_i + \overline{w}_i t + [\overline{\omega}_i \times (x - \overline{u}_{0i}t)] - \frac{1}{\overline{\omega}_i^2} \overline{\omega}_i (\overline{\omega}_i, \widehat{V}_i) \right)
\end{aligned}$$

$$
\begin{aligned}
&+2\beta_i\psi_i\left(\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right)+2\psi_i\sqrt{\beta}_i(p,\overline{\omega}_i)(\overline{\omega}_i,x)\\
&+2\beta_i\psi_i\left\{\overline{\omega}_i^2\left(x-\overline{u}_{0i}t,\widetilde{V}_i-\overline{u}_{0i}\right)-\frac{1}{\overline{\omega}_i^2}(\overline{w}_i,\overline{\omega}_i)(\overline{\omega}_i,\widetilde{V}_i)\right.\\
&+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\widetilde{V}_i+\overline{w}_it,\overline{w}_i-\overline{\omega}_i^2(x-\overline{u}_{0i}t)\right.\\
&\left.\left.+\left[\overline{\omega}_i\times\left(\widetilde{V}_i-\overline{u}_{0i}\right)\right]+(\overline{w}_i,[\overline{\omega}_i\times(x-\overline{u}_{0i}t)])\right)\right\}\Bigg|\\
&+\frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2}\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+\widetilde{V}_1-\widetilde{V}_2\right.\\
&+(\overline{w}_1-\overline{w}_2)t\left[\overline{\omega}_1\times(x-\overline{u}_{01}t)\right]-\left[\overline{\omega}_2\times(x-\overline{u}_{02}t)\right]\\
&\left.+\frac{1}{\overline{\omega}_2^2}\overline{\omega}_2(\overline{\omega}_2,\widetilde{V}_2)-\frac{1}{\overline{\omega}_1^2}\overline{\omega}_1(\overline{\omega}_1,\widetilde{V}_1)\right|.
\end{aligned}
$$

In the last inequality, let us turn to the supremum of both parts, the existence of which follows from conditions (14) of Theorem 2.1:

$$
\begin{aligned}
\Delta&=\sup_{(t,x)\in R^4}\int_{R^3}dV\int_{R^3}\omega\left|D(f)-Q(f,f)\right|\\
&\leqslant\frac{1}{\sqrt{\pi^3}}\sum_{i=1}^{2}\rho_{0i}\int_{R^3}dpe^{-p^2}\sup_{(t,x)\in R^4}\left|\frac{\partial\psi_i}{\partial t}\right.\\
&+\left(\frac{\partial\psi_i}{\partial x},\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\widetilde{V}_i+\overline{w}_it+[\overline{\omega}_i\times(x-\overline{u}_{0i}t)]-\frac{1}{\overline{\omega}_i^2}\overline{\omega}_i(\overline{\omega}_i,\widetilde{V}_i)\right)\\
&+2\beta_i\psi_i\left(\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right)+2\psi_i\sqrt{\beta}_i(p,\overline{\omega}_i)(\overline{\omega}_i,x)\\
&+2\beta_i\psi_i\left\{\overline{\omega}_i^2\left(x-\overline{u}_{0i}t,\widetilde{V}_i-\overline{u}_{0i}\right)-\frac{1}{\overline{\omega}_i^2}(\overline{w}_i,\overline{\omega}_i)(\overline{\omega}_i,\widetilde{V}_i)\right.\\
&+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\widetilde{V}_i+\overline{w}_it,\overline{w}_i-\overline{\omega}_i^2(x-\overline{u}_{0i}t)\right.\\
&\left.\left.+\left[\overline{\omega}_i\times\left(\widetilde{V}_i-\overline{u}_{0i}\right)\right]+(\overline{w}_i,[\overline{\omega}_i\times(x-\overline{u}_{0i}t)])\right)\right\}\Bigg|\\
&+\frac{4d^2\rho_{01}\rho_{02}}{\pi^2}\sup_{(t,x)\in R^4}\psi_1\psi_2\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}\right.\\
&+\widehat{V}_1-\widehat{V}_2+\widetilde{V}_1-\widetilde{V}_2+(\overline{w}_1-\overline{w}_2)t\\
&\left.+[\overline{\omega}_1\times(x-\overline{u}_{01}t)]-[\overline{\omega}_2\times(x-\overline{u}_{02}t)]+\frac{1}{\overline{\omega}_2^2}\overline{\omega}_2(\overline{\omega}_2,\widetilde{V}_2)-\frac{1}{\overline{\omega}_1^2}\overline{\omega}_1(\overline{\omega}_1,\widetilde{V}_1)\right|,
\end{aligned}
$$

which implies the representation for the value of Δ':

