Bifurcations and a chaos strip in states of long Josephson junctions

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Stationary and nonstationary, in particular, chaotic states in long Josephson junctions are investigated. Bifurcation lines on the parametric bias current-external magnetic field plane are calculated. The chaos strip along the bifurcation line is observed. It is shown that transitions between stationary states are the transitions from metastable to stable states and that the thermodynamical Gibbs potential of these stable states may be larger than for some metastable states. The definition of a dynamical critical magnetic field characterizing the stability of the stationary states is given.

Introduction

Dynamical chaos in long Josephson junctions is of great interest because it can be a source of a dynamical noise in devices based on them, in particular, in SQUIDs, limiting the sensitivity of these devices. Furthermore, dynamical chaos in long Josephson junctions (LJJ) is a very interesting physical phenomenon taking place in nonlinear systems in the absence of an external stochastic force [1−9]. Dynamical chaos in a LJJ is easily excited and therefore it may also be investigated experimentally rather easily [10,11].

In our previous works [12,13] we have shown that among a set of solutions of the Ferrell-Prange equation describing stationary states of the LJJ in an external magnetic field [14] are both stable and unstable ones. At the same time, these stationary states are asymptotic solutions of the nonstationary sine-Gordon equation and we have also shown that a selection of the stable solutions can be governed by the initial perturbation damping rapidly in time and entering into the nonstationary sine-Gordon equation through the boundary conditions. Changing the intensity of this perturbation at fixed shape, we can obtain various stationary states for the LJJ without a bias current or three clusters of states (stationary, and time dependent regular and chaotic) in the presence of a bias current. It turned out that asymptotic states are very sensitive to an external perturbation, its value and shape define the state (stationary, regular or chaotic) to which the system will tend at $t \rightarrow \infty$ (we have called this influence on the selection of asymptotic states of the small initial perturbation damping rapidly in time an effect of memory). The fact of coexistence of all these three characteristic asymptotic states selected only by the form of the initial perturbation seems to be astonishing. It is evidently enough that the Ferrell-Prange equation will not have solutions at a large bias current β. Therefore the question arises: at which values of β do stationary states of a LJJ disappear or what will be a boundary in the parametric $\beta - H_0$ plane (H_0 is an external magnetic field) that separates this plane on the regions where stationary states do and do not exist? Since the number of solutions of the Ferrell-Prange equation changes at variation of the parameters (H_0, β), another question arises: what is the form of bifurcation lines in the plane $\beta - H_0$ that separate the parametric plane on the regions with a different number of stationary solutions of the Ferrell-Prange equation?

The existence of several stable solutions of the Ferrell-Prange equation is equivalent to the fact that thermodynamical Gibbs potential G associated with the distribution of the magnetic field along the junction has minima, and each minimum corresponds to a certain solution of the Ferrell-Prange equation. Does a global minimum of G correspond to the most stable state (e.g., in the Lyapunov sense)? In the case of the junction of the finite length both Meissner and one-fluxon states are

thermodynamically advantageous simultaneously, so it is interesting to investigate dynamical properties of these states. Answering this question, we introduce a dynamical critical field that describes the stability characteristic of the junctions.

In Sec. 1 bifurcation lines on the parametric $\beta - H_0$ plane are calculated. In Sec. 2 the definition of the dynamical critical magnetic field is given and the dependence of this field on β and the length of the junction L is calculated. In Sec. 3 transitions between states are described. It is shown in Sec. 4 that a chaos strip arises along the bifurcation line on the parametric β–H_0 plane. The last Sec. 5 contains the discussion of our calculation and brief conclusions.

1. Bifurcation lines

Stationary states of a LJJ are investigated using the numerical integration of the Ferrell-Prange equation:

$$\varphi_{xx}(x) = \sin \varphi(x) - \beta , \qquad (1)$$

where $\varphi(x)$ is the stationary Josephson phase variable, β is the dc bias current density normalized to the critical current j_c, x is the distance along the junction normalized to the Josephson penetration length $\lambda_J = \sqrt{C\Phi_0/8\pi^2 j_c\,d}$, Φ_0 is the flux quantum, $d = 2\lambda_L + b$, λ_L is the London penetration length, b is the thickness of the dielectric barrier. The boundary conditions for Eq. (1) have the form

$$\varphi_x(x)\big|_{x=0} = \varphi_x(x)\big|_{x=L} = H_0 , \qquad (2)$$

where L is the total length of the junction normalized to λ_J and H_0 is the external magnetic field perpendicular to the junction and normalized to $\tilde{H} = \Phi_0/2\pi\lambda_J\,d$.

