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A truncated indefinite Stieltjes moment problem

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(*Presented by M. M. Malamud*)

Abstract. A truncated indefinite Stieltjes moment problem in the class $\mathbf{N}_\kappa^k$ of generalized Stieltjes functions is studied. The set of solutions of Stieltjes moment problem is described by Schur step-by-step algorithm, which is based on the expansion of the solutions in a generalized Stieltjes continued fraction. The resolvent matrix is represented in terms of generali-zed Stieltjes polynomials. A factorization formula for the resolvent matrix is found.

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1. Introduction

The classical Stieltjes moment problem was studied in [23]. It consists in the following:

Given a sequence of real numbers $\{s_i\}_{i=0}^{\infty}$, find a positive measure σ with a support on $\mathbb{R}_+$, such that

$$s_i = \int_{\mathbb{R}_+} t^i d\sigma(t), \qquad i \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{1.1}$$

The problem (1.1) with a finite data set $\{s_i\}_{i=0}^{2n}$ is called the truncated Stieltjes moment problem. The following inequalities

$$S_{n+1} := (s_{i+j})_{i,j=0}^{n} \geq 0, \quad S_n^+ := (s_{i+j+1})_{i,j=0}^{n-1} \geq 0 \tag{1.2}$$

are necessary for solvability of the truncated Stieltjes moment problem. If, additionally, the matrices S_{n+1} and S_n^+ are nondegenerate, then the inequalities

$$S_{n+1} > 0 \quad \text{and} \quad S_n^+ > 0$$

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are also sufficient for solvability of the truncated moment problem (1.1) with the data set $\{s_i\}_{i=0}^{2n}$ (see [16]). The degenerate case of the truncated Stieltjes moment problem was studied in [3].

Recall that a function f holomorphic on $\mathbb{C}\backslash\mathbb{R}$ is said to belong to the class $\mathbf{N}$ (see [1, Section 3.1]) [22, Appendix]), if $\mathrm{Im} f(z) \geq 0$ and $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}_+$. Clearly, the Stieltjes transform of σ

$$f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t-z} \qquad z \in \mathbb{C}\backslash\mathbb{R}_+ \tag{1.3}$$

belongs to $\mathbf{N}$. Moreover, f belongs to the Stieltjes class $\mathbf{S}$ consisting of functions $f \in \mathbf{N}$ which admit holomorphic and nonnegative continuations to $\mathbb{R}_-$. By M.G. Krein's criterion [15]

$$f \in \mathbf{S} \Longleftrightarrow f \in \mathbf{N} \quad \text{and} \quad zf \in \mathbf{N}. \tag{1.4}$$

By the Hamburger–Nevanlinna Theorem (see [1]) the truncated Stieltjes moment problem can be reformulated in terms of the Stieltjes transform (1.3) of σ as the following interpolation problem at ∞: Find $f \in \mathbf{S}$ such that

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \qquad z\widehat{\to}\infty. \tag{1.5}$$

The notation $z\widehat{\to}\infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

A function f meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if for every set $z_i \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($j = 1, \ldots, n$) the form

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \xi_i \overline{\xi}_j$$

has at most κ and for some choice of z_i ($i = 1, \ldots, n$) it has exactly κ negative squares. For $f \in \mathbf{N}_\kappa$ let us write $\kappa_-(f) = \kappa$. In particular, if $\kappa = 0$ then the class $\mathbf{N}_0$ coincides with the class $\mathbf{N}$ of Nevanlinna functions.

A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_\kappa^+$ (see [17, 18]) if $zf \in \mathbf{N}$ and to the class $\mathbf{N}_\kappa^k$ ($k \in \mathbb{N}$) if $zf \in \mathbf{N}_\kappa^k$ (see [5,6]). In particular, if $k = 0$, then $\mathbf{N}_\kappa^0 := \mathbf{N}_\kappa^+$, and if $\kappa = 0$, $k \neq 0$ N_0^k coincides with the generalized Stieltjes class S_κ^+ introduced in [12, 13].

In the present paper the following indefinite moment problem in the classes $\mathbf{N}_\kappa^k$ is studied.

Problem $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$. Given ℓ, κ, $k\in\mathbb{Z}_+$, and a sequence $\mathbf{s}=\{s_i\}_{i=0}^{\ell}$ of real numbers, describe the set $\mathcal{M}_\kappa^k(\mathbf{s})$ of functions $f\in\mathbf{N}_\kappa^k$, which have the following asymptotic expansion

$$f(z)=-\frac{s_0}{z^1}-\frac{s_1}{z^2}-\cdots-\frac{s_\ell}{z^{\ell+1}}+o\left(\frac{1}{z^{\ell+1}}\right),\quad z\widehat{\rightarrow}\infty. \tag{1.6}$$

Indefinite moment problems in the classes $\mathbf{N}_\kappa$ were studied in [4, 5, 14, 19]. Indefinite moment problems in the classes $\mathbf{N}_\kappa^+$ and $\mathbf{N}_\kappa^k$ were studied in [19, 20] and [7, 10], respectively.

This paper is a continuation of [10], where a Schur type algorithm for the moment problem $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$ was elaborated. We restrict ourselves to the case of a nondegenerate problem. Namely, if $\ell=2n-1$ the even moment problem $MP_\kappa^k(\mathbf{s},2n-1)$ is called *nondegenerate* if $\det S_n\neq 0$.

Recall ([9]), that a number $n_j\in\mathbb{N}$ is called a *normal index* of the sequence $\mathbf{s}$, if $\det S_{n_j}\neq 0$. The ordered set of all normal indices

$$n_1<n_2<\cdots<n_N$$

of the sequence $\mathbf{s}$ is denoted by $\mathcal{N}(\mathbf{s})$. With this notation the even moment problem $MP_\kappa^k(\mathbf{s},2n-1)$ is nondegenerate if $n\in\mathcal{N}(\mathbf{s})$. Let us set $n:=n_N$ and $\ell=2n_N-1$. In Theorem 4.2 we show that the nondegenerate even moment problem $MP_\kappa^k(\mathbf{s},2n_N-1)$ is solvable if and only if

$$\kappa_N^+:=\nu_-(S_{n_N})\leq\kappa\quad\text{and}\quad k_N^+:=\nu_-(S_{n_N}^+)\leq k,$$

where $\nu_-(S_{n_N})$ denotes the number of negative eigenvalues of S_{n_N} with account of multiplicities. Every solution f of the even moment problem $MP_\kappa^k(\mathbf{s},2n_N-1)$ admits the following representation

$$f(z)=\cfrac{1}{-zm_1(z)+\cfrac{1}{l_1(z)+\cdots+\cfrac{1}{-zm_N(z)+\cfrac{1}{l_N(z)+\tau(z)}}}}, \tag{1.7}$$

where $m_i(z)$ and $l_i(z)$ are some polynomials determined by the data $\mathbf{s}=\{s_i\}_{i=0}^{2n_N-1}$, and $\tau\in\mathbf{N}_{\kappa-\kappa_N^+}^{k-k_N^+}$ and $\tau(z)^{-1}=o(1)$, as $z\widehat{\rightarrow}\infty$.

Furthermore, the continued fraction (1.7) is associated with the following system of difference equations

$$\begin{cases} y_{2i+1}-y_{2i-1}=-zm_{i+1}(z)y_{2i}, \\ y_{2i+2}-y_{2i}=l_{i+1}(z)y_{2i+1}. \end{cases} \tag{1.8}$$

see [24, Section 1]. The polynomials $P_i^+(z)$ and $Q_i^+(z)$, which satisfy the system (1.8) and the following initial conditions

$$P_{-1}^+(z) \equiv -1, \quad P_0^+(z) \equiv 0; \qquad Q_{-1}^+(z) \equiv 0, \quad Q_0^+(z) \equiv 1$$

are called *generalized Stieltjes polynomials.*

In Theorem 5.5 it is shown that the formula (1.7) can be rewritten in terms of the polynomials Q_{2N-1}^+, Q_{2N}^+, P_{2N-1}^+ and Q_{2N}^+ as follows

$$f(z) = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N}^+(z)}{P_{2N-1}^+(z)\tau(z) + P_{2N}^+(z)}. \tag{1.9}$$

The resolvent matrix of the even moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$

$$W_{2N}(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N}^+(z) \\ P_{2N-1}^+(z) & P_{2N}^+(z) \end{pmatrix} \tag{1.10}$$

admits the following factorization

$$W_{2N}(z) = M_1(z)L_1(z)\dots M_N(z)L_N(z), \tag{1.11}$$

where the matrices $M_j(z)$ and $L_j(z)$ are defined by (4.12).