$$
\begin{aligned}
\Delta'&=\frac{1}{\sqrt{\pi^3}}\sum_{i=1}^{2}\rho_{0i}\int_{R^3}dpe^{-p^2}\sup_{(t,x)\in R^4}\left|\frac{\partial\psi_i}{\partial t}\right.\\
&+\left(\frac{\partial\psi_i}{\partial x},\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\widetilde{V}_i+\overline{w}_it+[\overline{\omega}_i\times(x-\overline{u}_{0i}t)]-\frac{1}{\overline{\omega}_i^2}\overline{\omega}_i(\overline{\omega}_i,\widetilde{V}_i)\right)
\end{aligned}
$$

$$+\,2\beta_i\psi_i\left(\overline{\omega}_i^2(x,\overline{u}_{0i})-\overline{u}_{0i}^2\overline{\omega}_i^2t-\left(\overline{\omega}_i\times\widetilde{V}_i,\overline{u}_{0i}\right)\right)+2\psi_i\sqrt{\beta}_i(p,\overline{\omega}_i)(\overline{\omega}_i,x)$$
$$+\,2\beta_i\psi_i\left\{\overline{\omega}_i^2\left(x-\overline{u}_{0i}t,\widetilde{V}_i-\overline{u}_{0i}\right)-\frac{1}{\overline{\omega}_i^2}(\overline{w}_i,\overline{\omega}_i)(\overline{\omega}_i,\widetilde{V}_i)\right.$$
$$+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\widetilde{V}_i+\overline{w}_it,\overline{w}_i-\overline{\omega}_i^2(x-\overline{u}_{0i}t)\right.$$
$$\left.\left.+\left[\overline{\omega}_i\times\left(\widetilde{V}_i-\overline{u}_{0i}\right)\right]+(\overline{w}_i,[\overline{\omega}_i\times(x-\overline{u}_{0i}t)])\right)\right\}\Bigg|$$
$$+\,\frac{4d^2\rho_{01}\rho_{02}}{\pi^2}\sup_{(t,x)\in R^4}\psi_1\psi_2\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}\right.$$
$$+\,\widehat{V}_1-\widehat{V}_2+\widetilde{V}_1-\widetilde{V}_2+(\overline{w}_1-\overline{w}_2)t$$
$$\left.+\,[\overline{\omega}_1\times(x-\overline{u}_{01}t)]-[\overline{\omega}_2\times(x-\overline{u}_{02}t)]+\frac{1}{\overline{\omega}_2^2}\overline{\omega}_2(\overline{w}_2,\widetilde{V}_2)-\frac{1}{\overline{\omega}_1^2}\overline{\omega}_1(\overline{w}_1,\widetilde{V}_1)\right|.$$

Using condition (15) of Theorem 2.1 and passing to the low-temperature limit, the validity of which follows from the lemma proved in [7], we have

$$\lim_{\beta_i\to+\infty}\Delta'=\frac{1}{\sqrt{\pi^3}}\sum_{i=1}^{2}\rho_{0i}\sup_{(t,x)\in R^4}\int_{R^3}dpe^{-p^2}\left|\frac{\partial\psi_i}{\partial t}\right.$$
$$\left.+\left(\frac{\partial\psi_i}{\partial x},\widehat{V}_i+\widetilde{V}_i-\frac{1}{\overline{\omega}_{0i}^2}\overline{\omega}_{0i}\left(\overline{\omega}_{0i},\widetilde{V}_i\right)\right)\right|$$
$$+\,\frac{4d^2\rho_{01}\rho_{02}}{\pi^2}\sup_{(t,x)\in R^4}(\psi_1\psi_2)\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\widehat{V}_1-\widehat{V}_2\right.$$
$$\left.+\,\widetilde{V}_1-\widetilde{V}_2-\frac{1}{\overline{\omega}_{01}^2}\overline{\omega}_{01}\left(\overline{\omega}_{01},\widetilde{V}_1\right)+\frac{1}{\overline{\omega}_{02}^2}\overline{\omega}_{02}\left(\overline{\omega}_{02},\widetilde{V}_2\right)\right|.$$

Calculating the integrals in the right-hand side of the last equality, we get that assertion (16) of Theorem 2.1 holds. □

Corollary 2.2. *Let all the conditions of Theorem 2.1 be valid and the functions ψ_i be of the form*

$$\psi_i(t,x)=C_i\left(x-t\left(\widehat{V}_i+\widetilde{V}_i-\frac{1}{\overline{\omega}_{0i}^2}\overline{\omega}_{0i}\left(\overline{\omega}_{0i},\widetilde{V}_i\right)\right)\right),\tag{23}$$

where C_i are nonnegative, smooth and bounded functions on R^4. In addition, one of the conditions:

$$d\to 0\tag{24}$$

or

$$\widehat{V}_1+\widetilde{V}_1-\frac{1}{\overline{\omega}_{01}^2}\overline{\omega}_{01}\left(\overline{\omega}_{01},\widetilde{V}_1\right)=\widehat{V}_2+\widetilde{V}_2-\frac{1}{\overline{\omega}_{02}^2}\overline{\omega}_{02}\left(\overline{\omega}_{02},\widetilde{V}_2\right)\tag{25}$$

is required to be fulfilled.

