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Non-Fermi-liquid behavior: Exact results for ensembles of magnetic impurities

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In this work we consider several exactly solvable models of magnetic impurities in critical quantum antiferromagnetic spin chains and multichannel Kondo impurities. Their ground state properties are studied and the finite set of nonlinear integral equations, which exactly describe the thermodynamics of the models, is constructed. We obtain several analytic low-energy expressions for the temperature, magnetic field, and frequency dependences of important characteristics of exactly solvable disordered quantum spin models and disordered multichannel Kondo impurities with essential many-body interactions. We show that the only low-energy parameter that gets renormalized is the velocity of the low-lying excitations (or the effective crossover scale connected with each impurity); the others appear to be universal. In our study several kinds of strong disorder important for experiments were used. Some of them produce low divergences in certain characteristics of our strongly disordered critical systems (compared with finite values for the homogeneous case or a single impurity). For weak disorder, or for narrow distributions of the local Kondo temperatures, our exact results reveal the presence of Kondo screening of disordered ensembles of magnetic impurities by low-lying excitations of the host. We point out that our results qualitatively coincide with the data of experiments on real disordered quasi-one-dimensional antiferromagnetic systems and with the similar behavior of some heavy metallic alloys.

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1. Introduction

The study of the behavior of magnetic impurities coupled to paramagnetic hosts remains one of the most interesting problems of the many-body physics. The Kondo effect [1], which describes the exchange interaction between the spin of a magnetic impurity and the spins of itinerant electrons, is, perhaps, the best known example in which modern theoretical methods like renormalization group (RG) theory, Bethe ansatz, bosonization, conformal field theory, etc. have manifested their strength [2–4]. The crossover from the strong coupling to the weak coupling regime is one of the most famous examples of nonperturbative effects in condensed matter theory.

In the last few years the interest in the non-Fermi-liquid (NFL) behavior of magnetic systems and metallic alloys has grown considerably. A large class of conducting nonmagnetic materials does not behave as usual Fermi liquids (FL) at low temperatures. One of the best-known examples of such behavior is the Kondo effect for multi (n) channel electron systems: For an impurity spin less than $n/2$ a NFL critical behavior results [5]. The critical behavior of a single magnetic impurity can also be connected with a quadrupolar Kondo effect or non-magnetic two-channel Kondo effect [6]. However, for most dirty metals and alloys in which the NFL behavior has been observed (see, e.g., the recent reviews [7–10] and Refs. 11–20), the magnetic susceptibility χ and low-temperature specific heat c usually manifest logarithmic or weak power law behavior with temperature T. The resistivity decreases linearly with temperature, showing a large residual resistivity. That is different from the predictions of the theory of the overscreened Kondo effect [4,5].

The last property together with the alloy nature of compounds suggests that the disorder (a random distribution of localized f electrons or a random coupling to the conducting electron host) may play the main role in the low-temperature NFL character of such systems. The idea of (nonscreened) local moments existing in disordered metallic systems has already been formulated recently [21–23]. It was proposed that near metal–insulator transitions (or for sufficiently alloyed systems far from the quantum critical point) disordered correlated metals contain localized moments. The change in the interactions between impurity sites and host spins can be considered as a modification of the characteristic energy scale, the Kondo temperature T_K. At that scale the behavior of the magnetic impurity manifests the crossover from the strong coupling regime (for $T, h << T_K$, where h is the external magnetic field) to the weak coupling regime $T_K << h,T$. The impurity spin behaves asymptotically free in the weak coupling case, and it is screened by the host spins in the strong coupling case. The random distribution of magnetic characteristics of the impurities (i.e., their Kondo temperatures) may be connected either with the randomness of exchange couplings of itinerant electrons with the local moments [22], or with the randomness of the densities of conduction electron states [21]. In fact, both types of randomness renormalize the single universal parameter — the Kondo temperature — which characterizes the state of the magnetic impurity. In Ref. 16 the results of the measurements of the magnetic susceptibility, nuclear magnetic resonance (NMR) Knight shift, and low-temperature specific heat have been reported. To explain the observed features it was necessary to assume some disorder, with a Gaussian distribution of the Kondo temperatures. However, the model used for the explanation of the experiment was oversimplified by an inadequate representation of the Kondo magnetization by the simple replacement $T \to T + bT_K$ in the Brillouin function, $B(ah/T + bT_K)$, with which the magnetization of a single magnetic moment was approximated (a, b are some constants). It was noted [16] that the data for the specific heat and Knight shift did not agree with the predictions of that simple theory, especially for nonzero values of the magnetic field. The inhomogeneous magnetic susceptibility was confirmed recently [24] by muon spin rotation experiments. The role of the long-range Ruderman–Kittel–Kasuya–Yosida (RKKY) coupling between the local moments was taken into account recently [25,26] (Griffiths phase theory), and the model was found to exhibit properties qualitatively similar to those of models with noninteracting local moments [24]. In addition, the presence of the spin–orbit interaction in some disordered heavy fermion alloys demands the study of magnetic anisotropy, which can play an essential role in the physics of disordered spin interactions [25,26].

Another interesting topic of research, which is related to the one mentioned above, is the behavior of disordered magnetic impurities in one-dimensional (1D) antiferromagnetic (AF) spin chains. Here we can mention several experiments on spin chains [27–30]. The theoretical works devoted to the description of disordered magnetic impurities in critical spin chains have mostly involved the approximate RG treatment of the problem [31–35]. Recently, however, we proposed an exact solution to the problem of the behavior of spin-1/2 AF quantum spin chain coupled to disordered magnetic impurities [36–38], which was later generalized to the description of disordered magnetic impurities in correlated electron chains [39].

It is known that the physics of a single magnetic impurity in a 1D AF Heisenberg spin $S = 1/2$ chain and that of a single Kondo impurity in a 3D free electron host are described by similar Bethe ansatz theories [2,3,40,41], e.g., the magnetization and the low-temperature magnetic specific heat of the impurity for the two models coincide. The Heisenberg model is the seminal model for correlated many-body systems. Most of its static properties are exactly known. A single spin-1/2 magnetic impurity in the AF spin chain and the Kondo impurity manifest total screening with the (marginal) FL-like low-temperature behavior of the magnetic susceptibility and specific heat, i.e., the finite values of $\chi(T)$ and $c(T)/T$ in the low-temperature limit [2,3,41,42]. In other words, the moment of the impurity is quenched by the localized host spins or by the spins of the conduction electrons, respectively. The magnetic anisotropy of the Kondo exchange interaction between the impurity spin and the spins of the free electron host was also taken into account exactly for the single Kondo impurity [3,43] and for a magnetic impurity in a AF spin chain [44,45]. It was pointed out that the magnetic anisotropy does not change drastically the Kondo effect of a single impurity. On the other hand, for the integrable lattice models one can incorporate a finite concentration of magnetic impurities [46,47] without destroying the exact solvability. Hence, for the random distribution of magnetic impurities we can suppose that low dimensionality is not essential for the

Kondo screening. The absence of magnetic ordering in the NFL Kondo systems [7,9,10] also confirms this assumption.

The goal of our present study is to find exactly the ground state and thermodynamic characteristics of disordered ensembles of spin-S' magnetic impurities in magnetically uniaxial spin-S chains in the critical region, i.e., in the domain of values of the magnetic anisotropy where excitations of the homogeneous host are gapless. As a byproduct, we find the exact solution to the behavior of random ensembles of multichannel Kondo impurities, coupled locally to the free-electron host with an «easy-plane» magnetic anisotropy of that coupling. We allow for various *random* distributions of the impurity—host couplings for arbitrary values of external magnetic field and temperature. The magnetic anisotropy parameter is assumed to be homogeneous for the host spins and for the impurity spins. In this paper we show that for several kinds of strong disorder of the impurity—host couplings the (Kondo) screening is absent, but for a weaker disorder the quenching persists, but with a NFL temperature behavior of the magnetic characteristics. We also show that the magnetic field lifts the degeneracy and effectively enhances the quenching of the impurity spins, hence decreasing the effect of disorder.

This paper is organized as follows. After the introduction in Sec. 1, the Hamiltonians for the spin chains studied are introduced in Sec. 2. Section 3 is devoted to the standard Bethe ansatz equations of the problem, and to the connection with the multichannel Kondo case. In Sec. 4 we present the ground state properties of the systems considered. The thermodynamic Bethe ansatz is introduced in Sec. 5 for random ensembles of magnetic impurities in the «easy-plane» spin chains and the multichannel Kondo situation, by use of the «quantum transfer matrix» approach. In Sec. 6 we present our results for the temperature and magnetic field dependence of the magnetic susceptibility and the specific heat obtained analytically and compare them with numerical calculations of the nonlinear integral equations. Section 7 contains concluding remarks.