Numerical integration of Eqs. (1)−(2) allows us to find the regions with a certain number of solutions on the parametric β–H_0 plane (Fig. 1). It is easy to show that the set of points corresponding to the even number of solutions forms two-dimensional domains on this plane, whereas the set corresponding to the odd ones may form just one-dimensional curves. Mostly, the lines corresponding to the odd number of the solutions of the Ferrell-Prange boundary problem coincide with the bifurcation lines. Using the shooting method for solving of the boundary problem one can prove that the 2π-periodicity of the function $H(\varphi_0)$ expressing the dependence of the magnetic field at the right side of the junction ($x = L$) on the phase taken at the left side ($x = 0$) results in the appearing of the odd number of solutions only when the $H(\varphi_0)$ touches the line $H = H_0$ in an extreme point, i.e., $\partial H(\varphi_0)/\partial\varphi_0 = 0$. As an illustration, we have plotted in Fig. 2 the function $H(\varphi_0)$ at $H_0 = 0.5$, $L = 5$, $\beta = 0.25$ and $\beta = 0.45$.

Boundaries between the regions − bifurcation lines − define an essential modification of the system. The bifurcation lines in Fig. 1 are obtained for $L = 5$; here a step by β is equal to $5\cdot10^{-3}$ and a step by H_0 is equal to $2.5\cdot10^{-3}$. In this figure the numbers of solutions of Eq. (1)−(2) are pointed out, the numbers of stable solutions are given in the brackets, and M and $1f$ denote a stable Meissner and one-fluxon states, respectively. It is seen that a Meissner state is stable at small values of H_0 and at large values of H_0 a one fluxon state is stable. It should be noted that the region where there are no
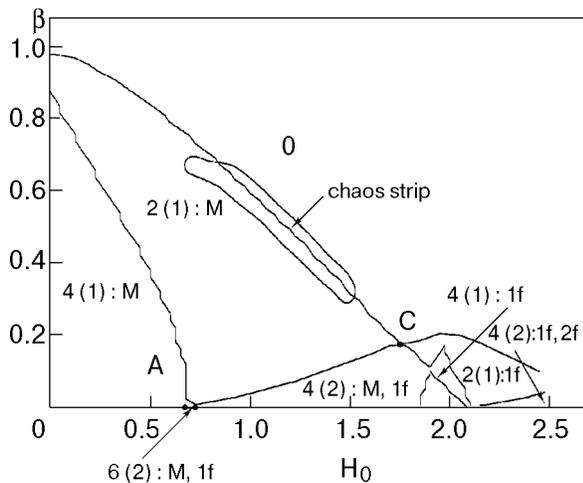

Fig. 1. Bifurcation lines. The number of solution of the Ferrell-Prange equation (1)−(2) are pointed out. The number of stable states is indicated in brackets. *M* denotes a stable Meissner state and 1*f* denotes a stable one-fluxon state. *L* = 5.
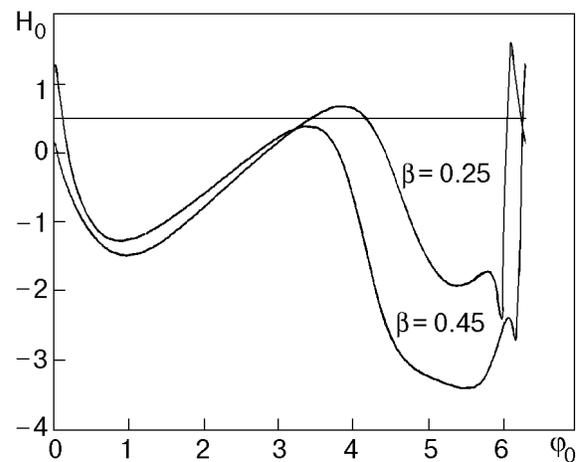

Fig. 2. Dependence of the magnetic field at $x = L$ on the phase taken at the left side of junction $x = 0$ at $H_0 = 0.5$, $L = 5$, $\beta = 0.25$ and $\beta = 0.45$.

stationary solutions (region 0) bounds with the region having a minimum of stationary solutions, being equal to 2 (region 2). In approaching the boundary of region 0 and 2 the number of stationary solutions decreases: $6 \to 4 \to 2 \to 0$, on the other hand, a number of nonstationary states which are the asymptotic solutions of the sine-Gordon equation, increases. Our calculations have shown that one of two stationary solutions in region 2 is stable, and another is unstable (metastable). We noted earlier [12] that the stable states are symmetrical. The presence of bias current β leads to a symmetry violation that results, evidently, in the instability of the states.