Analogous results for odd moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ are presented in Theorem 4.1 and Theorem 5.2. Sequences $\mathbf{s} = \{s_i\}_{i=0}^\ell$ which satisfy the condition

$$\det S_{n_j}^+ \neq 0 \quad j = 1, \dots, N, \tag{1.12}$$

are called regular, [10]. The moment problem $MP_\kappa^k(\mathbf{s}, \ell)$ in the class of regular sequences $\mathbf{s} = \{s_i\}_{i=0}^\ell$ was studied in [10]. As was shown in [10] the polynomials $l_j(z)$ in this case are reducing to constants and the resolvent matrices $L_j(z)$ are changing accordingly.

2. Preliminaries

2.1. Generalized Nevanlinna and Stieltjes classes

Every real polynomial $P(t) = p_\nu t^\nu + p_{\nu-1}t^{\nu-1} + \dots + p_1 t + p_0$ of degree ν belongs to a class $\mathbf{N}_\kappa$, where the index $\kappa = \kappa_-(P)$ can be evaluated by (see [17, Lemma 3.5])

$$\kappa_-(P) = \begin{cases} \left[\frac{\nu+1}{2}\right], & \text{if } p_\nu < 0; \text{ and } \nu \text{ is odd }; \\ \left[\frac{\nu}{2}\right], & \text{otherwise }. \end{cases} \tag{2.1}$$

Proposition 2.1. *([17]) Let $f \in \mathbf{N}_\kappa$, $f_1 \in \mathbf{N}_{\kappa_1}$, $f_2 \in \mathbf{N}_{\kappa_2}$. Then*

(1) $-f^{-1} \in \mathbf{N}_\kappa$*;*

(2) $f_1 + f_2 \in \mathbf{N}_{\kappa'}$*, where* $\kappa' \le \kappa_1 + \kappa_2$*;*

(3) If, in addition, $f_1(iy) = o(y)$ *as* $y \to \infty$ *and* f_2 *is a polynomial, then*

$$f_1 + f_2 \in \mathbf{N}_{\kappa_1+\kappa_2}. \tag{2.2}$$

(4) If a function $f \in \mathbf{N}_\kappa$ *has an asymptotic expansion* (1.6)*, then there exists* $\kappa' \le \kappa$*, such that* $\{s_j\}_{j=0}^{\ell} \in \mathcal{H}_{\kappa',\ell}$*.*

Proposition 2.2. *([10]) The following equivalences hold:*

(1) $f \in \mathbf{N}_\kappa^k \Longleftrightarrow -\frac{1}{f} \in \mathbf{N}_\kappa^{-k}$*;*

(2) $f \in \mathbf{N}_\kappa^k \Longleftrightarrow zf \in \mathbf{N}_k^{-\kappa}$*, in particular,* $f \in \mathbf{N}_\kappa^+ \Longleftrightarrow zf \in \mathbf{S}_\kappa^-$*;*

(3) If a function $f \in \mathbf{N}_\kappa^k$ *has an asymptotic expansion* (1.6) *then*

$$\{s_j\}_{j=0}^{\ell} \in \mathbf{H}_{\kappa',\ell}^{k'} \quad \text{with } \kappa' \le \kappa, \quad k' \le k. \tag{2.3}$$

2.2. Normal indices

Recall that the set $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=1}^N$ of normal indices of the sequence $\mathbf{s} = \{s_j\}_{j=0}^{\ell}$ is defined by

$$\mathcal{N}(\mathbf{s}) = \{n_j : D_{n_j} \ne 0, j = 1, 2, \dots, N\}, \quad D_{n_j} := \det(s_{i+k})_{i,k=0}^{n_j-1}. \tag{2.4}$$

Let us set $D_n^+ := \det(s_{i+j+1})_{i,j=0}^{n-1}$. By the Sylvester identity (see [9, Proposition 3.1] or [7, Lemma 5.1] for detail), the set $\mathcal{N}(\mathbf{s})$ is the union of two not necessarily disjoint subsets

$$\mathcal{N}(\mathbf{s}) = \{\nu_j\}_{j=1}^{N_1} \cup \{\mu_j\}_{j=1}^{N_2}, \tag{2.5}$$

which are selected by

$$D_{\nu_j} \ne 0 \quad \text{and} \quad D_{\nu_j-1}^+ \ne 0, \quad \text{for all } j = \overline{1, N_1} \tag{2.6}$$

and

$$D_{\mu_j} \ne 0 \quad \text{and} \quad D_{\mu_j}^+ \ne 0, \quad \text{for all } j = \overline{1, N_2}. \tag{2.7}$$

Moreover, the normal indices ν_j and μ_j satisfy the following inequalities

$$0 < \nu_1 \le \mu_1 < \nu_2 \le \mu_2 < \dots \tag{2.8}$$

For every $n_j \in \mathcal{N}(s)$ polynomials of the first and the second kind $P_{n_j}(z)$ and $Q_{n_j}(z)$ can be defined by standard formulas

$$\begin{aligned} P_{n_j}(z) &= \tfrac{1}{D_{n_j}} \det \begin{pmatrix} s_0 & s_1 & \cdots & s_{n_j} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n_j-1} & s_{n_j} & \cdots & s_{2n_j-1} \\ 1 & z & \cdots & z^{n_j} \end{pmatrix}, \\ Q_{n_j}(z) &= \mathfrak{S}_t\Big(\tfrac{P_{n_j}(z)-P_{n_j}(t)}{z-t}\Big), \end{aligned} \tag{2.9}$$

where $\mathfrak{S}_t$ is the linear functional on the set of polynomial of formal degree ℓ, defined by

$$\mathfrak{S}_t(t^i) = s_i, \quad i = 0, 1, \ldots, \ell.$$

Definition 2.3. *The sequence* $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ *is called regular* ($\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,reg}$) *if and only if one of the following equivalent conditions holds ([9, Lemma 3.1])*

(1) $P_{n_j}(0) \neq 0$ *for every* $j \leq N$*;*

(2) $D_{n_j-1}^{+} \neq 0$ *for every* $j \leq N$*;*

(3) $D_{n_j}^{+} \neq 0$ *for every* $j \leq N$*;*

(4) $\nu_j = \mu_j$ *for all* j*, such that* $\nu_j, \mu_j \in \mathcal{N}(\mathbf{s})$*.*

2.3. Class $\mathcal{U}_\kappa(J)$ and linear fractional transformations

Let $\kappa_1 \in \mathbb{N}$ and let J be the 2×2 signature matrix

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

A 2×2 matrix valued function $W(z) = (w_{i,j}(z))_{i,j=1}^{2}$ that is meromorphic in $\mathbb{C}_+$ is said to belong to the class $\mathcal{U}_\kappa(J)$ of *generalized J-inner* matrix valued functions if (see [2], [8]):

(i) the kernel

$$\mathsf{K}_\omega^W(z) = \frac{J - W(z)JW(\omega)^*}{-i(z-\bar{\omega})} \tag{2.10}$$

has κ negative squares in $\mathfrak{H}_W^+ \times \mathfrak{H}_W^+$ and

(ii) $J - W(\mu)JW(\mu)^* = 0$ for a.e. $\mu \in \mathbb{R}$, where $\mathfrak{H}_W^+$ denotes the domain of holomorphy of W in $\mathbb{C}_+$.

Consider the linear fractional transformation

$$T_W[\tau] = (w_{11}\tau(z) + w_{12})(w_{21}\tau(z) + w_{22})^{-1} \tag{2.11}$$

associated with the matrix valued function $W(z)$. The linear fractional transformation associated with the product W_1W_2 of two matrix valued function $W_1(z)$ and $W_2(z)$, coincides with the composition $T_{W_1} \circ T_{W_2}$.

As is known, if $W \in \mathcal{U}_{\kappa_1}(J)$ and $\tau \in \mathbf{N}_{\kappa_2}$ then $T_W[\tau] \in \mathbf{N}_{\kappa'}$, where $\kappa' \leq \kappa_1 + \kappa_2$, cf. [17, Satz 4.1]

In the present paper two partial cases, in which the preceding inequality becomes equality, will be needed.

Lemma 2.4. *([10]) Let $m(z)$ be a real polynomial $\kappa_1 = \kappa_-(zm)$, $k_1 = \kappa_-(m)$, let M be a 2×2 matrix valued function*

$$M(z) = \begin{pmatrix} 1 & 0 \\ -zm(z) & 1 \end{pmatrix} \tag{2.12}$$

and let τ be a meromorphic function, such that $\tau(z)^{-1} = o(z)$ as $z\widehat{\rightarrow}\infty$. Then $M \in \mathcal{U}_{\kappa_1}(J)$ and the following equivalences hold:

$$\tau \in \mathbf{N}_{\kappa_2} \Longleftrightarrow T_M[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2}, \tag{2.13}$$

$$\tau \in \mathbf{N}_{\kappa_2}^{k_2} \Longleftrightarrow T_M[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2}^{k_1+k_2}. \tag{2.14}$$

Lemma 2.5. *([10]) Let $l(z)$ be a real polynomial and indices $\kappa_1 = \kappa_-(l)$, $k_1 = \kappa_-(zl(z))$, let L be a 2×2 matrix valued function*

$$L(z) = \begin{pmatrix} 1 & l(z) \\ 0 & 1 \end{pmatrix} \tag{2.15}$$

and let τ be a meromorphic function, such that $\tau(z)^{-1} = o(1)$ as $z\widehat{\rightarrow}\infty$. Then $L \in \mathcal{U}_{k_1}(J)$ and the following equivalences hold:

$$\tau \in \mathbf{N}_{\kappa_2} \Longleftrightarrow T_L[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2},$$

$$\tau \in \mathbf{N}_{\kappa_2}^{k_2} \Longleftrightarrow T_L[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2}^{k_1+k_2}.$$

3. Basic moment problem in $\mathbf{N}_\kappa^k$

In this section we expose some material from [10] concerning the basic odd and even moment problems in generalized Stieltjes class $\mathbf{N}_\kappa^k$ and describe their solutions.