2. Bethe-ansatz solvable Hamiltonians

In our treatment we shall use the Bethe ansatz method (for a review, see, e.g., the monograph [48] and references therein). Let us start with $R^{\mu_i\mu_{i+1}}_{\alpha_i\beta_i}(u)$, the standard R matrix of a spin S chain with uniaxial «easy-plane» anisotropy (see, e.g., [44,45,49]). The indices α_i and β_i denote states of the spin at site i (acting in the Hilbert space V_i), and μ denotes states in the auxiliary space (Hilbert space V_a). The R matrix has the form

$$R = P\sum_{j=0}^{2S}\prod_{l=0}^{j-1}\frac{\sinh\gamma[i2(2S-l)-u]}{\sinh\gamma[i2(2S-l)]}\times$$

$$\times\prod_{i=j}^{2S-1}\frac{\sinh\gamma[i2(2S-l)+u]}{\sinh\gamma[i2(2S-l)]}\times$$

$$\times\prod_{\substack{p=0\\p\neq j}}^{2S}\frac{2\sin^2\gamma\hat{X}_{ia}-\sin\gamma p\sin\gamma(p+1)}{\sin\gamma(j-p)\sin\gamma(j+p+1)}, \qquad (1)$$

where u is the spectral parameter, γ is the parameter of the («easy-plane») magnetic anisotropy, the operator P permutes the spaces V_i and V_a, and

$$\hat{X}_{ia} = e^{i\gamma S_i^z}\Big(\frac{1}{2}[S_i^+S_a^- + S_i^-S_a^+] +$$

$$+\frac{\cos\gamma S\cos\gamma(S+1)}{\sin\gamma}S_i^zS_a^z +$$

$$+\frac{\sin\gamma S\sin\gamma(S+1)}{\sin^2\gamma}\cos\gamma S_i^z\cos\gamma S_a^z\Big)e^{-i\gamma S_a^z}, \qquad (2)$$

where $S^{\pm} = S^x \pm iS^y$, and which in the limit of the SU(2)-symmetric system ($\gamma \to 0$) simplifies to $\mathbf{S}_i\mathbf{S}_a + S(S+1)$. The R matrices satisfy the Yang—Baxter (triangle) relations [3,48]. The row-to-row (from the viewpoint of the associated statistical 2D problem), «standard» transfer matrix $\tau^{\beta}_{\alpha}(u)$ has the form of the trace over the auxiliary space of the product of R matrices with the same values of the spins (S) in sites i

$$\tau^{\beta}_{\alpha}(u,\{\theta\}_{i=1}^{L}) = \sum_{\mu}\prod_{i=1}^{L}R^{\mu_i\mu_{i+1}}_{\alpha_i\beta_i}(u,\theta_i), \qquad (3)$$

where L is the length of the quantum chain and θ_i are the inhomogeneity parameters, which are shifts of the spectral parameter. The R matrices satisfy the Yang—Baxter equations, and, hence, the transfer matrices with different spectral parameters commute [48]. The Hamiltonian of the uniaxial spin-S quantum chain with impurities of the same spin S is obtained as the derivative of the logarithm of the transfer matrix with respect to the spectral parameter (taken at $u = 0$) [49].

The Hamiltonian of the uniaxial spin-S chain with spin-S' impurities is obtained as the derivative of the logarithm of the transfer matrix, which is the trace over the auxiliary space of the product of R matrices with *different* values of spin (S for host sites and S' for impurity sites) in the spaces V_i, with respect to the spectral parameter (at $u = 0$). Notice that R matrices with different values of the spins for the quantum and auxiliary spaces mutually satisfy the Yang—Baxter relations. The Hamiltonian has the form

$$H = \sum_j 2H_{j,j+1} + H_{\rm imp} + H_{\rm imp-imp}$$

(the host exchange constant J is set to 2). In general, the form of the lattice Hamiltonian is very complicated; it depends on S, S', θ_j, and the anisotropy γ. We can directly write down several important limiting cases of the Hamiltonian to clarify the situation. For example, for a spin-S' impurity introduced into the spin-1/2 Heisenberg chain we have $H_{j,j+1} = \mathbf{S}_j\mathbf{S}_{j+1}$. The impurity part of the Hamiltonian has the form, say for the jth impurity situated between sites m and m+1 of the host [36,41,50]

$$H_{\rm imp} = [\theta_j^2 + (S' + \frac{1}{2})^2]^{-1}\{(H_{m,{\rm imp}} + H_{{\rm imp},m+1}) + \\ + \{H_{m,{\rm imp}}, H_{{\rm imp},m+1}\} - (\frac{1}{4} + 3S'(S'+1))H_{m,m+1} - \\ -2i\theta_j[H_{m,{\rm imp}}, H_{{\rm imp},m+1}]\}, \qquad (4)$$

where [.,.] ({.,.}) denotes the (anti)commutator. One can see that $\theta_j = 0$ and $S' = 1/2$ corresponds to the simple inclusion of an additional site coupled to the system by the bulk interaction. On the other hand, for $\theta_j \to \infty$ one obtains an impurity spin totally decoupled from the host. For the «easy-plane» spin-1/2 chain with spin-1/2 impurities we have [38]

$$H_{\rm imp} = \frac{\sin^2\gamma}{\sinh^2\theta_j + \sin^2\gamma}\{\hat{B}^j(H_{m,{\rm imp}}, H_{{\rm imp},m+1}) - \\ - H_{m,m+1} - i2\frac{\tanh\theta_j}{\sin\gamma}[H_{m,{\rm imp}}, H_{{\rm imp},m+1}]\}, \quad (5)$$

where the operator $\hat{B}^j$ modifies the Heisenberg-like interaction by multiplying the transverse terms by $\cosh\theta_j$. For the isotropic SU(2)-symmetric spin-S host the structure of the Hamiltonian is more complicated (without the impurity it corresponds to the Takhtajan—Babujian chain [51,52]) with

$$\mathcal{H}_{a,b} = \sum_{j=|S-S'|+1}^{S+S'} \sum_{k=|S-S'|+1}^{j} \frac{k}{k^2+\theta^2} \prod_{l=|S-S'|}^{S+S'} \frac{x - x_l}{x_j - x_l}, \qquad (6)$$

$x = \mathbf{S}_a\mathbf{S}_b (a, b = m, m+1, {\rm imp})$, and $2x_j = j(j+1) - S(S+1) - S'(S'+1)$. Note that in this case the overall multiplier is $[\theta^2 + (S+S')^2]^{-1}$ and the coefficient in front of $H_{m,m+1}$ becomes $-2S'(S'+1) - (S'+S)^2$. For the anisotropic case one has to replace x by $\hat{X}_{m,m+1}$, cf. Eq. (2), and x_j by the appropriate coefficients from Eq. (1).

If two impurities are situated between two neighboring host sites, they can interact *directly*, e.g., for the isotropic case the impurity—impurity part of the Hamiltonian is $H_{\rm imp-imp} = \sum_j \theta_j^2 J^{\rm j}_{\rm imp}\mathcal{H}_{j,j+1}$; for illustration, see Fig. 1. These impurity—impurity couplings can model, e.g., a RKKY interaction (being short-range though) between the impurities in concentrated metallic alloys. In the following we shall study the case with a *small* number of such neighboring impurities.

One can independently incorporate any number of impurities, described above, into the host spin chain. Each of them will be characterized by its own coupling to the host, i.e., by its own θ_j. The lattice Hamiltonian has additional terms, which renormalize the coupling between the neighboring sites of the host, and three-spin terms. However, it has been shown [50] that in the long-wave length limit such a lattice form of the impurity Hamiltonian yields the well-known form of the contact impurity—host interaction, similar to that of the usual Kondo problem [2,3]. The contact impurity coupling in this (conformal) limit is also determined by the same constant θ_j. We also point out that it was shown that the magnetic behavior of the impurities in the bulk and the magnetic behavior of the impurity situated at the edge of the chain (where the renormalization of the coupling between the neighboring sites of the host and three-spin terms can be eliminated, and the only interaction between the impurity and the host is the standard two-spin exchange interaction) coincide [41,53]. Finally, we would like to note that all the impurities considered

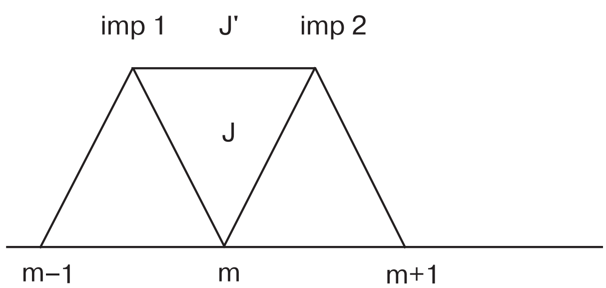

Fig. 1. Illustration of the impurity—host and impurity—impurity interactions. Here in the simplest case of the isotropic Heisenberg interaction the local impurity—host exchange constant $J = 2[\theta_j^2 + (S+S')^2]^{-1}$ and $J' \sim 2\,\theta_j^2 J$ in units of the host exchange constant 2.

in our work are *elastic* scatterers, i.e., each excitation only changes its phase when scattering off each impurity, but is not reflected. It is worthwhile to note the same property holds for the theory of a standard Kondo impurity in a free-electron host [2,3]. Equally important to mention is that we are studying a *lattice* model, hence all two-particle scattering processes, in particular, from one Fermi point to the other (backscattering), are taken into account in our work. However, we emphasize again that our model (as well as the exact solution for the Kondo problem in metals [2,3]) does not describe reflecting impurities.

The Hamiltonian and other integrals of motion, which can be constructed in a similar way as higher-order logarithmic derivatives, commute with the transfer matrix.

3. Bethe ansatz and relation to the Kondo problem

The eigenvalues and eigenstates of the above mentioned problem are parametrized by the quantum numbers (*rapidities*) $\{u_j\}_{j=1}^{M}$, where M is related to the z projection of the total spin as $S^z = (S + S'c_{\mathrm{imp}})L - M$, where L is the total length of the chain (including the impurity sites) and c_{imp} is the concentration of impurities. Notice that the Hamiltonian commutes with the z projection of the total spin; hence, M enumerates all possible states. We shall consider not very large concentrations of impurities. Those rapidites are the solutions of the Bethe equations

$$\prod_{j=1}^{L} e_{2s}(u_j - \theta_j) = -\prod_{k=1}^{M} e_2(u_j - u_k), \quad (7)$$

where $j = 1,\ldots,M$,

$$e_n(x) = \frac{\sinh\gamma(x + in)}{\sinh\gamma(x - in)},$$

$s = S$ for the host sites (with $\theta_f = 0$) and $s = S'$ for impurity sites, where θ_f can be nonzero. The energy of the state with the z projection of the total spin, characterized by M, is equal to

$$E_0 \equiv Le_0 = -i\frac{\sin(2\gamma S)}{4S}\sum_{j=1}^{M}\frac{d}{du_j}\ln e_s(u_j). \quad (8)$$

This formula is valid for the cases in which lengths of the clusters of neighboring impurities, which interact with each other, are small.