The problem of the stability of stationary states $\varphi(x)$ was solved in the following way [13]: the sine-Gordon equation was linearized in the vicinity of stationary solution: $\varphi(x, t) = \varphi(x) + \theta(x, t)$, where $\theta(x, t)$ is the infinitesimal perturbation. The equation for $\theta(x, t)$ — the linearized sine-Gordon equation — we can solve by means of the expansion of this function in terms of a complete system of eigenfunctions of the Schrödinger operator with potential $\cos [\varphi(x)]$:

$$\theta(x, t) = \sum_n e^{\lambda_n t} u_n(x) , \qquad (3)$$

where $u_n(x)$ are eigenfunctions of the Schrödinger operator of the problem:

$$- u_{xx}(x) + u(x) \cos \varphi(x) = Eu(x) , \qquad (4)$$

$$u_x(x)\big|_{x=0} = u_x(x)\big|_{x=L} = 0 ,$$

and

$$\lambda_n = - \gamma \pm \sqrt{\gamma^2 - E_n} , \qquad (5)$$

where γ is the dissipative coefficient in the sine-Gordon equation. We note that values of λ coincide with corresponding values of Lyapunov exponents in the case when perturbations are considered with respect to the stationary solutions. In general case, Lyapunov exponents are calculated in the same way as in Ref. 13. Thus, in the presence of a bias current we have the different picture of a LJJ states than at $\beta = 0$ (this case has been examined in Ref. 12). For example, at $H_0 = 1.9$ the increasing of β from 0 to 0.22 leads to the changing of the stationary states number $6 \to 4 \to 2 \to 0$, i.e., to a consecutive losing of the stationary solutions. Simultaneously, an increasing of the number of nonstationary states occurs that we found by directly solving the nonstationary sine-Gordon equation.

2. Dynamic critical field

In the literature the critical magnetic field H_{c1} in a LJJ is defined as a field value, at which an existence of a Josephson vortex (fluxon, soliton) becomes advantageous thermodynamically for the first time (see, for example, Refs. 10,11). In the case of an infinitely long junction the critical field is $H_{c1}(\infty) = 4/\pi \simeq 1.274$. Essentially, this field corresponds to the global minimum of the thermodynamic Gibbs potential for the one-fluxon state. However, in a junction of finite length there are some local minima that coexist with the global one and the every minimum corresponds to the solution of Eqs. (1)−(2). Some of these solutions are stable, another unstable in the sense discussed in Sec. 1.

We write down the thermodynamic Gibbs potential in the form

$$G = \int_0^L dx \ [\tfrac{1}{2} \varphi_x^2(x) + 1 - \cos \varphi(x) - \beta\varphi(x) - H_0 \varphi_x(x)].$$

$$(6)$$

Here G is the thermodynamic Gibbs potential per unit length along an external magnetic field and normalized to $\tilde{G} = \Phi_0/16\pi^3\lambda_j d$. The Ferrell-Prange equation is an extremal of the functional (6). An investigation of the second variation of G shows that all extrema of this functional satisfy to the necessary and sufficient conditions of a strong minimum [15]. Thus, all solutions of Eqs. (1)−(2) (both stable and unstable ones) correspond to minima of the thermodynamic Gibbs potential; one of them is global, the others are local. Our calculations of the thermodynamic Gibbs potential (6) show that, for example, at $\beta = 0$, $L = 5$ and $H_0 = 0.67$ the Meissner state has a global minimum ($G_M = - 0.44$), but the stable one-fluxon state has a local one ($G_{1f} = 4.03$). The one-fluxon state has a global minimum of G starting at $H_0 = 1.57$ ($G_{1f} = - 2.582$) and at the same value of β and L. At this value of H_0 a Meissner state has a local minimum $G_M = - 2.58$. At $H_0 \geq 2.09$ the Meissner state disappears. Thus, at a field less than the critical one H_{c1}, the stable one-fluxon state exists. We shall further call the minimum value of a magnetic field at given L and β, at which the stable one-fluxon state appeares for the first time and which corresponds to the local minimum of the thermodynamic Gibbs potential as the dynamical critical field H_{dc} . It is interesting that the dynamical critical field H_{dc} makes up on the parametric plane a line that coincides with the bifurcation line BC (see Fig. 1). Our calculations show that the
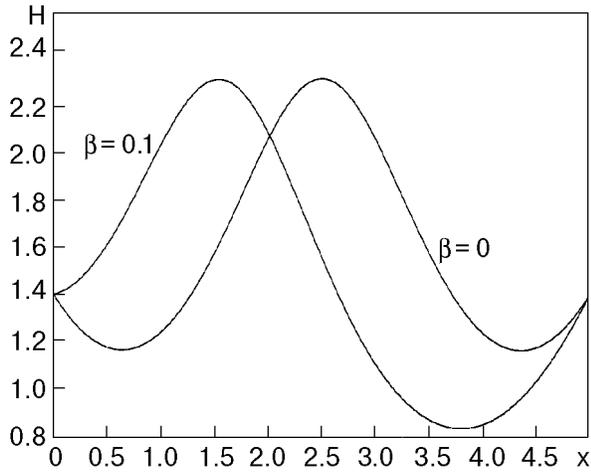
Fig.3. One-fluxon states at $H_0 = 1.4$ and $L = 5$ for $\beta = 0$ and $\beta = 0.1$.