3.1. Basic odd moment problem $MP_\kappa^k(\mathbf{s}, 2\nu_1 - 2)$

An odd moment problem $MP_\kappa^k(\mathbf{s}, 2n - 2)$ is called nondegenerate if

$$D_n \neq 0 \quad \text{and} \quad D_{n-1}^+ \neq 0. \tag{3.1}$$

If, in addition, $n = \nu_1 \in \mathcal{N}(\mathbf{s})$, then the nondegenerate moment problem $MP_\kappa^k(\mathbf{s}, 2\nu_1 - 2)$ is called basic. In this case

$$\mathcal{N}(\mathbf{s}) = \{\nu_1\} \quad \text{and} \quad s_0 = \ldots = s_{\nu_1-2} = 0, \quad s_{\nu_1-1} \neq 0. \tag{3.2}$$

The basic moment problem $MP_\kappa^k(\mathbf{s}, 2\nu_1 - 2)$ can be reformulated as follows:
Given a sequence $\mathbf{s} = \{s_i\}_{i=0}^{2\nu_1-2}$, such that (3.2) holds, or equivalently $\mathcal{N}(\mathbf{s}) = \{\nu_1\}$. Find all functions $f \in \mathbf{N}_\kappa^k$, which admit the asymptotic expansion

$$f(z) = -\frac{s_{\nu_1-1}}{z^{\nu_1}} - \cdots - \frac{s_{2\nu_1-2}}{z^{2\nu_1-1}} + o\left(\frac{1}{z^{2\nu_1-1}}\right), \quad z\widehat{\to}\infty. \tag{3.3}$$

Let $\mathbf{s} = \{s_i\}_{i=0}^{2\nu_1-2}$ be a sequence of real numbers from $\mathcal{H}$ and let (3.2) hold. Then $\mathbf{s} \in \mathcal{H}_{\kappa_1, 2\nu_1-2}^{k_1}$, where κ_1 and k_1 are defined by

$$\kappa_1 = \nu_-(S_{\nu_1}) = \begin{cases} \left[\frac{\nu_1+1}{2}\right], & \text{if } \nu_1 \text{ is odd and } s_{\nu_1-1} < 0; \\ \left[\frac{\nu_1}{2}\right], & \text{otherwise.} \end{cases} \tag{3.4}$$

$$k_1 = \nu_-(S_{\nu_1-1}^+) = \begin{cases} [\frac{\nu_1}{2}], & \text{if } \nu_1 \text{ is even and } s_{\nu_1-1} < 0; \\ [\frac{\nu_1-1}{2}], & \text{otherwise.} \end{cases} \tag{3.5}$$

Let us define the polynomial m_1, associated with the sequence $\mathbf{s}= \{s_i\}_{i=0}^{2\nu_1-2}$, by

$$m_1(z) = \frac{(-1)^{\nu_1+1}}{D_{\nu_1}} \begin{vmatrix} 0 & \ldots & 0 & s_{\nu_1-1} & s_{\nu_1} \\ \vdots & & \ldots & \ldots & \vdots \\ s_{\nu_1-1} & \ldots & \ldots & \ldots & s_{2\nu_1-2} \\ 1 & z & \ldots & z^{\nu_1-2} & z^{\nu_1-1} \end{vmatrix} \quad (D_{\nu_1} := \det S_{\nu_1}). \tag{3.6}$$

Obviously, the leading coefficient of m_1 is

$$(-1)^{\nu_1+1}\frac{D_{\nu_1-1}^+}{D_{\nu_1}} = \frac{1}{s_{\nu_1-1}} \tag{3.7}$$

and by Proposition 2.1, $m_1 \in \mathbf{N}_{k_1}^{\kappa_1}$, i.e. the indices κ_1 and k_1 are connected with m_1 by

$$\kappa_1 = \kappa_-(zm_1), \quad k_1 = \kappa_-(m_1). \tag{3.8}$$

Lemma 3.1. *(cf. [4, 10]) Let a function* $f \in \mathbf{N}_\kappa^k$ *admit the asymptotic expansion* (3.3) *and let* ν_1 *be the first normal index of the sequence* $\boldsymbol{s} = \{s_i\}_{i=0}^{2\nu_1-2}$*, let polynomial* m_1*, indices* κ_1 *and* k_1 *be defined by* (3.6) *and* (3.8)*, respectively. Then* f *admits the following representation*

$$f(z) = T_{M_1}[\tau] = \frac{\tau(z)}{-zm_1(z)\tau(z)+1}, \tag{3.9}$$

where

$$\tau \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1} \quad \text{and} \quad \tau^{-1} = o(z), \qquad z\widehat{\to}\infty. \tag{3.10}$$

Furthermore, the matrix valued function

$$M_1(z) = \begin{pmatrix} 1 & 0 \\ -zm_1(z) & 1 \end{pmatrix} \tag{3.11}$$

belongs to the class $\mathcal{U}_{\kappa_1}(J)$*.*

Conversely, if τ *satisfies* (3.10) *and* f *is defined by* (3.9)*, then* $f \in \mathbf{N}_\kappa^k$*.*

Proof. Assume that $f \in \mathbf{N}_\kappa^k$ and f admits the asymptotic expansion (3.3). Then by [10, Lemma 3.1]

$$f(z) = -\frac{1}{zm_1(z)+g(z)}, \tag{3.12}$$

where the polynomial m_1 is defined by (3.6), $g \in \mathbf{N}_{\kappa-\kappa_1}$ and $g(z) = o(z)$ as $z\widehat{\to}\infty$. On the other hand, we can rewrite (3.12) as follows

$$-1/f(z) = zm_1(z) + g(z). \tag{3.13}$$

Replacing g by $-\tau^{-1}$ in (3.13), we obtain $\tau \in \mathbf{N}_{\kappa-\kappa_1}$. Due to the assumption $zf \in \mathbf{N}_k$ one gets $-\frac{1}{zf} \in \mathbf{N}_k$ and hence the equality

$$-1/zf(z) = m_1(z) - 1/z\tau(z), \tag{3.14}$$

Proposition 2.1 and (3.8) imply $-(z\tau(z))^{-1} \in \mathbf{N}_{k-k_1}$. Therefore, $\tau \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1}$ and $\tau^{-1} = o(z)$ as $z\widehat{\to}\infty$. Replacing g by $-\tau^{-1}$ in (3.12) one obtains (3.9). Furthermore, by Lemma 2.4 $M_1 \in \mathcal{U}_{\kappa_1}(J)$. This completes the proof. □

A sequence $(c_0, \dots, c_n)$ of real numbers determines an upper triangular Toeplitz matrix $T(c_0, \dots, c_n)$ of order $(n+1)\times(n+1)$ with entries $t_{i,j} = c_{j-i}$ for $i \le j$ and $t_{i,j} = 0$ for $i > j$:

$$T(c_0, \dots, c_n) = \begin{pmatrix} c_0 & \dots & c_n \\ & \ddots & \vdots \\ & & c_0 \end{pmatrix}. \tag{3.15}$$

Theorem 3.2. *([10]) Let ν_1 be the first normal index of the sequence $\boldsymbol{s}=\{s_i\}_{i=0}^{2\nu_1-2}$, let m_1, κ_1 and k_1 be defined by (3.6), (3.4) and by (3.5), respectively, and let $\ell \ge 2\nu_1-2$. Then:*

(1) The problem $MP_\kappa^k(\boldsymbol{s},\ell)$ is solvable if and only if

$$\kappa_1 \le \kappa \quad \text{and} \quad k_1 \le k. \tag{3.16}$$

(2) $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, 2\nu_1-2)$ if and only if f admits the representation

$$f = T_{M_1}[\tau], \tag{3.17}$$

where τ satisfies the following conditions

$$\tau \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z), \quad z\widehat{\to}\infty. \tag{3.18}$$