It is easy to show that the behavior of the ensemble of multichannel (with $2S$ channels) Kondo impurities, each of which with coupled to the free-electron gas via its own «easy-plane»-anisotropic local exchange interaction, with the Hamiltonian

$$H_K = \sum_{k,l,\sigma}\epsilon_k c^{\dagger}_{k,l,\sigma}c_{k,l,\sigma} + \frac{1}{2}\sum_j \delta(x - x_j)\times$$
$$\times\sum_{l=1}^{2S}\{\Delta_j[(S')^z_j]^2 + J^{\mathrm{par}}_j(S')^z_j \times$$
$$\times(c^{\dagger}_{x,l,\uparrow}c_{x,l,\uparrow} - c^{\dagger}_{x,l,\downarrow}c_{x,l,\downarrow}) +$$
$$+J^{\mathrm{perp}}_j[(S')^+_j c^{\dagger}_{x,l,\downarrow}c_{x,l,\uparrow} + (S')^-_j c^{\dagger}_{x,l,\uparrow}c_{x,l,\downarrow}]\}, \quad (9)$$

where $c^{\dagger}_{k,l,\sigma}$ creates an electron of channel l with the spin σ and impurities are situated at sites x_j, can be also described within the Bethe ansatz scheme. In the scaling limit for small magnetic anisotropy one has $J^{\mathrm{par}}_j = 2\gamma/\theta_j\rho$, $J^{\mathrm{perp}}_j = J^{\mathrm{par}}_j(1 - \delta_j/3)$, and $\Delta_j = -J^{\mathrm{par}}_j\delta_j/3$, where $\delta_j = (\theta_j^2/2) + (\gamma^2/8)$ and ρ is the density of states of conduction electrons at the Fermi level. In this case the low-energy spin behavior (which is the most important one for the Kondo impurities) is determined by the solution of Eqs. (7), while the energy is determined via

$$E_K \equiv L_{e_0K} = -i\frac{\sin(2\gamma S)}{4S}\sum_{j=1}^{M}\ln e_s(u_j). \quad (10)$$

The condition of the applicability of the Bethe ansatz scheme for ensembles of disordered impurities is the presence of large enough numbers of magnetic impurities with equal exchange constants, while those constants for other impurities can be randomly distributed. For small enough impurity concentrations the probability of having long clusters of impurities connected by the direct impurity—impurity interactions is small, and in the thermodynamic limit $L \to \infty$ one can neglect the contribution of such clusters. In this case the contribution of each impurity (or of each small cluster of directly coupled impurities) is *additive*, and we can solve the problem for each impurity (cluster), determined by the local exchange coupling constant, related to θ_j, and, then, introducing the distribution of θ_j over the chain (in the volume of the metal for the case of the Kondo impurities), average the answers for thermodynamic characteristics. Such an additive property is the consequence of the exact integrability of the problem and is strictly connected with the structure of the Hamiltonians considered. It turns out, however, that in the long-wavelength limit the «triangular» structure of the impurity—host interaction actually produces a local contact impurity—host interaction [36–38,50], and the «fine-tuning» structure of the couplings between magnetic impurities and the host becomes nonessential.

4. The ground state behavior

Let us first study the ground state behavior of the systems considered. In the absence of magnetic field the ground state energy of the impurity is

$$e_0(\theta_j) = -\frac{\pi \sin(2\gamma S)}{4\gamma S} \int d\omega e^{i\frac{\pi\omega\theta_j}{\gamma}} \times$$

$$\times \frac{\sinh\left[\frac{\pi\omega}{\gamma}\min(S,S')\right]\sinh\left[\left(\frac{\pi^2}{2\gamma} - \pi\max(S,S')\right)\omega\right]}{\sinh(\pi\omega)\sinh\left(\frac{\pi\omega}{2}\right)}, \tag{11}$$

and the total ground state energy is equal to

$$E_0 = \sum_j e_0\left(\frac{\pi}{\gamma}\theta_j\right), \tag{12}$$

where the sum is taken over all the sites (for sites without impurities we get $e_0(0)$). Notice that for $\theta_j = 0$ and for $S' = S$ the impurity is just an additional site of the host, the ground state energy per site of which is

$$e_0(0) = -\frac{\pi \sin(2\gamma S)}{4\gamma S} \times$$

$$\times \int d\omega \frac{\sinh\left(\frac{\pi\omega}{\gamma}S\right)\sinh\left[\left(\frac{\pi^2}{2\gamma} - \pi S\right)\omega\right]}{\sinh(\pi\omega)\sinh\left(\frac{\pi\omega}{2}\right)}. \tag{13}$$

For the Kondo impurities Eq. (11) can be used with the overall multiplier 1/2.

The coupling of the impurity to the host ($J^j_{\rm imp}$) is determined by the constant θ_j. We can show (see also [41,50]) that precisely this constant determines the effective Kondo temperature of the impurity in a spin chain via $T_{jK} \propto \exp(-\pi|\theta_j|)$. For energies higher than this crossover Kondo scale one has the asymptotically free impurity spin S, while for lower energies the impurity spin is underscreened for $S' > S$ (with the Curie-like behavior of the remnant effective spin $S' - S$), totally screened for $S' = S$ (with the usual marginal FL-like behavior persisting with the finite susceptibility and linear temperature dependence of the specific heat at low temperature, and, hence, finite Wilson ratio in the ground state) and overscreened for $S' < S$ (with the critical non-FL behavior of a single spin [5]). It is similar to the findings in the theory of a Kondo impurity in a free-electron matrix [2,3]. In other words, θ measures the shift off the Kondo resonance (higher values of $|\theta_j|$ correspond to lower values on the Kondo scale) of the impurity level with the host spin excitations, similar to the standard picture of the Kondo effect in the electron host. The difference between the two models is that in the free-electron host the spins of free electrons screen the magnetic impurity, while in the spin chain the low-lying spin excitations (spinons for the AF chain) quench the spin of the impurity.

Let us illustrate this with the help of the ground state behavior of impurities in small magnetic field h. The ground state energy per site is equal to (we shall consider small enough $\gamma < \pi/2S$)

$$e_0(\theta_j, h) = e_0(\theta_j) - \int d\omega\, e^{\frac{i\omega\pi\theta_j}{\gamma}} \times$$

$$\times \frac{y^+\left(\frac{\pi\omega}{\gamma}\right)\sinh(\omega\pi S')}{2\cosh\left(\frac{\pi\omega}{2}\right)\sinh(\omega\pi S)} \tag{14}$$

for $S' \le S$ and

$$e_0(\theta_j, h) = e_0(\theta_j) - \frac{\pi(S' - S)h}{\pi - 2S\gamma} -$$

$$-\int \frac{d\omega}{2\gamma} e^{\frac{i\omega\pi\theta_j}{\gamma}} \frac{y^+\left(\frac{\pi\omega}{\gamma}\right)\sinh\left[\omega\left(\frac{\pi^2}{2\gamma} - \pi S'\right)\right]}{2\cosh\left(\frac{\pi\omega}{2}\right)\sinh\left[\omega\left(\frac{\pi^2}{2\gamma} - \pi S\right)\right]} \tag{15}$$

for $S' \ge S$. Here $y^+(\omega)$ is the positive part of the solution of the equation

$$y(u) + \int_0^\infty du' y(u') J(u - u') - \frac{h}{2} +$$

$$+ \frac{\pi \sin(2\gamma S)}{4\gamma S \cosh\left[\frac{\pi(u + B)}{\gamma}\right]} = -\int_0^\infty du' y(u') J(u + u' + 2B), \tag{16}$$

where the Fourier transform of $J(x)$ is

$$J(\omega)=\frac{\sinh\left(\frac{\gamma\omega}{2}\right)\sinh\left(\frac{\pi\omega}{2}\right)}{2\cosh\left(\frac{\gamma\omega}{2}\right)\sinh(\gamma\omega S)\sinh\left[\omega\left(\frac{\pi}{2}-\gamma S\right)\right]}\quad(17)$$

and B is connected with the value of the external magnetic field. Notice that for the Kondo problem the right-hand side of Eq. (16) is small and is usually dropped (see, however, [54]). Equation (16) for small fields can be solved as the sequence of Wiener—Hopf equations. It also gives the connection between h and B:

$$h=\frac{\pi^2\sin(2\gamma S)}{2\gamma S}\,e^{-\frac{(B+a)\pi}{\gamma}}\,\frac{\Gamma\left(1+\frac{\pi}{2\gamma}\right)}{\Gamma(1+S)\,\Gamma\left(1-S+\frac{\pi}{2\gamma}\right)}+\ldots,\quad(18)$$

where a is some nonuniversal constant.

For $S'=S$ we close the contour of integration in Eq. (15) through the upper half plane (the main pole is of $\cosh(\pi\omega/2)$) and have

$$e_0(\theta_j,h)=e_0(\theta_j)-\frac{2S^2\gamma(\pi-2\gamma S)h^2}{2\pi^3\sin(2\gamma S)T_{jK}}-\frac{Ah^{2+2\gamma(\pi-2\gamma S)}}{T_{jK}}-\ldots,\quad(19)$$

for $h \ll T_{jK}$, where for small γ, $T_{jK}=v\,e^{-\pi|\theta_j|/\gamma}$ ($v=\pi\sin(2\gamma S)/2\gamma S$ is the Fermi velocity of low-lying excitations) plays the role of the «local» Kondo temperature, and A is some nonuniversal constant. Each single magnetic impurity is *totally compensated* for $h\le T_{jK}$. The susceptibility of a single impurity is *finite* as $h\to 0$ and is renormalized by a factor of T_{jK} with respect to the host susceptibility. Again, the total ground state energy is the sum of energies of all impurities and host spins (the latter with $\theta_j=0$, i.e., $T_{jK}=v$). We shall show below (see Sec. 4) that the strong disorder in the distribution of the local Kondo temperatures can lead to the divergent magnetic susceptibility for $h\to 0$, i.e., to the NFL behavior.

It turns out that some studies connect the multiplier $(1-2\gamma S/\pi)$ with the renormalization of the effective g factor of the spins [3], while other works relate such a change to the NFL behavior caused by the magnetic anisotropy [43,44].

For $S'>S$ the main contribution to the integral arises from the poles at $\omega=i\pi/\gamma$ (and then $\omega=2\pi/(\pi-2\gamma S)$) which produces for $h \ll T_{jK}$

$$e_0(\theta_j,h)=e_0(\theta_j)-\frac{(S'-S)\pi h}{(\pi-2\gamma S)}-Ch\left(\frac{h}{T_{jK}}\right)^{2\gamma/(\pi-2\gamma S)}+\ldots,\quad(20)$$

where C is a nonuniversal constant. We can see that for $h\to 0$ the magnetization is *finite* for a single impurity, and spins of single impurities are *underscreened* to the value $S'-S$ by host low-lying excitations.