Fig. 4. Stationary states of LJJ at $H_0 = 2.035$, $\beta = 0.001$ and $L = 5$. States 1, 2, 3, 5 are unstable, states 4 and 6 are stable.

bias current increases the dynamical critical field H_{dc}. Evidently, it is connected with a symmetry violation of a state by the bias current β. In Fig. 3 two stable one-fluxon states at $\beta = 0$ and $\beta = 0.1$ ($L = 5$, $H_0 = 1.4$) are shown. It is seen that the state with $\beta = 0.1$ is asymmetric. The dynamical critical field at $L = 5$ are $H_{dc} = 0.67$ at $\beta = 0$ and $H_{dc} = 1.4$ at $\beta = 0.1$. Upon increasing L the value of H_{dc} is changed ($\beta = 0$): $H_{dc}(L = 5) = 0.66$, $H_{dc}(L = 6) = 0.4$, $H_{dc}(L = 7) = 0.26$, $H_{dc}(L = 8) = 0.15$, $H_{dc}(L = 10) = 0.06$, i.e. the H_{dc} decreases. In this case the critical field H_{c1} has the values: $H_{c1}(L = 5) = 1.57$, $H_{c1}(L = 6) = 1.45$, $H_{c1}(L = 7) = 1.38$, $H_{c1}(L = 8) = 1.34$, $H_{c1}(L = 10) = 1.28$, i.e., the H_{c1} decreases also approaching to the value of $H_{c1}(L = \infty) = 1.274$.

3. Transitions between states

As it has been shown in the previous section, every stationary state of LJJ, i.e., the solution of Eqs. (1)−(2), corresponds to a minimum of the thermodynamic Gibbs potential and these minima are not equivalent with respect to the problem of instability. For example, in Fig. 4 stationary states of LJJ at $H_0 = 2.035$, $\beta = 0.001$ and $L = 5$ are shown. The values of the Gibbs potential calculated using Eq. (6) are as follows: $G_1 = -5.03$, $G_2 = -4.52$, $G_3 = -4.61$, $G_4 = -4.64$, $G_5 = -4.61$, $G_6 = -6.7$. States 4 (Meissner) and 6 (one-fluxon) are stable, the other ones are metastable. It should be noted that unstable state 1 corresponds to deeper minimum than the stable state 4. This property contradicts the naive idea that more stable states occur at deeper minima. Now we shall consider this question in detail.
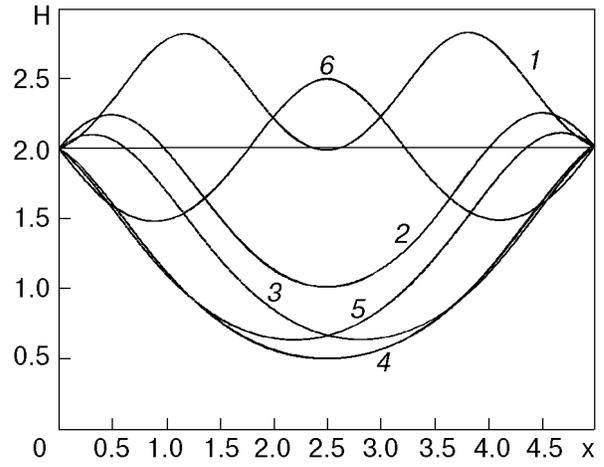
The sine-Gordon equation with dissipation and bias current describing an evolution of the initial state has the form:

$$\varphi_{tt}(x, t) + 2\gamma\varphi_t(x, t) - \varphi_{xx}(x, t) = -\sin \varphi(x, t) + \beta ,$$