(3) If $\ell > 2\nu_1-2$, then $f \in \mathcal{M}_\kappa^k(\boldsymbol{s},\ell)$ if and only if f admits the representation $f = T_{M_1}[\tau]$, where $\tau \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1}$ and τ admits the following asymptotic expansion

$$-\tau^{-1}(z) = -\mathfrak{s}_{-1}^{(1)} - \frac{\mathfrak{s}_0^{(1)}}{z} - \cdots - \frac{\mathfrak{s}_{\ell-2\nu_1}^{(1)}}{z^{\ell-2\nu_1+1}} + o\left(\frac{1}{z^{\ell-2\nu_1+1}}\right), \quad z\widehat{\to}\infty, \tag{3.19}$$

where the sequence $\left\{\mathfrak{s}_i^{(1)}\right\}_{i=-1}^{\ell-2\nu_1}$ is determined by the matrix equation

$$T(m_{\nu_1-1}^{(1)},\dots,m_0^{(1)},-\mathfrak{s}_{-1}^{(1)},\dots,-\mathfrak{s}_{\ell-2\nu_1}^{(1)})\,T(s_{\nu_1-1},\dots,s_\ell) = I_{\ell-\nu_1+2}. \tag{3.20}$$

Remark 3.3. On the other hand, the sequence $\{\mathfrak{s}_i^{(1)}\}_{i=-1}^{n-2\nu_1}$ can be found by the following equivalent formulas (see [4, Proposition 2.1])

$$\mathfrak{s}_{-1}^{(1)} = \frac{(-1)^{\nu_1+1}}{s_{\nu_1-1}}\frac{D_{\nu_1}^+}{D_{\nu_1}}, \tag{3.21}$$

$$\mathfrak{s}_i^{(1)} = \frac{(-1)^{i+\nu_1}}{s_{\nu_1-1}^{i+\nu_1+2}} \begin{vmatrix} s_{\nu_1} & s_{\nu_1-1} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & s_{\nu_1-1} \\ s_{2\nu_1+i} & \dots & \dots & \dots & s_{\nu_1} \end{vmatrix} \quad i=\overline{0,n-2\nu_1}. \tag{3.22}$$

3.2. Basic even moment problem $MP_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$

An even moment problem $MP_\kappa^k(\mathbf{s}, 2n-1)$ is called nondegenerate, if the following conditions hold

$$D_n \neq 0 \quad \text{and} \quad D_n^+ \neq 0. \tag{3.23}$$

The nondegenerate even moment problem $MP_\kappa^k(\mathbf{s}, 2n-1)$ is called basic, if n is the smallest normal index of the sequence $\{s_i\}_{i=0}^{2n-1}$ such that (3.23) holds. In view of the classification of normal indices in (2.6) and (2.7), the basic even moment problem coincides with the problem $MP_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$. In this case

$$\text{either } \mathcal{N}(\mathbf{s}) = \{\nu_1\} \text{ or } \mathcal{N}(\mathbf{s}) = \{\nu_1, \mu_1\},$$

regarding to the conditions

$$\nu_1 = \mu_1 \quad \text{or} \quad \nu_1 < \mu_1.$$

The basic even moment problem $MP_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$ can be reformulated as follows:

Given a sequence $\mathbf{s} = \{s_i\}_{i=0}^{2\mu_1-1} \in \mathcal{H}$, where μ_1 is the smallest index n such that (3.23) holds, find all functions $f \in \mathbf{N}_\kappa^k$, such that

$$f(z) = -\frac{s_{\nu_1-1}}{z^{\nu_1}} - \cdots - \frac{s_{2\mu_1-1}}{z^{2\mu_1}} + o\left(\frac{1}{z^{2\mu_1}}\right), \quad z\widehat{\rightarrow}\infty.$$

Solution of the basic even moment problem will be splitted into two steps. On the first step one applies Lemma 3.1 to construct a sequence $\{\mathfrak{s}_i^{(1)}\}_{i=-1}^{2(\mu_1-\nu_1)-1}$ from the asymptotic expansion of the function $-\tau^{-1}$. If $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$ then by Theorem 3.2 f admits the representation (3.9) which can be rewritten as

$$-\frac{1}{f(z)} = zm_1(z) - \frac{1}{g_1(z)}, \tag{3.24}$$

where we use g_1 instead of τ and $-g_1^{-1}$ has the following asymptotic expansion

$$-\frac{1}{g_1(z)} = -\mathfrak{s}_{-1}^{(1)} - \frac{\mathfrak{s}_0^{(1)}}{z} - \cdots - \frac{\mathfrak{s}_{2(\mu_1-\nu_1)-1}^{(1)}}{z^{2(\mu_1-\nu_1)}} + o\left(\frac{1}{z^{2(\mu_1-\nu_1)}}\right), \quad z\widehat{\rightarrow}\infty, \tag{3.25}$$

with $\mathfrak{s}_i^{(1)}$ defined by (3.20). By Lemma 2.5

$$\begin{aligned} \kappa - \kappa_-(zm_1) = \kappa_-(g_1) \geq \kappa_-(l_1) + \kappa_-(\tau), \\ \kappa - \kappa_-(m_1) = \kappa_-(zg_1) \geq \kappa_-(zl_1) + \kappa_-(z\tau). \end{aligned} \tag{3.26}$$

Therefore, $f \in \mathbf{N}_\kappa^k$ if and only if $g_1 \in \mathbf{N}_{\kappa-\kappa_-(zm_1)}^{k-\kappa_-(m_1)}$ and g_1 is represented as

$$g_1(z) = T_{L_1}[\tau] := l_1(z) + \tau(z), \tag{3.27}$$

where $\tau \in N_{\kappa-\kappa_-(zm_1)-\kappa_-(l_1)}^{k-\kappa_-(m_1)-\kappa_-(zl_1)}$ and $l_1(z)$ is calculated as follows:

(1) if $\nu_1 = \mu_1$, then

$$l_1 = \frac{1}{\mathfrak{s}_{-1}^{(1)}} = (-1)^{\nu_1+1} s_{\nu_1-1} \frac{D_{\nu_1}}{D_{\nu_1}^+}; \tag{3.28}$$

(2) if $\nu_1 < \mu_1$, then

$$l_1(z) = \frac{1}{\mathfrak{s}_{\mu_1-\nu_1-1}^{(1)} \det(\mathcal{S}_{\mu_1-\nu_1}^{(1)})} \begin{vmatrix} \mathfrak{s}_0^{(1)} & \dots & \mathfrak{s}_{\mu_1-\nu_1-1}^{(1)} & \mathfrak{s}_{\mu_1-\nu_1}^{(1)} \\ \dots & \dots & \dots & \dots \\ \mathfrak{s}_{\mu_1-\nu_1-1}^{(1)} & \dots & \mathfrak{s}_{2\mu_1-2\nu_1-2}^{(1)} & \mathfrak{s}_{2\mu_1-2\nu_1-1}^{(1)} \\ 1 & \dots & z^{\mu_1-\nu_1-1} & z^{\mu_1-\nu_1} \end{vmatrix}, \tag{3.29}$$

where the matrix $\mathcal{S}_{\mu_1-\nu_1}^{(1)}$ is defined as in (1.2), i.e.

$$\mathcal{S}_{\mu_1-\nu_1}^{(1)} = (\mathfrak{s}_{i+j-1}^{(1)})_{i,j=0}^{\mu_1-\nu_1-1}.$$

Theorem 3.4. *([10]) Let $\boldsymbol{s} = \{s_i\}_{i=0}^{2\mu_1-1}$ be a sequence from $\mathcal{H}_\kappa^k$, such that $\mathcal{N}(\mathbf{s}) = \{\nu_1, \mu_1\}$ ($\nu_1 \le \mu_1$), and let m_1, l_1 be defined by (3.6), (3.28) and (3.29), respectively. Then:*

(1) The problem $MP_\kappa^k(\boldsymbol{s}, 2\mu_1 - 1)$ is solvable if and only if

$$\kappa_1^+ := \nu_-(S_{\mu_1}) \le \kappa \quad \textit{and} \quad k_1^+ := \nu_-(S_{\mu_1}^+) \le k. \tag{3.30}$$

(2) $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, 2\mu_1 - 1)$ if and only if f admits the following representation

$$f = T_{M_1 L_1}[\tau] = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1(z) + \tau(z)}}, \tag{3.31}$$

where

$$\tau \in \mathbf{N}_{\kappa-\kappa_1^+}^{k-k_1^+} \quad \textit{and} \quad \tau(z) = o(1) \quad \textit{as} \quad z \widehat{\rightarrow} \infty. \tag{3.32}$$

The indices κ_1^+ and k_1^+ can be expressed in terms of m_1 and l_1 by

$$\kappa_1^+ = \kappa_-(zm_1) + \kappa_-(l_1), \quad k_1^+ = \kappa_-(m_1) + \kappa_-(zl_1). \tag{3.33}$$