Finally, for $S'<S$ and $h \ll T_{jK}$ we have

$$e_0(\theta_j,h)=e_0(\theta_j)-C'h\left(\frac{h}{T_{jK}}\right)^{1/S}+\ldots,\quad(21)$$

for $S>1$, where C' is a nonuniversal constant, and for $S=1$, $S'=1/2$ we have

$$e_0(\theta_j,h)=e_0(\theta_j)-\frac{2\gamma(\pi-2\gamma)h^2}{4\pi^3\sin(2\gamma)T_{jK}}\ln(T_{jK}/h)+\ldots\quad(22)$$

Hence, for $S'<S$ the spins of single impurities are *overscreened*, and that produces the NFL behavior.

5. Thermodynamics and the «quantum transfer matrix» approach

For our 1D inhomogeneous quantum spin system at finite temperature we choose a suitable lattice path integral representation by a mapping which preserves integrability. For a general formulation of the Trotter—Suzuki decompositions used in our approach we refer to Refs. 55–58. As usual, we study the associated 2D classical vertex model instead of the direct treatment of the 1D quantum system.

One can introduce R matrices of different types, related to the initial one by a counterlockwise rotation $\overline{R}^{\mu\nu}_{\alpha\beta}(u)=R^{\alpha\beta}_{\nu\mu}(u)$ and $\tilde{R}^{\mu\nu}_{\alpha\beta}(u)=R^{\beta\alpha}_{\mu\nu}(u)$ by a clockwise rotation. The transfer matrix $\overline{\tau}(u,\{\theta\}_{i=1}^{L})$ can be constructed in a way similar to the case of τ. Then we substitute $u=-J\sin\gamma/NT$, where N is the Trotter number. We find

$$[\tau(u)\overline{\tau}(u)]^{N/2}=e^{-H/T}+O(1/N).\quad(23)$$

Hence, the partition function of the quantum 1D system is identical to the partition function of an inhomogeneous classical vertex model with alternating rows on a square lattice of size $L\times N$:

$$Z=\lim_{N\to\infty}\mathrm{Tr}\,[\tau(u)\overline{\tau}(u)]^{N/2}.\quad(24)$$

The interactions on the 2D lattice are four-spin interactions with coupling parameters depending on $(NT)^{-1}$ and interaction parameters θ_i, where i is the number of the column to which that particular vertex of the lattice belongs. Note that the interactions are homogeneous in each column, but vary from column to column. This is similar to the McCoy—Wu model [59], which is the Ising model with disorder. (However in its 1D realization the Hamiltonian of the McCoy—Wu model can be mapped on the quadratic fermion form by means of the Jordan—Wigner transformation, i.e., there are *no* interactions in that model. Our models definitely reveal an essential coupling between particles.) We study this system in the thermodynamic limit $N, L \to \infty$ using an approach which is based on a transfer matrix describing transfers in the horizontal direction. The corresponding column-to-column transfer matrices are referred to as «quantum transfer matrices» (QTMs) (where an external magnetic field h is included by means of twisted boundary conditions):

$$\tau_{QTM}(\theta_j, u) = \sum_{\mu} e^{\mu_1 h/T} \times$$

$$\times \prod_{i=1}^{N/2} R^{\mu_{2i-1}\mu_{2i}}_{\alpha_{2i-1}\beta_{2i-1}}(u + i\theta_j)\tilde{R}^{\mu_{2i}\mu_{2i+1}}_{\alpha_{2i}\beta_{2i}}(u - i\theta_j). \quad (25)$$

See Fig. 2 for an illustration of the transfer matrices of the associated 2D statistical model.

In general all QTMs corresponding to the L columns are different. However, all these operators commute pairwise. Therefore, the free energy per lattice site of our 1D quantum system can be calculated from the largest eigenvalues of the quantum transfer matrices (corresponding to only one eigenstate). The free energy per site f of the 1D inhomogeneous quantum spin chain is given by only the largest eigenvalue of the quantum transfer matrix Λ_{QTM} as

$$f = -\lim_{L\to\infty} \frac{T}{L} \sum_{i=1}^{L} \lim_{N\to\infty} \ln \Lambda_{QTM}(\theta_i, u), \quad (26)$$

where $u = -J \sin\gamma / TN$ and the dependence on N is understood implicitly.

Let us consider the hierarchy of QTMs acting on the subspace $\otimes^N V_{2S}$ (the index specifies the spins of the scatterers) with T_n being a member of such hierarchy with the auxiliary subspace V_n (here the index n specifies the spin of the auxiliary particle, i.e., the auxiliary particle with spin $n/2 = S'$ scatters off N spins S). By means of a Bethe ansatz pro-

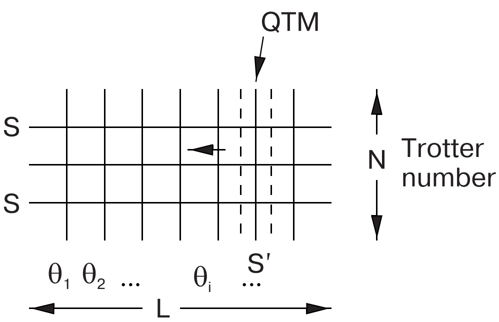

Fig. 2. The classical 2D model with four-spin interaction around vertices and alternating coupling parameters from column to column, related to the quantum 1D chain.

cedure we find the eigenvalue of the quantum transfer matrix to be given by

$$\Lambda_{QTM}(\theta_i) = \frac{\Lambda_{2S'}\left(\frac{2\theta_i}{\gamma}\right)}{\prod_{p=1}^{2S'} [\sinh(ip\gamma)]^{N/2}} \quad (27)$$

and

$$\Lambda_p(x) = \sum_{l=1}^{p+1} \lambda_l^{(p)}(x),$$

$$\lambda_l^p(x) = \psi_l^{(p)}(x)\, e^{h(p+2-2l)/T} \times$$

$$\times \frac{Q[x + i(p+1)]Q[x - i(p+1)]}{Q[x + i(2l - p - 1)]Q[x + i(2l - p - 3)]},$$

$$\psi_l^{(p)}(x) = \prod_{z=1}^{p-l+1} \phi_-[x - i(p - 2S - 2z)] \times$$

$$\times \phi_+[x + i(p - 2S + 2 - 2z)] \times$$

$$\times \prod_{z=1}^{l-1} \phi_-[x - i(p - 2S + 2 - 2z)] \times$$

$$\times \phi_+[x + i(p - 2S - 2z)] \quad (28)$$

with $p \geq 2$, $\Lambda_0 = 1$ and

$$\Lambda_1(x) = \phi_+[x - i(2S-1)]\phi_-[x - i(2S+1)]\, e^{h/T} \times$$

$$\times\frac{Q(x+2i)}{Q(x)}+\phi_-[x+i(2S-1)]\phi_+[x+i(2S+1)]\times$$
$$\times e^{-h/T}\frac{Q(x-2i)}{Q(x)}. \quad (29)$$

Here we have dropped the dependence on u and θ_i, which are fixed, and consider the dependence on the spectral parameter x explicitly. We have used

$$\phi_\pm(x)=\sinh^{N/2}\left(\gamma\frac{x\pm iu'}{2}\right),$$
$$(30)$$
$$Q(x)=\prod_{j=1}^{m}\sinh\left(\gamma\frac{x-x_j}{2}\right)$$

with «renormalized» $u'=2u/\gamma$. Here $\{x_j\}_{j=1}^m$ is the set of Bethe ansatz rapidities which are subject to the «local» Bethe ansatz equations

$$\frac{\phi_-[x_j+i(2S-1)]\phi_+[x_j+i(2S+1)]}{\phi_+[x_j-i(2S-1)]\phi_-[x_j-i(2S+1)]}=$$
$$=e^{-2h/T}\frac{Q(x_j+2i)}{Q(x_j-2i)}, \quad (31)$$

where m is the number of the roots of the «local» Bethe ansatz equations, being different for different eigenstates of the QTM. For the largest eigenvalue we have to take $m=NS$. However we shall not solve Eqs. (31) directly, but rather shall be interested in the functional properties of the eigenvalue of the transfer matrix. Note that $\Lambda_0=1$ and

$$\Lambda_p(x+i)\Lambda_p(x-i)=f_p(x)+\Lambda_{p-1}(x)\Lambda_{p+1}(x), \quad (32)$$

where $p\geq 1$ and

$$f_n(x)=\prod_{j=1}^{n}\prod_{\pm}\phi_\pm[x\pm i(n-2S-2j+1)]\times$$
$$\times\phi_\pm[x\pm i(2S-n+2j+1)]. \quad (33)$$

For this purpose we introduce auxiliary functions $y_n(x)$, $Y_n(x)=1+y_n(x)$, $b(x)$, $\bar b(x)$, $B(x)=1+b(x)$, and $\bar B(x)=1+\bar b(x)$ by

$$y_n(x)=\Lambda_{n-1}(x)\Lambda_{n+1}(x)/f_n(x),\ \ n\geq 1,$$
$$b(x)=\frac{\lambda_1^{(2S')}(x+i)+\ldots+\lambda_{2S'}^{(2S')}(x+i)}{\lambda_{2S'+1}^{(2S')}(x+i)},$$
$$\bar b(x)=\frac{\lambda_2^{(2S')}(x-i)+\ldots+\lambda_{2S'+1}^{(2S')}(x-i)}{\lambda_1^{(2S')}(x-i)}, \quad (34)$$

where $n\geq 1$. Then one can straightforwardly check that ($y_0=0$)

$$y_n(x+i)y_n(x-i)=Y_{n-1}(x)Y_{n+1}(x),$$
$$\Lambda_{2S'}(x+i)=B(x)\lambda_{2S'+1}^{(2S')}(x+i)=$$
$$=e^{-2S'/T}\prod_{\pm}\prod_{j=1}^{2S'}\phi_\pm[x+i(2j+2S-2S'-\pm 1)]\times$$
$$\times\frac{Q(x-2iS')}{Q(x+2iS')},$$
$$\Lambda_{2S'}(x-i)=\bar B(x)\lambda_1^{(2S')}(x-i)=$$
$$=e^{-2S'h/T}\prod_{\pm}\prod_{j=1}^{2S'}\phi_\pm[x-i(2j+2S-2S'\pm 1)]\times$$
$$\times\frac{Q(x+2iS')}{Q(x-2iS')}. \quad (35)$$