Then we have the statement:

$$\forall\varepsilon>0\quad\exists\beta_0\quad\forall\beta_i>\beta_0\quad\Delta<\varepsilon.\tag{26}$$

The validity of this corollary obviously follows from the inequality $\Delta \leqslant \Delta'$. If we substitute the functions ψ_i of the form (23) into (16), then its first term vanishes. If we use any of additional conditions (24) or (25), then the last term of (16) also vanishes.

Corollary 2.3. *As a function ψ_i, one can consider an arbitrary function of the form*

$$\psi_i(t,x) = C_i\left(\left[x \times \left(\widehat{V}_i + \widetilde{V}_i - \frac{1}{\overline{\omega}_{0i}^2}\overline{\omega}_{0i}\left(\overline{\omega}_{0i}, \widetilde{V}_i\right)\right)\right]\right),$$

where C_i are also nonnegative, smooth and bounded functions on R^4. If additionally one of the conditions (24) or (25) is satisfied, then (26) remains true.

In this case, the functions ψ_i depend only on spatial coordinates and naturally the first sum on the right-hand side of (16) vanishes. If one of the conditions (24), (25) is satisfied, then (26) remains true.

Theorem 2.4. *Suppose that the coefficient functions $\varphi_i(t,x)$ are of the form*

$$\varphi_i(t,x) = \psi_i(t,x)e^{-\beta_i\overline{\omega}_i^2 r_i^2}, \tag{27}$$

where the same conditions as in Theorem 2.1 are imposed on the functions ψ_i, but expressions (14) remain bounded even after multiplying them by the factor $e^{-2\beta_i\overline{w}_i x}$. Then, if condition (15) is satisfied, the statement (16) of Theorem 2.1 is also valid.

Proof. Estimation (20) remains true, so let us calculate the derivatives of the coefficient functions:

$$\frac{\partial\varphi_i}{\partial t} = e^{-\beta_i\overline{\omega}_i^2 r_i^2}\left(\frac{\partial\psi_i}{\partial t} + 2\beta_i\psi_i\left[\overline{\omega}_i^2(x,\overline{u}_{0i}) - \overline{u}_{0i}^2\overline{\omega}_i^2 t - \left(\overline{\omega}_i \times \widetilde{V}_i, \overline{u}_{0i}\right)\right]\right),$$

$$\frac{\partial\varphi_i}{\partial x} = e^{-\beta_i\overline{\omega}_i^2 r_i^2}\left(\frac{\partial\psi_i}{\partial x} + 2\beta_i\psi_i\left[\overline{\omega}_i(\overline{\omega}_i,x) - \overline{\omega}_i^2(x - \overline{x}_{0i} - \overline{u}_{0i}t)\right]\right).$$

Next, substituting the functions $\varphi_i(t,x)$ (27), the obtained derivatives, the density (7) and the mass velocity $\overline{V}_i$ in the right-hand side of inequality (20), we have:

$$\begin{aligned}
&\int_{R^3} dV \int_{R^3} d\omega\,|D(f) - Q(f,f)| \leqslant \frac{1}{\sqrt{\pi^3}}\sum_{i=1}^{2}\rho_{0i}e^{-2\beta_i\overline{w}_i x}\int_{R^3} dp e^{-p^2}\left|\frac{\partial\psi_i}{\partial t}\right.\\
&\quad + 2\beta_i\psi_i\left(\overline{\omega}_i^2(x,\overline{u}_{0i}) - \overline{u}_{0i}^2\overline{\omega}_i^2 t - \left(\overline{\omega}_i \times \widetilde{V}_i, \overline{u}_{0i}\right)\right)\\
&\quad + \left(\frac{\partial\psi_i}{\partial x} + 2\beta_i\psi_i\left(\overline{\omega}_i(\overline{\omega}_i,x) - \overline{\omega}_i^2(x - \overline{x}_{0i} - \overline{u}_{0i}t)\right),\right.\\
&\qquad\qquad\qquad\qquad \left.\left.\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i + \overline{w}_i t + [\overline{\omega}_i \times (x - x_{0i} - \overline{u}_{0i}t)]\right)\right|\\
&\quad + \frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2}e^{-2\beta_1\overline{w}_1 x - 2\beta_2\overline{w}_2 x}\int_{R^3} dq\int_{R^3} dq_1 e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}}\right.
\end{aligned}$$

$$+\widehat{V}_1-\widehat{V}_2+(\overline{w}_1-\overline{w}_2)t+[\overline{\omega}_1\times(x-x_{01}-\overline{u}_{01}t)]-[\overline{\omega}_2\times(x-x_{02}-\overline{u}_{02}t)]\Big|.$$