(3) If $\ell > 2\mu_1 - 1$, then $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, \ell)$, if and only if f admits the representation (3.31)*, where*

$$\tau \in \mathcal{M}_{\kappa-\kappa_1}^{k-k_1^+}(\boldsymbol{s}^{(1)}, \ell - 2\mu_1), \tag{3.34}$$

κ_1^+ and k_1^+ are determined by (3.30) *and the sequence $\{s_i^{(1)}\}_{i=-1}^{\ell-2\mu_1}$ is determi-ned by the matrix equation*

$$T(l_1, -s_0^{(1)}, \dots, -s_{\ell-2\mu_1}^{(1)})\, T(\mathfrak{s}_{-1}^{(1)}, \dots, \mathfrak{s}_{\ell-2\mu_1}^{(1)}) = I_{\ell-2\mu_1+2}, \tag{3.35}$$

if $\mu_1 = \nu_1$, and if $\nu_1 < \mu_1$ by the following equation

$$T(l_{\mu_1-\nu_1}^{(1)}, \dots, l_0^{(1)}, -s_0^{(1)}, \dots, -s_{\ell-2\mu_1}^{(1)})\, T(\mathfrak{s}_{\mu_1-\nu_1-1}^{(1)}, \dots, \mathfrak{s}_{\ell-2\nu_1}^{(1)}) = I_{\ell-\mu_1-\nu_1+2}. \tag{3.36}$$

Proof. (1)–(3) are implied by the above considerations, in particular, (3.30) follows from (3.26) and (3.33) follows from (3.27) and Proposition 2.1. □

Remark 3.5. The sequence $\{s_i^{(1)}\}_{i=0}^{\ell-2\mu_1}$ can also be found by the following formula (see [4, Proposition 2.1], [10, (3.38)])

$$s_i^{(1)} = \frac{(-1)^{i+\mu_1-\nu_1}}{(\mathfrak{s}_{\mu_1-\nu_1-1}^{(1)})^{i+\mu_1-\nu_1+2}} \begin{vmatrix} \mathfrak{s}_{\mu_1-\nu_1}^{(1)} & \mathfrak{s}_{\mu_1-\nu_1-1}^{(1)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \mathfrak{s}_{\mu_1-\nu_1-1}^{(1)} \\ \mathfrak{s}_{2(\mu_1-\nu_1)+i}^{(1)} & \dots & \dots & \dots & \mathfrak{s}_{\mu_1-\nu_1}^{(1)} \end{vmatrix}, \tag{3.37}$$

where $i = \overline{0, \ell - 2\mu_1}$.

Remark 3.6. The resolvent matrix of the basic even moment problem $\mathcal{M}_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$ takes the form

$$W_2(z) = \begin{pmatrix} 1 & l_1(z) \\ -zm_1(z) & -zm_1(z)l_1(z) + 1 \end{pmatrix}. \tag{3.38}$$

Furthermore, $W_2(z)$ admits the following factorization

$$W_2(z) = M_1(z)L_1(z), \tag{3.39}$$

where the matrices $M_1(z)$ and $L_1(z)$ are defined by (2.12), (2.15) and the corresponding linear fractional transform is defined by

$$T_{W_2}[f_1] = \frac{f_1(z) + l_1(z)}{-zm_1(z)f_1(z) - zm_1(z)l_1 + 1}. \tag{3.40}$$

Remark 3.7. If the sequence $\mathbf{s} = \{s_i\}_{i=0}^{2\mu_1-1}$ belongs to $\mathcal{H}_{\kappa,2\mu_1-1}^{k,reg}$, then $l_1(z)$ is a constant,

$$l_1 = \frac{1}{\mathfrak{s}_{-1}^{(1)}} \qquad \text{and} \qquad L_1 = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}.$$

In this case, the resolvent matrix $W_2(z)$ of the basic even moment prob-lem $\mathcal{M}_\kappa^k(\mathbf{s}, 2\mu_1 - 1)$ admits the factorization

$$W_2(z) = M_1(z)L_1.$$

and (3.38) takes the form

$$W_2(z) = \begin{pmatrix} 1 & l_1 \\ -zm_1(z) & -zm_1(z)l_1 + 1 \end{pmatrix}. \tag{3.41}$$

4. The Schur algorithm

In this section we study a step-by-step algorithm, which describes all solutions of the general nondegenerate indefinite moment problem in the class $\mathbf{N}_\kappa^k$. This algorithm is based on the elementary steps introduced in the previous section.

4.1. Odd moment problem

Let $MP_\kappa^k(\mathbf{s}, 2\nu_N - 2)$ be a nondegenerate odd moment problem, i.e.

$$D_{\nu_N} \neq 0 \quad \text{and} \quad D_{\nu_N-1}^+ \neq 0. \tag{4.1}$$

Theorem 4.1. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2\nu_N-2} \in \mathcal{H}_{\kappa,2\nu_N-2}^k$, *let* $\mathcal{N}(\mathbf{s}) = \{\nu_j\}_{j=1}^N \cup \{\mu_j\}_{j=1}^{N-1}$, *and let* $m_j(z)$ *and* $l_j(z)$ *be defined by* (4.10) *and* (4.11), *respectively. Then:*

(1) A nondegenerate odd moment problem $MP_\kappa^k(\boldsymbol{s}, 2\nu_N - 2)$ *is solvable if and only if*

$$\kappa_N := \nu_-(S_{\nu_N}) \leq \kappa \quad \text{and} \quad k_N := \nu_-(S_{\nu_N-1}^+) \leq k. \tag{4.2}$$

(2) $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, 2\nu_N - 2)$ *if and only if* f *admits the following representation*

$$f = T_{W_{2N-1}}[\tau], \tag{4.3}$$

where

$$W_{2N-1}(z) := M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z) \tag{4.4}$$

and $\tau(z)$ *satisfies the conditions*

$$\tau \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z), \quad z\widehat{\rightarrow}\infty. \tag{4.5}$$

(3) *The representation* (4.3) *can be rewritten as a continued fraction expansion*

$$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1(z) + \cfrac{1}{-zm_2(z) + \cdots + \cfrac{1}{-zm_N(z) + \cfrac{1}{\tau(z)}}}}}. \tag{4.6}$$

(4) *The indices* κ_N *and* k_N *are related to* m_j *and* l_j *by*

$$\kappa_N = \sum_{j=1}^{N} \kappa_-(zm_j) + \sum_{j=1}^{N-1} \kappa_-(l_j), \quad k_N = \sum_{j=1}^{N} \kappa_-(m_j) + \sum_{j=1}^{N-1} \kappa_-(zl_j).$$

Proof. Let $f \in \mathbf{N}_\kappa^k$ and f have the asymptotic

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2\nu_N-2}}{z^{2\nu_N-1}} + o\left(\frac{1}{z^{2\nu_N-1}}\right), \quad z\widehat{\to}\infty.$$

Then by Theorem 3.4, the function f can be represented as follows

$$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1(z) + f_1(z)}},$$

where the polynomials m_1 and l_1 are defined by (3.6) and (3.29), respectively, and

$$\kappa_1^+ = \kappa_-(zm_1) + \kappa_-(l_1) \le \kappa \quad \text{and} \quad k_1^+ = \kappa_-(m_1) + \kappa_-(zl_1) \le k. \tag{4.7}$$

In this case $f_1 \in \mathbf{N}_{\kappa-\kappa_1^+}^{k-k_1^+}$ and f_1 has the following asymptotic expansion

$$f_1(z) = -\frac{s_0^{(1)}}{z} - \frac{s_1^{(1)}}{z^2} - \cdots - \frac{s_{2(\nu_N-\nu_1)-2}^{(1)}}{z^{2(\nu_N-\nu_1)-1}} + o\left(\frac{1}{z^{2(\nu_N-\nu_1)-1}}\right), \quad z\widehat{\to}\infty,$$

where the sequence $\mathbf{s}^{(1)} = \{s_i^{(1)}\}_{i=1}^{2(\nu_N-\nu_1)-2}$ is found recursively by (3.20), (3.35) and (3.36). Moreover, by [10, Lemma 2.5] the set of normal indices of the sequence $\mathbf{s}^{(1)}$ is

$$\mathcal{N}(\mathbf{s}^{(1)}) = \{n_j - \nu_1\}_{j=2}^{N}.$$

Continuing this process and applying Theorem 3.4 $N-1$ times, one const-ructs sequences of polynomials m_j, l_j and functions f_j, g_j, such that

$$-\frac{1}{f_{j-1}(z)} = zm_j(z) - \frac{1}{g_j(z)}, \qquad 1 \le j \le N,$$

$$g_j(z) = l_j(z) + f_j(z), \qquad 1 \le j \le N-1.$$

The indices κ_j^+ and k_j^+ are defined by

$$\begin{aligned} \kappa_j^+ &= \sum_{i=1}^{j} \kappa_-(zm_i) + \sum_{i=1}^{j} \kappa_-(l_i) \le \kappa, \\ k_j^+ &= \sum_{i=1}^{N} \kappa_-(m_i) + \sum_{i=1}^{j} \kappa_-(zl_i) \le k. \end{aligned} \tag{4.8}$$