Notice that the first set of equations is nothing else than the fusion hierarchy (so-called Y system). Let us use the first $2S'-2$ equations of the Y system as they are. In the equation for $y_{2S'-1}$ we replace $Y_{2S'}(x)$ by $B(x)\bar B(x)$, due to

$$Y_p(x)=B(x)\bar B(x), \quad (36)$$

i.e., we have

$$y_{2S'-1}(x-i)y_{2S'-1}(x+i)=Y_{2S'-2}B(x)\bar B(x). \quad (37)$$

Then we obviosly have

$$b(x)=e^{(2S'+1)h/T}\times$$
$$\times\prod_{\pm}\frac{\phi_\pm[x+i(2S-2S'\pm 1)]\Lambda_{2S'-1}(x)}{\prod_{j=1}^{2S'}\phi_\pm[x+i(2j+2S-2S'\pm 1)]}\times$$
$$\times\frac{Q[x+i(2S'+2)]}{Q(x-2iS')},$$
$$\bar b(x)=e^{-(2S'+1)h/T}\times$$
$$\times\prod_{\pm}\frac{\phi_\pm[x+i(2S-2S'\pm 1)]\Lambda_{2S'-1}(x)}{\prod_{j=1}^{2S'}\phi_\pm[x-i(2j+2S-2S'\pm 1)]}\times$$

$$\times \frac{Q[x - i(2S' + 2)]}{Q(x + 2iS')} \quad (38)$$

and

$$\Lambda_{k-1}(x - i)\Lambda_{k-1}(x + i) = Y_{k-1}(x) f_{k-1}(x), \quad (39)$$

which are consequences of the definitions.

What do these additional functions describe? We can understand it by taking into consideration only the SU(2)-symmetric case, i.e., $\gamma = 0$. Here according to Ref. 60 the scattering matrix of excitations of the quantum spin-1/2 system factorizes into the matrix of the spin-1/2 SU(2)-symmetric model and the matrix of the level-$2S$ sl_2-symmetric RSOS (restricted solid-on-solid) model [61,62] (consistent with the quantum field theory prediction that the conformal field theory (CFT) is the level-$2S$ Wess—Zumino—Novikov—Witten (WZNW) model, which can be approximately presented by the sum of a Gaussian sector with the central charge $c = 1$ and the Z_{2S} parafermionic sector with $c = (2S - 1)/(S + 1)$ [63]). In the scaling limit Z_k parafermionic theory is approximately equivalent to the sl_2 RSOS model. Hence the functions b, $\bar{b}$, B, and $\bar{B}$ describe the spinon sector (spinons of the spin $S = 1/2$ model), which pertains to the Gaussian for the SU(2)-symmetric case, while the y_j functions (with the additional condition $y_k = 0$) describe the RSOS sector.

One can see that these auxiliary functions are analytic, nonzero, and have constant asymptotic behavior for the strip $-1 < \mathrm{Im}\, x \le 0$ for $b(x)$ and $B(x)$, for the strip $0 \le \mathrm{Im}\, x < 1$ for $\bar{b}(x)$ and $\bar{B}(x)$, and for the strip $-1 \ge \mathrm{Im}\, x \ge 1$ for y_n and Y_n. Introducing $a(x) = b(2(x + i\epsilon)/\pi)$ and $\bar{a}(x) = \bar{b}(2(x - i\epsilon)/\pi)$ (infinitesimal $\epsilon > 0$), taking the logarithmic derivative of these functions, then Fourier transforming the equations, eliminating the functions $Q(x)$, and finally inverse-Fourier transforming, we obtain the final set of nonlinear integral equations. Eventually, we take the limit $N \to \infty$. Proceeding in this way we find for our system the following set of nonlinear integral equations for the «energy density» functions of spinons $a, \bar{a}$, $A = 1 + a$, $\bar{A} = 1 + \bar{a}$, y_n and Y_n in dependence on the spectral parameter x:

$$\ln y_1(x) = \int k'(x - y) \ln Y_2(y) dy,$$

$$\ln y_j(x) = \int k'(x - y) \ln [Y_{j-1}(y) Y_{j+1}(y)] dy,$$

$$2 \le j \le 2S' - 1,$$

$$\int [k'(x - y) \ln Y_{2S'-2}(y) + k'(x - y + i\epsilon) \ln A(y) + $$
$$+ k(x - y - i\epsilon) \ln \bar{A}(y)] dy = \ln y_{2S'-1}(x),$$

$$\int [k(x - y) \ln \bar{A}(y) - k(x - y - i\pi + i\epsilon) \ln A(y) + $$
$$+ k'(x - y + i\epsilon) \ln Y_{2S'-1}(y)] dy =$$
$$= \ln a(x) + \frac{v}{T \cosh x} + \frac{\pi h}{2(\pi - \gamma) T},$$

$$\int [k(x - y) \ln A(y) - k(x - y + i\pi - i\epsilon) \ln \bar{A}(y) + $$
$$+ k'(x - y - i\epsilon) \ln Y_{2S'-1}(y)] dy =$$
$$= \ln \bar{a}(x) + \frac{v}{T \cosh x} + \frac{\pi h}{2(\pi - \gamma) T} \quad (40)$$

with kernel functions

$$k(x) = \frac{1}{2\pi} \int d\omega \frac{\sinh [(\pi^2 S/\gamma - (2S + 1)\pi)\omega] \cos (x\omega)}{2 \cosh (\pi\omega/2) \sinh (S\pi\omega(\pi - \gamma)/\gamma)} \quad (41)$$

and

$$k'(x) = \frac{1}{2\pi} \int d\omega \frac{\cos (x\omega)}{2 \cosh (\pi\omega/2)}. \quad (42)$$

The free energy per site f is given by

$$f(x) = e_0(x) - \frac{T}{2\pi} \int \frac{\ln A(y)\, dy}{\cosh (x - y + i\epsilon)} -$$
$$- \frac{T}{2\pi} \int \frac{\ln \bar{A}(y)\, dy}{\cosh (x - y - i\epsilon)}, \quad (43)$$

where e_0 is the ground state energy. The free energy of the total chain with impurities is

$$F = \sum_j f\left[\frac{\pi}{\gamma} \theta_j + i\pi(S' - S)\right], \quad (44)$$

where the sum is taken over all the sites (for sites without impurities we get $f(0)$).

The free energy per impurity of the multichannel Kondo problem of the ensemble of disordered impurities f_K is given by

$$f_K(x) = e_{0K}(x) - \frac{T}{2\pi}\int \frac{\sinh(x-y+i\epsilon)\ln A(y)\,dy}{\cosh^2(x-y+i\epsilon)} -$$

$$-\frac{T}{2\pi}\int \frac{\sinh(x-y-i\epsilon)\ln \overline{A}(y)\,dy}{\cosh^2(x-y-i\epsilon)}, \quad (45)$$

where e_{0K} is the ground state energy. Notice that for $S' < S$ one has to put $\ln \overline{Y}_{2S'}$ into Eqs. (43)–(45) instead of $\ln A\overline{A}$, see Eq. (36). The free energy of the total ensemble of Kondo impurities is equal to

$$F_K = \sum_j f_K\left[\frac{\pi}{\gamma}\theta_j + i\pi(S'-S)\right], \quad (46)$$

where the sum is taken over all the impurities. It turns our that for large arguments, which are important for the low-temperature characteristics, the behaviors of the kernels of Eqs. (43) and (45) are similar. The difference appears to be important for the energies of order of the values of local exchange constants and higher, cf. Fig. 3.

These equations can be easily solved numerically for arbitrary magnetic field values and temperatures. The random distribution of the values θ_j can be described by a distribution function $P(\theta_j)$. It is worthwhile to emphasize here the simplicity of the

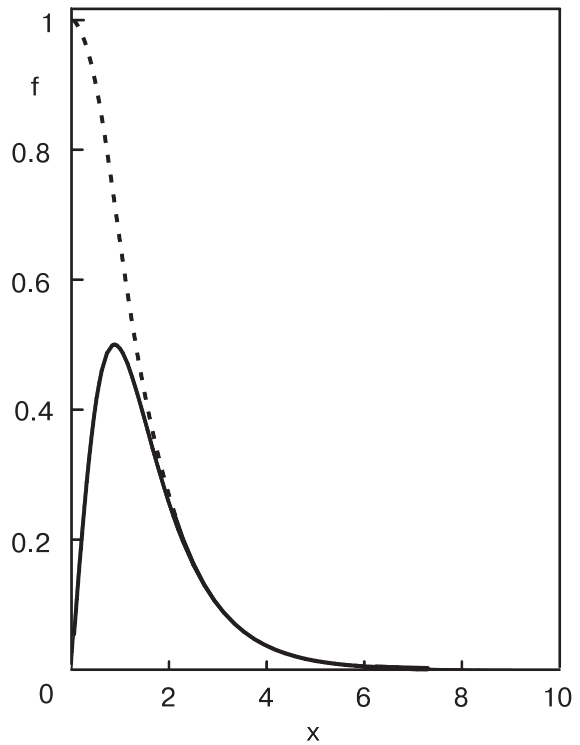

Fig. 3. The behavior of the kernels of Eq. (43) (dashed line) and Eq. (45) (solid line).

derived equations. For each impurity there are only two parameters, the real and imaginary shifts of the spectral parameter in the formula for the free energy per site, Eqs. (43),(45). Then the exact solvability of the problem for any number of impurities permits one to introduce the distribution of these shifts (the strengths of the impurity—host couplings pertinent to the local Kondo temperatures and the spins of the impurities). We have only $2S' + 1$ nonlinear integral equations, Eqs. (40), to solve, and the answer can in principle be obtained for arbitrary temperature and magnetic field ranges.

6. Disordered ensembles of magnetic impurities

One can see from Eqs. (40)–(46) that for low T the temperature behavior of the magnetic susceptibility and specific heat of single impurities strongly depends on relative values of the host spins S and impurity spin S'.