(7)

where t is a time normalized to the inverse of the Josephson plasma frequency $\omega_J = \sqrt{2\pi c j_c / C\Phi_0}$, C is the junction capacitance per unit area, $\gamma = \Phi_0 \omega_J / 4\pi c R j_c$ is the dissipative coefficient per unit area, R is the resistance of junction per unit area. We write down the boundary conditions for Eq. (7) in the form

$$\varphi_x(x, t)\big|_{x=0} \equiv H(0, t) = \varphi_x(x, t)\big|_{x=L} \equiv$$

$$\equiv H(L, t) = H_0(1 - a\, e^{-t/2t_0} \cos 0.5t) .$$ (8)

The integration of Eqs. (7)−(8) for $H_0 = 2.035$, $\beta = 0.001$, $L = 5$ (the same as in Fig. 4) and $\gamma = 0.26$ gives: the metastable state 1 passes to the stable state 6 at any values of perturbation parameter a, $2 \rightarrow 4$ at $a = 0$, $2 \rightarrow 6$ at $a = 1$, $3 \rightarrow 4$ at $a = 0.05$, $3 \rightarrow 6$ at $a = 0.07$, $4 \rightarrow 6$ at $a = 0.5$ and so on. Every transition from the metastable state to the stable one, $m \rightarrow n$, is a transition from the state with the certain value of local minimum G_m to other state with smaller value of minimum G_n. These transitions $m \rightarrow n$ with $G_m > G_n$ are realized by certain values of the parameter of the initial perturbation a in expression (8). One can say that the local minima of G_l are connected with each other by a certain disintegration channel along the coordinate a. From this point of view one can say also that stationary states contain a specific «la-

tent» parameter, by which a connection with different local minima G_l may be realized. In particular, the perturbation parameter a appears here as a «latent» parameter. It is possible, there are several «latent» parameters connecting the stationary states. One of the most important characteristics of «latent» parameters is that the stationary state does not depend on them directly; however, the form of the asymptotic state and the rate of disintegration depend essentially on them. The presence of a «latent» parameter apparently explains, a nonequivalence of the different local minima with respect to the stability, especially in the case when a stable local minimum is above a nonstable local one. In Table results of the integration of Eqs. (1), (2) and the calculation of G for the every of these solutions at $H_0 = 1.174$, $\beta = 0$, $L = 8$ are represented. The transitions between states $m \rightarrow n$ are defined as follows: the m-th solution of the stationary Ferrell-Prange equation (1)–(2) was taken as an initial condition of the sine-Gordon Eqs. (7), (8). If this m-th state was unstable so it fell into the n-th stable state.

Table

Transitions between states

Number of state	Stability	G	Transitions $m \rightarrow n$	Sort of stable states
1	unstable	2.34	$1 \rightarrow 10$	
2	unstable	2.78	$2 \rightarrow 8$	
3	unstable	0.64	$3 \rightarrow 8$	
4	unstable	14.69	$4 \rightarrow 10$	
5	unstable	14.98	$5 \rightarrow 10$	
6	unstable	14.69	$6 \rightarrow 10$	
7	unstable	13.53	$7 \rightarrow 10$	
8	stable	−1.42	$8 \rightarrow 8$	Meissner
9	unstable	0.64	$9 \rightarrow 8$	
10	stable	−0.44	$10 \rightarrow 10$	1 fluxon
11	unstable	2.34	$11 \rightarrow 10$	
12	stable	2.29	$12 \rightarrow 12$	2 fluxon

The scheme of the transitions between states $m \rightarrow n$ is represented in Fig. 5. It is seen that $G_m > G_n$ for the all transitions (we note that G_3 and G_9 for the metastable states 3 and 9 are less than G_{12}; the state 12 is stable). The stable states — Meissner, one-fluxon, and two-fluxon — are shown in Fig. 6 at the same parameters as in Fig. 5.
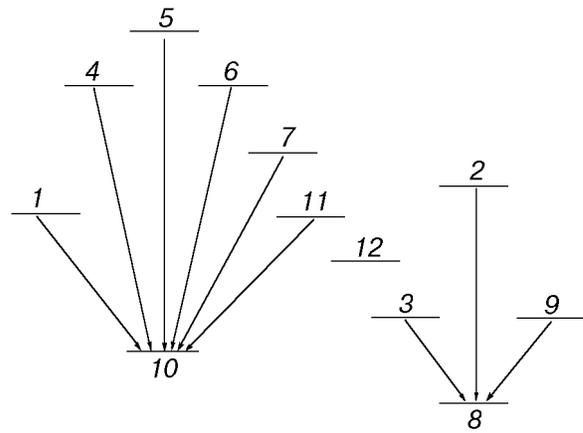

Fig. 5. The scheme of transitions between states $m \rightarrow n$. States 8, 10 and 12 are stable (8 — Meissner, 10 — one fluxon, 12 — two fluxon), others are unstable. $H_0 = 1.174$, $L = 8$, $\beta = 0$.