Hence

$$g_j \in \mathbf{N}_{\kappa-\kappa_j}^{k-k_j} \quad \text{and} \quad f_j \in \mathbf{N}_{\kappa-\kappa_j^+}^{k-k_j^+}, \qquad 1 \le j \le N-1.$$

Moreover, g_j and f_j have the following induced asymptotic expansions

$$g_j(z) = -\mathfrak{s}_{-1}^{(j)} - \frac{\mathfrak{s}_0^{(j)}}{z} - \frac{\mathfrak{s}_1^{(j)}}{z^2} - \cdots - \frac{\mathfrak{s}_{2(\nu_N-\nu_j)-2}^{(j)}}{z^{2(\nu_N-\nu_j)-1}} + o\left(\frac{1}{z^{2(\nu_N-\nu_j)-1}}\right), \quad z\widehat{\to}\infty,$$

$$f_j(z) = -\frac{s_0^{(j)}}{z} - \frac{s_1^{(j)}}{z^2} - \cdots - \frac{s_{2(\nu_N-\mu_j)-2}^{(j)}}{z^{2(\nu_N-\mu_j)-1}} + o\left(\frac{1}{z^{2(\nu_N-\mu_j)-1}}\right), \quad z\widehat{\to}\infty,$$

where the sequences $\{\mathfrak{s}_i^{(j)}\}_{i=-1}^{2(\nu_N-\nu_i)-2}$ and $\{s_i^{(j)}\}_{i=0}^{2(\nu_N-\mu_i)-2}$ are found from the equalities

$$T(m_{\nu_j-1}^{(j)},\ldots,m_0^{(j)},-\mathfrak{s}_{-1}^{(j)},\ldots,-\mathfrak{s}_{\ell_j-2\nu_j}^{(j)})T(s_{\nu_j-1}^{(j)},\ldots,s_{\ell_j}^{(j)}) = I_{\ell_j-\nu_1+2}, \ \ \ell_j = \ell-2\mu_{j-1},$$

$$T(l_{\mu_j-\nu_j}^{(j)},\ldots,l_0^{(j)},-s_0^{(j)},\ldots,-s_{\ell-2\mu_j}^{(j)})\,T(\mathfrak{s}_{\mu_j-\nu_j-1}^{(j)},\ldots,\mathfrak{s}_{\ell_j-2\nu_j}^{(j)}) = I_{\ell-\mu_j-\nu_j+2}.$$

Therefore, f_{j-1} takes the following representation in terms of f_j:

$$f_{j-1}(z) = \cfrac{1}{-zm_j(z) + \cfrac{1}{l_j(z)+f_j(z)}} \quad (j = 1,\ldots,N-1), \tag{4.9}$$

Here the sequence $\mathbf{s}^{(j)} = \{s_i^{(j)}\}_{i=0}^{2(\nu_N-\mu_j)-2}$ is determined recursively by (3.20) and (3.36) and polynomials m_j and l_j are defined by the formulas

$$m_j(z) = \frac{(-1)^{\nu+1}}{\det S_\nu^{(j)}} \begin{vmatrix} 0 & \dots & 0 & s_{\nu-1}^{(j-1)} & s_\nu^{(j-1)} \\ \vdots & & \dots & \dots & \vdots \\ s_{\nu-1}^{(j-1)} & \dots & \dots & \dots & s_{2\nu-2}^{(j-1)} \\ 1 & z & \dots & z^{\nu-2} & z^{\nu-1} \end{vmatrix}, \tag{4.10}$$

$$l_j(z) = \begin{cases} \frac{1}{\mathfrak{s}_{-1}^{(j)}} = (-1)^{\nu+1} s_{\nu-1}^{(j)} \frac{D_\nu^{(j)}}{D_\nu^{(j)+}}, & \text{if } \nu_j = \mu_j; \\ \frac{1}{\mathfrak{s}_{\mu-1}^{(j)} \det(\mathcal{S}_\mu^{(j)})} \begin{vmatrix} \mathfrak{s}_0^{(j)} & \dots & \mathfrak{s}_{\mu-1}^{(j)} & \mathfrak{s}_\mu^{(j)} \\ \dots & \dots & \dots & \dots \\ \mathfrak{s}_{\mu-1}^{(j)} & \dots & \mathfrak{s}_{2\mu-2}^{(j)} & \mathfrak{s}_{2\mu-1}^{(j)} \\ 1 & \dots & z^{\mu-1} & z^\mu \end{vmatrix}, & \text{if } \quad \nu_j < \mu_j. \end{cases}, \tag{4.11}$$

where $\nu = \nu_j - \mu_{j-1}$ and $\mu = \mu_j - \nu_j$ for all $j = 1, \dots, N-1$.

Let the matrix functions $M_j(z)$ and $L_j(z)$ be defined by

$$M_j(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} \text{ and } L_j(z) = \begin{pmatrix} 1 & l_j(z) \\ 0 & 1 \end{pmatrix}, \ (j = 1, \dots, N-1). \tag{4.12}$$

Then it follows from (4.9) that

$$f_{j-1}(z) = T_{M_j(z)L_j(z)}[f_j(z)] \quad (j = 1, \dots, N-1). \tag{4.13}$$

On the last step we get the function $f_{N-1}(z)$, which is a solution of the basic moment problem $MP_\kappa^k(\mathbf{s}^{(N-1)}, 2(\nu_N - \mu_{N-1}) - 2)$. By Theorem 3.2, the function $f_{N-1}(z)$ can be represented as

$$f_{N-1}(z) = \frac{1}{-zm_N(z) + \dfrac{1}{f_N(z)}} = T_{M_N(z)}[f_N(z)], \tag{4.14}$$

where the polynomial $m_N(z)$ is defined by (4.10) and $f_N(z)$ is a function from $\mathbf{N}_{\kappa-\kappa_N}^{k-k_N}$, such that $f_N(z)^{(-1)} = o(z)$ as $z\widehat{\rightarrow}\infty$ and

$$\kappa_N = \kappa_{N-1}^+ + \kappa_-(zm_N) \le \kappa \quad \text{and} \quad k_N = k_{N-1}^+ + \kappa_-(m_N) \le k. \tag{4.15}$$

Now (4.2) is implied by (4.8) and (4.15).

The converse statements of Theorem 4.1 are also implied by Theorem 3.2 and Theorem 3.4. Replacing $f_N(z)$ by $\tau(z)$, we get (2) and (3). Combining (4.9), (4.13) and Lemmas 2.4–2.5, we obtain the statement (4). □

4.2. Even moment problem

Let $\mathbf{s} = \{s_i\}_{i=0}^{2\mu_N-1} \in \mathcal{H}^k_{\kappa,2\mu_N-1}$, let the set of normal indices $\mathcal{N}(\mathbf{s}) = \{\nu_j\}_{j=1}^N \cup \{\mu_j\}_{j=1}^N$ and let $MP^k_\kappa(\mathbf{s}, 2\mu_N - 1)$ be a nondegenerate even moment problem, i.e.

$$D_{\mu_N} \neq 0 \quad \text{and} \quad D^+_{\mu_N} \neq 0. \tag{4.16}$$

Theorem 4.2. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2\mu_N-1} \in \mathcal{H}^k_{\kappa,2\mu_N-1}$ *and let* $\mathcal{N}(\mathbf{s}) = \{\nu_j\}_{j=1}^N \cup \{\mu_j\}_{j=1}^N$.