Thus, we have expression (21) with accuracy up to the factor $e^{-2\beta_i\overline{w}_ix}$ and the term $\overline{w}_i$. Further, in the same way as in the proof of Theorem 2.1, performing the same transformations, but imposing an additional condition of boundness on the functions (14) with the factor $e^{-2\beta_i\overline{w}_ix}$, due to condition (15), we get convinced of the correctness of assertions (16), which proves Theorem 2.4. □

Theorem 2.5. *Let the coefficient functions $\varphi_i(t,x)$ be of the form*

$$\varphi_i(t,x)=\psi_i(t,x)e^{2\beta_i\overline{w}_ix}, \tag{28}$$

where the same conditions as in Theorem 2.1, are imposed on the functions ψ_i, but expressions (14) remain bounded even after multiplying them by the factor $e^{\beta_i\overline{\omega}_i^2r_i^2}$. Then, if condition (15) remains true, the statement (16) of Theorem 2.1 is also valid.

Proof. Using again (20), calculate the derivatives of the coefficient functions (28):

$$\frac{\partial\varphi_i}{\partial t}=e^{2\beta_i\overline{w}_ix}\frac{\partial\psi_i}{\partial t},$$
$$\frac{\partial\varphi_i}{\partial x}=e^{2\beta_i\overline{w}_ix}\left(\frac{\partial\psi_i}{\partial x}+2\beta_i\psi_i\overline{w}_i\right).$$

Further, as in the proves of previous theorems, we substitute the expressions for φ_i of the form (28), the density (7) and the mass velocity $\overline{V}_i$ into inequality (20):

$$\int_{R^3}dV\int_{R^3}d\omega\,|D(f)-Q(f,f)|$$
$$\leqslant\frac{1}{\sqrt{\pi^2}}\sum_{i=1}^{2}\rho_{0i}e^{\beta_i\overline{\omega}_i^2r_i^2}\int_{R^3}dpe^{-p^2}\left|\frac{\partial\psi_i}{\partial t}+\left(\frac{\partial\psi_i}{\partial x}+2\beta_i\psi_i\overline{w}_i,\right.\right.$$
$$\left.\left.\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i+\overline{w}_it+[\overline{\omega}_i\times(x-x_{0i}-\overline{u}_{0i}t)]\right)\right|$$
$$+\frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2}{\pi^2}e^{\beta_1\overline{\omega}_1^2r_1^2+\beta_2\overline{\omega}_2^2r_2^2}\int_{R^3}dq\int_{R^3}dq_1e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}\right.$$
$$\left.+\widehat{V}_1-\widehat{V}_2+(\overline{w}_1-\overline{w}_2)t+[\overline{\omega}_1\times(x-x_{01}-\overline{u}_{01}t)]-[\overline{\omega}_2\times(x-x_{02}-\overline{u}_{02}t)]\right|.$$

The obtained expression is simpler than the estimation (21). Using the conditions of the theorem, we prove it in the same way as Theorem 2.1. □

So, in the paper, the bimodal distribution (11) with Maxwell modes M_i of the most general form is obtained for the model of rough spheres, which with arbitrary degree of accuracy minimizes the uniform-integral error (12) between the sides of the Bryan–Pidduck equation (1).

From the physical point of view, the obtained solution can be interpreted as follows: with descending of temperatures of flows, their rotational movement slows down and simultaneously their linear acceleration is reduced.

Supports. This work was partially supported by the NAS of Ukraine Project "Linear evolution equations in a Hilbert space and the Boltzmann equation".

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Received September 6, 2016, revised December 2, 2016.

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Взаємодія максвеллівських потоків загального виду для моделі Брайана–Піддака

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Вивчається взаємодія між двома максвеллівськими потоками загального вигляду в газі із шорсткуватих сфер. Наближений розв'язок рівняння Брайана–Піддака, яке описує цю взаємодію, є бімодальним розподілом зі спеціально підібраними коефіцієнтними функціями. Показано, що за певних додаткових умов, накладених на ці функції і на гідродинамічні параметри потоків, норма різниці між частинами рівняння Брайана–Піддака може бути якою завгодно малою.

Ключові слова: шорсткуваті сфери, рівняння Брайана–Піддака, відхил, максвеллівські потоки, бімодальний розподіл, гідродинамічні параметри.