For $S > S'$ the impurity is *underscreened* by low-lying excitations of the chain (in the case of the Kondo impurity — by spins of conduction electrons). The magnetic susceptibility χ_j of such an impurity is divergent at $h = 0$ for $T \to 0$. The specific heat c_j exhibits the Schottky anomaly, related to the undercompensated spin of the impurity. The entropy of a single impurity at $T = h = 0$ becomes nonzero, $\mathcal{S}_j = \ln[1 + 2\pi(S' - S)/(\pi - 2\gamma S)]$. A finite magnetic field lifts the degeneracy and the remnant entropy becomes zero. Naturally, the total low-temperature magnetic susceptibility of any disordered ensemble of such impurities is also divergent at low temperatures.

On the other hand, for $S' < S$ the spins of low-lying excitations of the antiferromagnetic critical chain (spins of itinerant electrons for the multichannel Kondo case) *overscreen* the spin of a single magnetic impurity. This yields the critical behavior, which reveals itself in the divergences of the $T \to 0$ magnetic susceptibility of a single magnetic impurity and of the low-T Sommerfeld coefficient of the specific heat c_j/T for $h = 0$. In this case one has a remnant $T = h = 0$ entropy of each impurity

$$\mathcal{S}_j = \ln\frac{\sin[\pi(2S'+1)/(2S'+2)]}{\sin[\pi/(2S+2)]},$$

which is removed by a finite magnetic field that lifts the spin degeneracy of the system. It is not difficult to show by solving Eqs. (40)–(43) that at low T one has $c_j \propto T\chi_j \sim (T/T_{jK})^{2/(S+1)}$ for $S > 1$, and $T_{jK}c_j/T \propto T_{jK}\chi_j \sim \ln(T_{jK}/T)$ for

$S = 1/2$ at zero magnetic field $h = 0$. The total low-temperature χ_j and the Sommerfeld coefficient of any disordered ensemble of such impurities are also divergent at low temperatures.

Here the disorder of the distributions of the impurity—host couplings (local exchanges between Kondo impurities and conduction electrons) does not yield any qualitative changes but introduces only specific additional features of the NFL behavior of the system, which is already present for a single magnetic impurity.

A more interesting situation arises in the case $S' = S$. Here the solution of Eqs. (40)–(43) can be obtained [62] analytically. We know that at sufficiently low temperatures the functions a and $\ln A$ manifest a sharp crossover behavior, reminiscent of a step function: $|a| << 1$ and $|\ln A| << 1$ for $x < \ln \alpha T_{jK}/T$, and $|a|, |\ln A| \sim O(1)$ for $x > \ln \alpha T_{jK}/T$, where α is some constant and T_{jK} was introduced in Sec. 4. We can introduce [37] the scaling functions

$$\ln a^{\pm} = \ln a\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

$$\ln \bar{a}^{\pm} = \ln \bar{a}\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

$$\ln A^{\pm} = \ln A\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

$$\ln \bar{A}^{\pm} = \ln \bar{A}\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

$$\ln y_p^{\pm} = \ln y_p\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

$$\ln Y_p^{\pm} = \ln Y_p\{\pm[x + \ln(\alpha T_{jK}/T)]\},$$

where $p = 1, \ldots, 2S' - 1$. In terms of those scaling functions Eqs. (40) are renormalized in such a way that the driving terms (those which do not depend on functions $a, \bar{a}, \ldots, y_p$ and Y_p) in the last two equations for $h = 0$ become proportional to $v \exp(-x \pm i\epsilon)$ (where small corrections of order $O(T)$ were neglected). Hence only the asymptotic behavior of A and $\bar{A}$ at large spectral parameter is essential [62]. Following the procedure described in [62] we obtain the low-temperature behavior of the free energy per site (for $h = 0$)

$$f(\theta_j) = e_0(\theta_j) - \\ - \frac{\pi S \gamma T^2}{2(S+1)\sin(2\gamma S) T_{jK}} \left[1 + \frac{3S^3}{[\ln(\alpha T_{jK}/T)]^3}\right] + \ldots \quad (47)$$

In the presence of a weak magnetic field $h << T$ we can calculate the temperature corrections to the free energy per site

$$f(\theta_j) = e_0^j(\theta_j, h) - \frac{\pi S \gamma T^2}{2(S+1)\sin(2\gamma S) T_{jK}} - \\ - \frac{h^2}{4\pi T_{jK}}\left[1 + \frac{1}{(2S+1)\ln(\alpha T_{jK}/T)} + \right. \\ \left. + \frac{\ln|\ln(\alpha T_{jK}/T)|}{(2S+1)^2 \ln^2(\alpha T_{jK}/T)}\right] + O(T^2). \quad (48)$$

For a single impurity $P(T_{jK}) = \delta(T_{jK} - T_K)$ we immediately recover the famous Kondo behavior of the asymptotically free spin (characteristic for a Kondo impurity in a free-electron host [2,3] and for a single impurity in a Heisenberg AF chain [41]). For the homogeneous case we put $\theta_j = 0$ (which means that $T_{jK} \to v$, where v is the Fermi velocity of spinons). Naturally our result in this case coincides with the Bethe ansatz solution [58] and with the field theoretical prediction [64]. It turns out that the central charge of the CFT is $c = 3S/(S+1)$ and does not depend on the parameter of the impurity θ_j. One can see that only one parameter gets renormalized in the disordered case — the Fermi velocity of the U(1)-symmetric low-lying excitations: spinons (the Kondo scale plays the role of a «local Fermi velocity» for an impurity [37]).

Our models permit averaging over a distribution of θ_j (or «local» Fermi velocities) because of the factorization of the free energy of the system. This is a consequence of the integrability of our models (i.e., of the only *elastic* scattering off impurities). Note that the θ_j dependence present in the low-energy characteristics results only in the universal scales T_{jK} (that is not so for higher energies, but the latter are irrelevant for the low-temperature disorder-driven divergences). Hence for low energies we can use distributions of T_{jK}, which are also more appropriate in connection to the experiments [11–20,27–29]. That is why the main features of the low-energy characteristics of our disordered spin chain are determined by the distributions of the effective Fermi velocities for the impurities. Let us consider the strong disorder distribution, which starts with the term $P(T_{jK}) \propto G^{-\lambda}(T_{jK})^{\lambda-1}$ $(\lambda < 1)$ valid till some energy scale G for the lowest values of T_{jK} (that distribution was shown to pertain to real disordered quantum spin chains [27–29] and some heavy fermion alloys [11–20,25], see Figs. 4, 5, curves *3*, for which we took $\lambda = 0.7$ and $G = 2$. Now we can calculate the low-temperature behavior of the average magnetic susceptibility χ, the Sommerfeld coefficient of the specific heat, and the correlation length of the form (the lower limit of the inte-

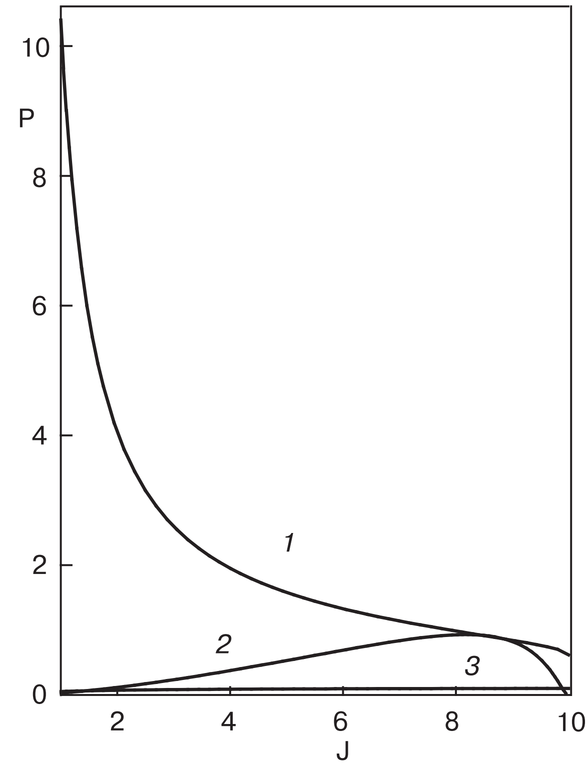

Fig. 4. Distributions, as functions of the local exchange constant (with $\gamma = 2$), used in this study, which produce the NFL behavior of the ensembles of spin-S impurities in the spin-S host: *1* — the Lorentzian distribution of θ_j; *2* — the log-normal distribution of θ_j; *3* — the power-law distribution. Very small values of the exchange constant are excluded.

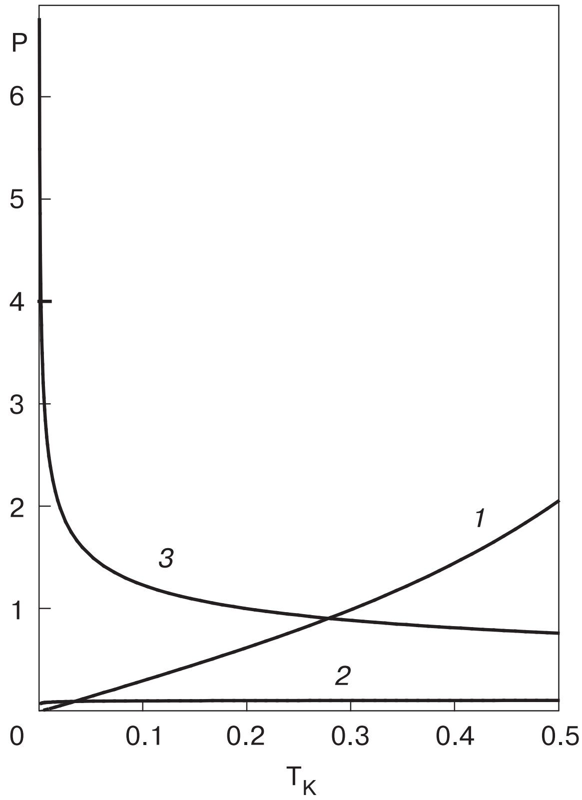

Fig. 5. Distributions, as function of T_K, used in this study, which produce the NFL behavior of the ensembles of spin-S impurities in the spin-S host: *1* — the Lorentzian distribution of θ_j; *2* — the log-normal distribution of θ_j; *3* — the power-law distribution. Notice that we used only small values of T_K.

gral over the distribution of T_{jK} gives a regular contribution)

$$\langle\chi\rangle \propto \frac{\langle c\rangle}{T} \sim G^{-\lambda}T^{\lambda-1} \; . \tag{49}$$

These formulas definitely manifest the low-T divergences of $\langle\chi\rangle$ and $\langle c\rangle/T$ and strong renormalization in the disordered spin chain as compared to the homogeneous situation. The ground state average magnetization displays $M^z \sim (h/G)^\lambda$ behavior, also different from the homogeneous case.