(1) *A nondegenerate odd moment problem* $MP^k_\kappa(\boldsymbol{s}, 2\mu_N-1)$ *is solvable, if and only if*

$$\kappa^+_N := \nu_-(S_{\mu_N}) \leq \kappa \quad \text{and} \quad k^+_N := \nu_-(S^+_{\mu_N}) \leq k; \tag{4.17}$$

(2) $f \in \mathcal{M}^k_\kappa(\boldsymbol{s}, 2\mu_N - 1)$ *if and only if* f *admits the representation*

$$f = T_{W_{2N}}[\tau], \tag{4.18}$$

where

$$W_{2N}(z) := W_{2N-1}(z)L_N(z) = M_1(z)L_1(z)\dots M_N(z)L_N(z) \tag{4.19}$$

and $\tau(z)$ *satisfies the following conditions*

$$\tau \in \mathbf{N}^{k-k^+_N}_{\kappa-\kappa^+_N} \quad \text{and} \quad \frac{1}{\tau(z)} = o(1), \quad z\widehat{\rightarrow}\infty; \tag{4.20}$$

(3) *The representation* (4.18) *can be rewritten as the continued fraction expansion*

$$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1(z) + \cdots + \cfrac{1}{-zm_N(z) + \cfrac{1}{l_N(z) + \tau(z)}}}}, \tag{4.21}$$

where $m_j(z)$ *and* $l_j(z)$ *are defined by* (3.6) *and* (4.11), *respectively;*

(4) *The indices* κ^+_N *and* k^+_N *can be found by*

$$\kappa^+_N = \sum_{j=1}^N \kappa_-(zm_j) + \sum_{j=1}^N \kappa_-(l_j), \quad k^+_N = \sum_{j=1}^N k_-(m_j) + \sum_{j=1}^N \kappa_-(zl_j).$$

Proof. Applying Theorem 3.4 $N-1$ times in the same way as in the odd case one obtains the sequence of $f_j \in \mathbf{N}_{\kappa-\kappa_j^+}^{k-k_j^+}$ and polynomials m_j and l_j defined by (4.10) and (4.11), respectively, such that (4.8) and (4.9) hold. On the last step we obtain the function $f_{N-1}(z)$, which is a solution of the basic even moment problem $MP_{\kappa-\kappa_{N-1}^+}^{k-k_{N-1}^+}(\mathbf{s}^{(N-1)}, 2(\mu_N - \mu_{N-1}) - 1)$. By Theorem 3.4, the function f_{N-1} can be represented as follows:

$$f_{N-1}(z) = \cfrac{1}{-zm_N(z) + \cfrac{1}{l_N(z) + f_N(z)}}, \tag{4.22}$$

the inequalities

$$\begin{aligned} \kappa_N^+ &= \kappa_{N-1}^+ + \kappa_-(zm_N) + \kappa_{-1}(l_N) \le \kappa, \\ k_N^+ &= k_{N-1}^+ + \kappa_-(m_N) + \kappa_-(zl_N) \le k \end{aligned} \tag{4.23}$$

hold and $f_N(z)$ is a function from $\mathbf{N}_{\kappa-\kappa_N^+}^{k-k_N^+}$, such that $f_N(z) = o(1)$ as $z\widehat{\rightarrow}\infty$.

Replacing f_N by τ and combining the statements (4.9) and (4.22) one obtains (2)–(4).

By (4.9) and (4.22) the inequality (4.17) is implied by (4.8), (4.23). Conversely, if (4.17) holds, one can apply Theorem 3.2 $N-1$ times and then Theorem 3.4. By these theorems the function f determined by (4.18) belongs to $MP_\kappa^k(\mathbf{s}, 2\mu_N - 1)$. This completes the proof. □

5. Resolvent matrices in odd and even cases

5.1. Odd moment problem

In the present section resolvent matrices W_{2N-1} and W_{2N} for odd and even moment problem will be studied.

Recall some facts concerning continued fractions

Proposition 5.1. *([24, Chapter I]) Let $a_1, a_2, \ldots, a_n, \omega \in \mathbb{C}$ and let*

$$f_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n + \omega}}}. \tag{5.1}$$

Then f_n can be represented as follows

$$\frac{A_{n-1}\omega + A_n}{B_{n-1}\omega + B_n}, \tag{5.2}$$

where the quantities A_i, B_i $(i \in \mathbb{N})$ are solutions of the following recurrence system

$$y_{i+1} - y_i = a_{i+1} y_{i-1}, \qquad i = \overline{0, n-1}, \tag{5.3}$$

subject to the initial conditions

$$A_{-1} = 1, \quad A_0 = 0, \quad B_{-1} = 0, \quad B_0 = 1. \tag{5.4}$$

Continued fractions (4.6) and (4.21) have partial denominators of two types

$$a_{2i-1} = -z m_i(z) \quad \text{and} \quad a_{2i} = l_i(z), \quad i = \overline{1, N}. \tag{5.5}$$

Therefore, it is reasonable to write (5.3) separately for odd and even indices. The numerator and denominator of the $n-$th convergent of (5.1) will be denoted by

$$Q_i^+(z) = A_i \quad \text{and} \quad P_i^+(z) = B_i. \tag{5.6}$$

Then the equality (5.3) takes the form

$$\begin{aligned} y_{2i+1} - y_{2i-1} &= -z m_{i+1}(z) y_{2i}, \\ y_{2i+2} - y_{2i} &= l_{i+1}(z) y_{2i+1}. \end{aligned} \tag{5.7}$$

By Proposition 5.1 $P_i^+(z)$ and $Q_i^+(z)$ are solutions of the system(5.7) subject to the initial conditions

$$P_{-1}^+(z) \equiv 0, \qquad P_0^+(z) \equiv 1, \qquad Q_{-1}^+(z) \equiv 1, \qquad Q_0^+(z) \equiv 0. \tag{5.8}$$

Polynomials $P_i^+(z)$ and $Q_i^+(z)$ will be called generalized Stieltjes polyno-mials of the first and the second kind, respectively. In the case of a regular sequence $\{s_i\}_{i=0}^{\ell} \in \mathcal{H}_\kappa^{k,reg}$ explicit formulas for $P_i^+(z)$ and $Q_i^+(z)$ were found in [10]. In the definite case (i.e. $\mathbf{s} \in \mathcal{H}_0^0$) see [22, v.4.2] and [11], [12, (10.29)].

The results of Theorems 4.1 and 4.2 can be reformulated in terms of generalized Stieltjes polynomials.

Theorem 5.2. *Let $\mathbf{s} \in \mathcal{H}_{\kappa, 2\nu_N - 2}^k$, let* (4.2) *hold and let polynomials $m_j(z)$ $(1 \le j \le N)$ and $l_j(z)$ $(1 \le j \le N-1)$ be defined by* (4.10) *and* (4.11), *respectively. Let $P_i^+(z)$ and $Q_i^+(z)$ be generalized Stieltjes polynomials of the first and the second kind, respectively. Then any solution of the moment problem $MP_\kappa^k(\boldsymbol{s}, 2\nu_N - 2)$ admits the following representation*

$$f(z) = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N-2}^+(z)}{P_{2N-1}^+(z)\tau(z) + P_{2N-2}^+(z)}, \tag{5.9}$$

where τ satisfies the conditions

$$\tau(z) \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N} \quad \textit{and} \quad \frac{1}{\tau(z)} = o(z), \quad (z\widehat{\to}\infty). \tag{5.10}$$

Furthermore, the resolvent matrix of the odd moment problem $MP_\kappa^k(\boldsymbol{s}, 2\nu_N - 2)$

$$W_{2N-1}(z) = \begin{pmatrix} Q_{2N-1}^+ & Q_{2N-2}^+ \\ P_{2N-1}^+ & P_{2N-2}^+ \end{pmatrix} \tag{5.11}$$

belongs to the $\mathcal{U}_{\kappa_N}(J)$ *and admits the following factorization*

$$W_{2N-1}(z) = M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z), \tag{5.12}$$

where the matrices $M_j(z)$ *and* $L_j(z)$ *are defined by* (4.12).

Proof. Assume f belongs to the Nevanlinna class $\mathbf{N}_\kappa^k$ and f has the asymptotic expansion

$$f(z) = -\frac{s_0}{z} - \dots - \frac{s_{2\nu_N-2}}{z^{2\nu_N-1}} + o\left(\frac{1}{z^{2\nu_N-1}}\right), \quad z\widehat{\to}\infty.$$

Then, by Proposition 4.1, the function f takes the following form

$$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1(z) + \dots + \cfrac{1}{l_{N-1}(z) + \cfrac{1}{-zm_N(z) + \cfrac{1}{\tau(z)}}}}}, \tag{5.13}$$

where (5.10) holds and by Proposition 5.1, we can rewrite f as follows

$$f(z) = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N-2}^+(z)}{P_{2N-1}^+(z)\tau(z) + P_{2N-2}^+(z)}, \tag{5.14}$$

where the polynomials Q_{2N-2}^+, Q_{2N-1}^+ and P_{2N-2}^+, P_{2N-1}^+ are defined by the recurrence relations (5.7)–(5.8).

Hence, the solution matrix $W_{2N-1}(z)$ is well defined by (5.11). Applying the induction, we show that $W_{2N-1}(z)$ admits the factorization (5.12), i.e.