4. Chaos strip

As we noted above, the number of stationary states decreases with approach to the bifurcation line 0–2, but the number of nonstationary asymptotic states is increased simultaneously. Changing the perturbation parameter a we can obtain three sorts of typical states: stationary, regular and chaotic [13]. These states are distinguished not only by a form of the field distribution in the junction and a variation in time, but also by values of the Lyapunov exponent λ: for the stationary states $\lambda < 0$, for the regular states $\lambda \leq 0$ and for the chaos states $\lambda > 0$. The Lyapunov exponents were calculated in the same way as in Ref. 13. However, as the calculations have shown, chaotic states may be excited not in the whole region 2 (see Fig. 1), but only in the bounded region in close to the bifurcation line 0–2. This region is extended in the form of
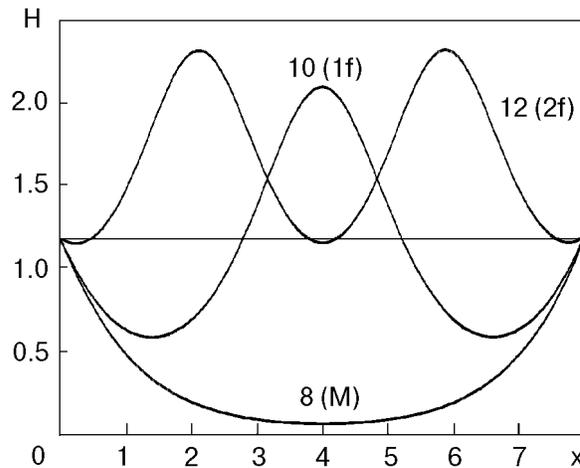

Fig. 6. The stable states: M — Meissner, $1f$ — one-fluxon, and $2f$ — two-fluxon at the same parameters as those in Fig. 5.

a narrow strip along the bifurcation line 0–2 approximately from 0.7 to 1.6 in H_0 and in the range of 0.002–0.015 in β. We note that the chaos strip is arranged mostly under the bifurcation line in the region 2, but not in the region 0, as it may be expected because of all states in the region 0 are nonstationary. The chaos strip is outlined on the parametric β–H_0 plane in Fig. 1.

This chaos strip along the bifurcation line 0–2 calls to mind (to a certain extent) the separatrix of a nonlinear oscillator, where a chaos motion is observed.

5. Discussion and conclusions

In the present work we have shown that the parametric β–H_0 plane of a LJJ is separated on series of regions with the different number of solutions of the stationary Ferrell-Prange equation. The boundaries between these regions — bifurcation lines — characterize an essential modification of the system. A chaos strip arises along the bifurcation line 0–2. We have found that the chaos strip is arranged in the main below the bifurcation line 0–2, where stationary states take place.

We have introduced the definition of a dynamical critical field as the lowest field at which the one-fluxon state becomes stable for the first time in the Lyapunov sense. In addition, the Meissner state may also be stable at same parameters. Because both the Meissner and the one-fluxon states may be thermodynamically advantageous simultaneously, our definition based on the stability in the Lyapunov sense characterizes an important feature of the stationary states of the LJJ.

We have shown that disintegration of the metastable states and the transition to some stable states $m \to n$ occur for $G_m > G_n$. A metastable state corresponds to the local minimum of the Gibbs potential, and also this minimum may be lower than this one of a stable state. A nonequivalence of these local minima we explain by means of existing of a

«latent» parameter not detecting in a stationary state, by which, for example, two local minima may be connected and a channel of the disintegration of the upper state may arise. In our case the perturbation parameter plays a role of a «latent» parameter, however, the number of these parameters may be much greater. We note the analogy between the quantum transitions and the transitions mentioned above, although the system is described by the classical Ferrell-Prange and sine-Gordon equations.

We are aware that we could not touch upon all questions concerning the properties of a LJJ. We hope to return to the problems of a LJJ in our next work.

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