Other important characteristics of our disordered spin chain, e.g., the dynamic magnetic susceptibility $\langle\chi''\rangle(\omega,T)$, can be calculated. We can use the standard ansatz for the relaxational form of the susceptibility of a single magnetic impurity [22,23]

$$\chi''(\omega,T) = \chi(T)\,\frac{\Gamma(T)\omega}{\omega^2 + \Gamma^2(T)} \; , \tag{50}$$

in which one supposes that the relaxation rate (proportional to the half-linewidth of the resonance line) Γ does not depend on the frequency ω. That ansatz automatically satisfies the Kramers—Kronig relation. At low temperatures the use of the Shiba approximation [22,23] determines the first ($T = 0$) term in the expansion of $\Gamma(T)$ via

$$\lim_{\omega\to\infty} \frac{\chi''(\omega,0)}{\pi\omega} = 2\chi^2(0) \; . \tag{51}$$

That gives the low-temperature dependence of the relaxation rate per site for the disordered spin chain $\Gamma(T) \sim T_{jK}$. Hence we get

$$\langle\chi''\rangle(\omega,T) \sim G^{-1}(G/T)^{\lambda-1}g(\omega/T) \tag{52}$$

with g being the universal scaling function determined by $g(x) = x\int_1^\infty dy/y^{\lambda-1}(x^2+y^2)$, which differs drastically from the homogeneous case. Similar calculations, e.g., for the variation of the Knight shift and for the NMR relaxation rate yield $\delta K/K \propto \delta\chi/\chi \sim T^{-\lambda/2}$ (where δA denotes the mean square deviation of A due to the distribution of T_{jK} and $T_1^{-1} \sim G^{-1}(G/T)^{\lambda-1}g(\omega/T)$.

For the important *marginal* case $\lambda = 1$ the logarithmic T divergences appear. Here one has the distribution $P(T_{jK} = 0) = P_0 \neq 0$ valid till G. Then averaging the low-temperature part of the susceptibility and Sommerfeld coefficient, we obtain

$$\langle\chi\rangle \propto \frac{\langle c\rangle}{T} \sim -\frac{P_0}{2\pi}[\ln\frac{G}{T} + \frac{1}{2}\ln\ln\alpha\frac{G}{T} + \dots]. \quad (53)$$

Here we again see the zero-temperature divergences of $\langle\chi\rangle$ and $\langle c\rangle/T$ (weaker, though, than in the previous case). We can also calculate the low-field ground state magnetization:

$$\left\langle M^z \right\rangle \sim hP_0[-\ln(h/G) - \ln(\ln(h/C'G)) + \dots].$$

We obtain for the dynamic magnetic susceptibility the scaling behavior

$$\langle\chi''\rangle(\omega,T) \sim P_0[(\pi/2) - \tan^{-1}(2GT/0.41\pi\omega)]$$

(which is again in drastic contrast to the homogeneous case).

The weak power law or logarithmic dependence pertains to the Griffiths singularities in the proximity of the critical point $T = 0$ (cf. [25,26]). For these distributions of T_{jK} the Wilson ratio at $T = 0$ is equal to $2\pi^2/3$, characteristic for a FL-like situation. It turns out that our above-mentioned results for low temperatures are also valid for random ensembles of $S' = n/2$ (where n is the number of channels) multichannel Kondo impurities with a local anisotropic, generally speaking, interaction of the latter with conduction electrons, because at low temperatures the difference between the energy of the spin chain and the spin subsystem of the Kondo system is small (cf. Fig. 3).

We can illustrate our analytic results by numerical calculations for the solutions of Eqs. (40)–(44) (for accurate numerical calculations see Ref. 38). In Fig. 6 the temperature dependences of the magnetic susceptibility and the Sommerfeld coefficient for the most usual AF spin magnetically isotropic spin $S = 1/2$ chain are depicted. The solid lines show the *finite* values of the low-T χ and c/T in this case. However, the dashed and dotted lines present the answers for the distributions of θ_j (which, in turn, corresponds to the distributions of either the impurity—host exchange constants, see curves *1* and *2* of Fig. 4, or local effective Kondo temperatures, see Fig. 5, which presents results for $\gamma = 2$) with strong disorder. The latter means that the wings of the distributions are large enough compared to the maxima of the distributions. The dotted line corresponds to the Lorentzian distribution $P(\theta_j) = [(2\theta_j/\gamma)^2 + \pi^2]^{-1}$. The dashed line

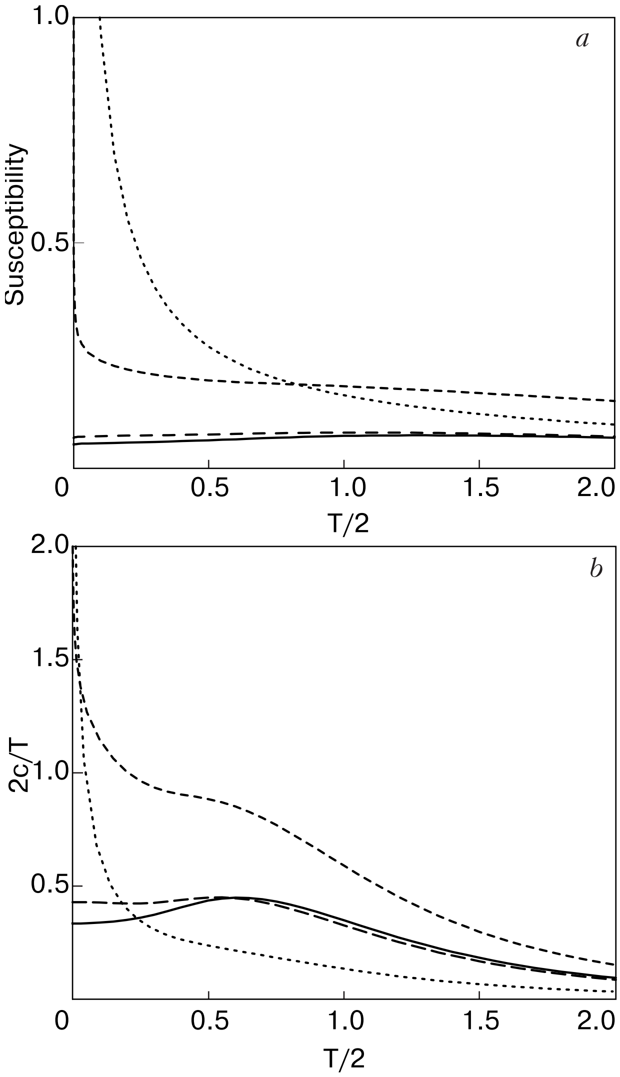

Fig. 6. Magnetic susceptibility (*a*) and the Sommerfeld coefficient c/T (*b*) at $h = 0$ for the isotropic spin-1/2 antiferromagnetic chain with spin-1/2 magnetic impurities. The exchange constant of the host is 2. The solid line shows the homogeneous chain; the long-dashed line — the Gaussian distribution; the dashed line — the log-normal distribution; the dotted line — the Lorentzian distribution of θ_j.

pertains to the so-called logarithmically normal distribution [65]

$$P(\theta_j) = \frac{\exp(-[\ln(|2\theta_j/\gamma| + 10^{-6}) + 1/4]^2)}{\sqrt{\pi}(|2\theta_j/\gamma| + 10^{-6})},$$

which is also characteristic for strong disorder. One can see the qualitative difference between the behavior of $S' = S$ magnetic impurities with the strong disorder of the distribution of their couplings to the host as compared to the isotropic spin chain. The magnetic susceptibility and the Sommerfeld coefficient diverge strongly at $T \to 0$ for the strongly randomly distributed parameters of the impurity—host couplings (note that in Ref. 37 we have shown that at low temperatures only the T_{jK} determine the scaling behavior of local impurities). This is in stark contrast with the homogeneous case. It turns out that the low-temperature asymptotics of the log-normal case of the disorder are [38]

$$c \sim \{\ln(1/T)\exp([\ln\ln(1/T)]^2\}^{-1},$$
$$\chi \sim \{T\ln\ln(1/T)\exp([\ln\ln(1/T)]^2\}^{-1}, \qquad (54)$$

while for the Lorentzian distribution one has

$$c \sim [\ln(1/T)]^{-2}, \ \ \chi \sim [T\ln(1/T)]^{-1} . \qquad (55)$$

The latter case is similar to the situation present for the so-called Griffiths phase [26] at very low temperatures.

In Fig. 7 similar behaviors are seen for the magnetic susceptibilities and Sommerfeld coefficients of the homogeneous case and the cases with the log-normal and Lorentzian distributions (strong disorder) and the Gaussian distribution (weak disorder, see below) for the mostly anisotropic «easy-plane» case $\gamma = \pi/2$ (for $S = 1/2$ this corresponds to the XX model, which for the homogeneous case pertains to the free spinless fermion gas). One can see that the changes due to the nonzero magnetic anisotropy of the «easy-plane» type are only qualitative. This is clear, because such an «easy-plane» magnetic anisotropy does not produce gaps for the low-energy excitations (i.e., it is a marginally irrelevant perturbation from the RG viewpoint), and, hence, the system remains in the critical regime.