(i) if $i = 1$, then $W_1(z) = M_1(z)$ and

$$W_1(z) = \begin{pmatrix} Q_1^+(z) & Q_0^+(z) \\ P_1^+(z) & P_0^+(z) \end{pmatrix}; \tag{5.15}$$

(ii) if $i = N-1$, then (4.12) and (5.12) hold (assumption of induction);

(iii) if $i = N$, then

$$\begin{aligned} W_{2N-1}(z) &= M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z) = W_{2N-3}(z)L_{N-1}M_N(z) \\ &= \begin{pmatrix} Q^+_{2N-3}(z) & Q^+_{2N-4}(z) \\ P^+_{2N-3}(z) & P^+_{2N-4}(z) \end{pmatrix} \begin{pmatrix} 1 & l_{N-1}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_N(z) & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q^+_{2N-3}(z) & l_{N-1}(z)Q^+_{2N-3}(z) + Q^+_{2N-4}(z) \\ P^+_{2N-3}(z) & l_{N-1}(z)P^+_{2N-3}(z) + P^+_{2N-4}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_N(z) & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q^+_{2N-3}(z) & Q^+_{2N-2}(z) \\ P^+_{2N-3}(z) & P^+_{2N-2}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -zm_N(z) & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q^+_{2N-3}(z) - zm_N(z)Q^+_{2N-2}(z) & Q^+_{2N-2}(z) \\ P^+_{2N-3}(z) - zm_N(z)P^+_{2N-2}(z) & P^+_{2N-2}(z) \end{pmatrix} \\ &= \begin{pmatrix} Q^+_{2N-1}(z) & Q^+_{2N-2}(z) \\ P^+_{2N-1}(z) & P^+_{2N-2}(z) \end{pmatrix}. \end{aligned} \tag{5.16}$$

So, (5.12) is proved.

By Lemmas 2.4 and 2.5 the matrices valued function $M_i(z)$ and $L_i(z)$ belong to the classes

$$M_i(z) \in \mathcal{U}_{\kappa_-(zm_i)}(J) \quad \text{and} \quad L_i(z) \in \mathcal{U}_{\kappa_-(l_i)}(J), \quad i = \overline{1, N}.$$

As is known the product of mvf's from the classes $\mathcal{U}_{\kappa_1}(J)$ and $\mathcal{U}_{\kappa_2}(J)$ belongs to the class $\mathcal{U}_{\kappa'}(J)$, where $\kappa' \le \kappa_1 + \kappa_2$.

Therefore

$$W_{2N-1}(z) = M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z) \in \mathcal{U}_{\kappa'}(J),$$

where

$$\kappa' \le \sum_{j=1}^{N} \kappa_-(zm_j) + \sum_{j=1}^{N-1} \kappa_-(l_j) = \kappa_N. \tag{5.17}$$

By [8, Lemma 3.4] the function $f = T_{W_{2N-1}}[1]$, corresponding to the parameter $\tau(z) \equiv 1$, belongs to the class $\mathbf{N}_{\kappa''}$, with

$$\kappa'' \le \kappa'. \tag{5.18}$$

On the other hand, by Theorem 4.1 $f = T_{W_{2N-1}}[1] \in \mathbf{N}_{\kappa_N}$, i.e.

$$\kappa'' = \kappa_N. \tag{5.19}$$

Comparing (5.17), (5.18) and (5.19) yields

$$\kappa' = \kappa'' = \kappa_N$$

and thus $W_{2N-1} \in \mathcal{U}_{\kappa_N}(J)$. This completes the proof. □

Remark 5.3. In the case, where $\mathbf{s} \in \mathcal{H}_\kappa^+$, $\deg(m_i) \leq 1$ and $l_i = \text{const} > 0$ in (4.6), the moment problem $MP_\kappa^+(\mathbf{s}, 2\nu_N - 2)$ was studied in [21] and these results are the special case of Theorem 5.2.

Remark 5.4. In the case. where $f \in \mathbf{N}_\kappa^{k,reg}$, the odd moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ was studied in [10]. These results are contained in previous Theorem. Moreover, the polynomials $l_j(z)$ are non-zero constants in (5.13), such that

$$l_j(z) = \frac{1}{s_{-1}^{(j)}}. \tag{5.20}$$

5.2. Even moment problem

Now we study the even moment problem $MP_\kappa^k(\mathbf{s}, 2\mu_N - 1)$. In this case we also find all solutions of $MP_\kappa^k(\mathbf{s}, 2\mu_N - 1)$ by the following statement

Theorem 5.5. *Let* $\mathbf{s} \in \mathcal{H}_{\kappa, 2\mu_N - 1}^k$, *let* (4.17) *hold and let polynomials* $m_j(z)$ *and* $l_j(z)$ $(1 \leq j \leq N)$ *be defined by* (4.10) *and* (4.11), *respectively. Let* $P_i^+(z)$ *and* $Q_i^+(z)$ *be generalized Stieltjes polynomials of the first and the second kind, respectively. Then any solution of the moment problem* $MP_\kappa^k(\boldsymbol{s}, 2\mu_N - 1)$ *admits the following representation*

$$f(z) = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N}^+(z)}{P_{2N-1}^+(z)\tau(z) + Q_{2N}^+(z)}, \tag{5.21}$$

where τ *satisfies the following conditions*

$$\tau(z) \in \mathbf{N}_{\kappa - \kappa_N^+}^{k - k_N^+} \quad \textit{and} \quad \tau(z) = o(1), \quad z \widehat{\to} \infty. \tag{5.22}$$

Furthermore, the resolvent matrix of the even moment problem $MP_\kappa^k(\boldsymbol{s}, 2\mu_N - 1)$

$$W_{2N}(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N}^+(z) \\ P_{2N-1}^+(z) & P_{2N}^+(z) \end{pmatrix} \tag{5.23}$$

belongs to the $\mathcal{U}_{\kappa_N^+}(J)$ *and admits the following factorization*

$$W_{2N}(z) = M_1(z)L_1(z)\dots M_N(z)L_N(z), \tag{5.24}$$

where the matrices $M_j(z)$ *and* $L_j(z)$ *are defined by* (4.12).

Proof. Suppose f belongs to the Nevanlinna class $\mathbf{N}_\kappa^k$ and f has the asymptotic expansion

$$f(z) = -\frac{s_0}{z} - \cdots - \frac{s_{2\mu_N-1}}{z^{2\mu_N}} + o\left(\frac{1}{z^{2\mu_N}}\right), \quad z\widehat{\to}\infty.$$

By Proposition 4.2, the function f takes the form (4.21), where (5.22) holds. By [24, chapter I], the function f can be rewritten in the form (5.21), where the polynomials $P_i^+(z)$ and $Q_i^+(z)$ can be found as the solutions of the recurrence relations (5.7)–(5.8).

Hence, the resolvent matrix of the even moment problem $MP_\kappa^k(\mathbf{s}, 2\mu_N - 1)$ takes the form (5.23). By Theorem 5.2 (see (5.11) and (5.12)), we obtain

$$\begin{aligned} M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z)L_N(z) &= \begin{pmatrix} Q_{2N-1}^+ & Q_{2N-2}^+ \\ P_{2N-1}^+ & P_{2N-2}^+ \end{pmatrix} \begin{pmatrix} 1 & l_N(z) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q_{2N-1}^+(z) & l_N Q_{2N-1}^+(z) + Q_{2N-2}^+(z) \\ P_{2N-1}^+(z) & l_N P_{2N-1}^+(z) + P_{2N-2}^+(z) \end{pmatrix} \\ &= \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N}^+(z) \\ P_{2N-1}^+(z) & P_{2N}^+(z) \end{pmatrix} = W_{2N}(z). \end{aligned}$$

By Lemmas 2.4 and 2.5 the mvf's $M_i(z)$ and $L_i(z)$ belong to the classes

$$M_i(z) \in \mathcal{U}_{\kappa_-(zm_i)}(J) \quad \text{and} \quad L_i(z) \in \mathcal{U}_{\kappa_-(l_i)}(J), \quad i = \overline{1, N}.$$

As is known the product of mvf's from the classes $\mathcal{U}_{\kappa_1}(J)$ and $\mathcal{U}_{\kappa_2}(J)$ belongs to the class $\mathcal{U}_{\kappa'}(J)$, where $\kappa' \le \kappa_1 + \kappa_2$. Hence

$$W_{2N}(z) = M_1(z)L_1(z)\dots L_{N-1}(z)M_N(z)L_N(z) \in \mathcal{U}_{\kappa'}(J),$$

where

$$\kappa' \le \sum_{j=1}^N \kappa_-(zm_j) + \sum_{j=1}^N \kappa_-(l_j) = \kappa_N^+. \tag{5.25}$$

By [8, Lemma 3.4] the function $f = T_{W_{2N}}[z]$, corresponding to the parameter $\tau(z) = z$, belongs to the class $\mathbf{N}_{\kappa''}$, with

$$\kappa'' \le \kappa'. \tag{5.26}$$

On the other hand, by Theorem 4.2 $f = T_{W_{2N}}[z] \in \mathbf{N}_{\kappa_N^+}$, i.e.

$$\kappa'' = \kappa_N^+. \tag{5.27}$$

Comparing (5.25), (5.26) and (5.27) yields

$$\kappa' = \kappa'' = \kappa_N^+$$

and thus $W_{2N} \in \mathcal{U}_{\kappa_N^+}(J)$. This completes the proof. □

Remark 5.6. The even moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$ in the class $\mathbf{N}_\kappa^{k,reg}$ was studied in [10] and the results in [10, Theorem 5.9] is the special case of Theorem 5.5.

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