On the other hand, the weak disorder does not produce such qualitative changes in the behavior of random ensembles of disordered magnetic impurities. By weak disorder we mean a narrow distribution of θ_j. The long-dashed lines of Figs. 6 and 7 depict the temperature behavior of the ensemble of magnetic impurities with the weak Gaussian distribution of θ_j (which is close to a single impurity distribution $P(\theta_j) = \delta(\theta_j)$). One can obviously see that such a narrow distribution (weak disorder) does not yield the divergences of the low-temperature magnetic susceptibility and the Sommerfeld coefficient of the specific heat.

The reason for such a different behavior of wide and narrow distributions of the parameters, which defines the impurity—host couplings (or strong—weak disorder, respectively), is clear. At low energies the local Kondo temperature define the crossover scale for the behavior of the magnetic impurity. For the case $S' = S$ a single magnetic impurity is screened by low-lying excitations of the host for $T < T_{jK}$, and is not screened for $T > T_{jK}$ (with the Curie-like behavior of the unscreened remnant spin). For the ensembles of magnetic impurities with the weak disorder the temperature is larger than the average Kondo temperature of the ensemble of impurities, and, hence, the total magnetic susceptibility and the Sommerfeld coefficient are finite for $T \to 0$. For the strong disorder, on the contrary, many local Kondo temperatures are *less* than the temperature. Those impurities remain unscreened by the low-lying excitations of the host, and, hence, the total magnetic susceptibility and the Sommerfeld coefficient become divergent for $T \to 0$.

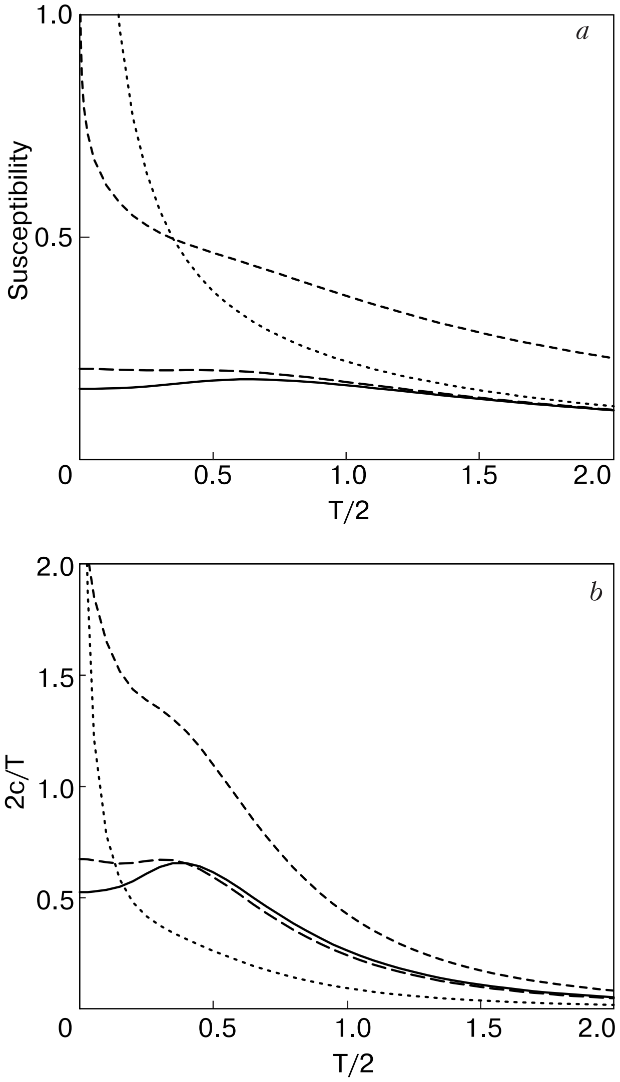

Fig. 7. Magnetic susceptibility (*a*) and the Sommerfeld coefficient c/T (*b*) at $h = 0$ for the anisotropic $\gamma = \pi/2$ spin-1/2 antiferromagnetic chain with spin-1/2 magnetic impurities. The solid line shows the homogeneous chain; the long-dashed line — the Gaussian distribution; the dashed line — the log-normal distribution; the dotted line — the Lorentzian distribution of θ_j.

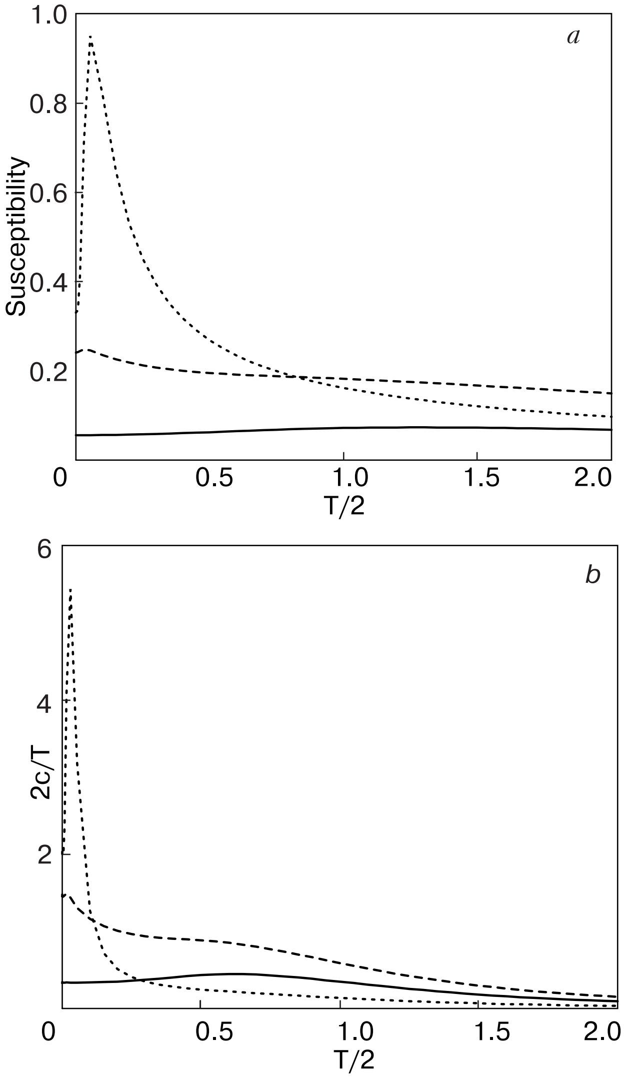

Fig. 8. Magnetic susceptibility (a) and the Sommerfeld coefficient c/T (b) at $h = 0.2$ for the isotropic spin-1/2 antiferromagnetic chain with spin-1/2 magnetic impurities. The solid line shows the homogeneous chain; the dashed line — the log-normal distribution; the dotted line — the Lorentzian distribution.

Finally we would like to show how the magnetic field lifts the degeneracy. In Fig. 8 the temperature behavior of magnetic susceptibilities and the Sommerfeld coefficients for the isotropic cases for the log-normal and Lorentzian distributions (cf. Fig. 6), but for the nonzero magnetic field $h = 0.2$ are depicted. One can clearly see that such a field removes the divergences in the low-T susceptibilities and Sommerfeld coefficients for the models with strong disorder. As an example, the temperature dependences of the same values for $h = 0.2$ are shown for the homogeneous chain. It turns out that the weak magnetic field does not yield any qualitative changes in the temperature behaviors, as expected.

For higher values of spins the changes, compared to the case $S' = S = 1/2$, are only quantitative. For example, the values of χ and c become larger for larger spin values. However, there are no drastic changes in the behavior of disordered ensembles of impurities, in comparison with the case discussed above. This seems to be natural, because only low-lying excitations (which have Dirac seas in the ground state) are responsible for the Kondo-like screening of spins of impurities, while other excitations (the quasienergies of which are described by y_p and Y_p), are higher-energy. In other words, spinons, which describe the SU(2) (or U(1)) symmetries of the system (or the Gaussian of the WZNW model) are essential for the process of screening of magnetic impurities, while excitations that describe the sl_2 symmetry of the RSOS sector (or parafermions of the WZNW model) do not play a qualitative role in that process.

We point out again that for low temperatures ($T < 2$) the numerical data are applicable to the behavior of the ensembles of Kondo impurities with «easy-plane» magnetic anisotropy (and, naturally, without it) of the local exchange interaction between magnetic impurities and conduction electrons.

7. Conclusions

Summarizing, in this work we have considered a number of exactly solvable models of magnetic impurities in critical quantum antiferromagnetic spin chains and multichannel Kondo impurities. We have studied their ground state properties and constructed the finite set of nonlinear integral equations which exactly describe the thermodynamics of the models. We have obtained several analytic low-energy expressions for the temperature, magnetic field, and frequency dependences of important characteristics of the exactly solvable disordered quantum spin models and disordered multichannel Kondo impurities with essential many-body interactions. We also have analyzed the data of numerical calculations of those nonlinear integral equations. We have shown that the only low-energy parameter that gets renormalized is the velocity of the low-lying excitations (or the effective crossover scale connected with each impurity); the others appear to be universal. [Note that the finite size corrections to the ground state behavior of our disordered spin chains can be obtained just by replacing $(G/T) \to L$.] We used several kinds of strong disorder important for experiments. Some of them produce low divergences in certain characteristics of our strongly disordered critical systems (compared with the finite values for the homogeneous case or a single impurity). They pertain to the wide distributions of the local Kondo temperatures, i.e., to a strong disorder in the system. On the other hand, for a weak disorder, or, in other words, for narrow distributions of the local Kondo temperatures, our exact results reveal the presence of the Kondo screening of the disordered ensembles of magnetic impurities by low-lying excitations of the host. We point out that our results qualitatively coincide with the data of experiments on real disordered quasi-1D quantum AF systems [27–29] with $\lambda \sim 0.26$–0.42. Also, qualitatively similar behavior has been observed in the 3D heavy metallic alloys [11–20] with $\lambda \sim 0.60$–0.85. It is interesting to note that similar results were recently obtained in which the distributions of Kondo temperatures used in this work were derived either from the proximity to a phase transition point in the Griffiths phase approximation (cluster percolation) [26] or from the Anderson localization effects in the infinite-dimensional statistical dynamical mean field theory approximation [66]. Also our results can be useful for the description of the Kondo necklace model [67,68]. All these similarities can be considered as the manifestation of generic features of the behavior of concentrated disordered magnetic systems for temperatures higher than a critical temperature in our effectively one-dimensional exactly integrable quantum models.

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