

## Selective survey on Subset Combinatorics of Groups

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**Abstract.** We survey recent results concerning the combinatorial size of subsets of groups. For a cardinal  $\kappa$ , according to its arrangement in a group  $G$ , a subset of  $G$  is distinguished as  $\kappa$ -large,  $\kappa$ -small,  $\kappa$ -thin,  $\kappa$ -thick and  $P_\kappa$ -small. By analogy with topology, there arise the following combinatorial cardinal invariants of a group: density, cellularity, resolvability, spread etc. The paper consists of 7 sections: Balleian context, Amenability, Ideals, Partitions, Packings, Around thin subsets, Colorings.

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### Introduction

Selectivity of the survey is in the choice of the subject: the combinatorial size of subsets of groups. We leave outside (or touch a little bit) the other aspects of the *Subset Combinatorics of Groups*, in particular, the *Additive Combinatorics* [52] and the *Ramsey Group Theory* [22, 41]. By the combinatorial size of a subset  $A$  of a group  $G$  we mean some (mainly, cardinal) characteristic which reflects an arrangement of  $A$  in  $G$ . The main results under consideration concern the groups, however, for our goal it is more convenient to use the terminology of  $G$ -spaces and, sometimes, much more general context of balleans.

Let  $G$  be a group with the identity  $e$ ,  $X$  be a  $G$ -space with the action  $G \times X \rightarrow X$ ,  $(g, x) \mapsto gx$ . For the subsets  $F \subseteq G$ ,  $A \subseteq X$ ,  $FA$  denotes the subset  $\{ga : g \in F, a \in A\}$  of  $X$ . For a cardinal  $\kappa$ ,  $[G]^{<\kappa}$  denotes the set  $\{F \subseteq G : |F| < \kappa\}$ .

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Let  $\kappa$  be an infinite cardinal,  $\kappa \leq |G|$ . We say that a subset  $A$  of  $X$  is

- $\kappa$ -large if there exists  $F \in [G]^{<\kappa}$  such that  $X = FA$ ;
- $\kappa$ -small if  $L \setminus A$  is  $\kappa$ -large for every  $\kappa$ -large subset  $L$ ;
- $\kappa$ -thick if, for every  $F \in [G]^{<\kappa}$ , there exists  $a \in A$  such that  $Fa \subseteq A$ ;
- $\kappa$ -thin if  $|gA \cap A| < \kappa$  for every  $g \in G$ ,  $g \neq e$ ;
- $P_\kappa$ -small if there exists a subset  $F$ ,  $|F| = \kappa$  such that the subsets  $\{gA : g \in F\}$  are pairwise disjoint.

In the case  $\kappa = \aleph_0$ , we omit  $\kappa$  and write, say, large instead of  $\aleph_0$ -large.

In what follows, a group  $G$  is considered as a left regular  $G$ -space with the action  $G \times G \rightarrow G$ ,  $(g, h) \mapsto gh$ .

These definitions are coming from many different sources, sometimes under different names, in particular, from group theory, topological algebra and topological dynamics. In this paper, we consider a lot of extensions, modifications and refinements of these basic notions.

The names large and small subset of a group were suggested by A. Bella and V. Malykhin in [9] with the question whether every countable group can be partitioned in two large subsets. This question was answered in [33]: every infinite group can be partitioned in countably many large subsets. The syndetic and piecewise syndetic subsets of a group from topological dynamics (see [22, Chapter IV]) are exactly large and not small subsets.

The thick subsets were used in [31] and [29] to show that every infinite group  $G$  can be partitioned in  $|G|$  subsets dense in every totally bounded group topology on  $G$ .

The thin subsets play an important role in algebra of the Stone-Čech compactifications [22]. By means of thin subsets, C. Chou [11] proved that every infinite amenable group  $G$  admits  $2^{2^{|G|}}$  distinct Banach measures.

The  $P$ -small subsets were introduced by I. Prodanov [30] as a technical tool in the study of the minimal topological groups.

In Section 1 we begin an exposition with some basic notions concerning ballean. A ballean is a set  $X$  endowed with some family of subsets which are called the balls. The properties of the family of balls are postulated in such a way that the ballean can be considered as a counterparts of the uniform topological spaces. Besides the metric spaces which have the natural ball structures, for every infinite cardinal  $\kappa$ , a  $G$ -space  $X$  determines a ballean if we call the set  $Fx \cup \{x\}$ ,  $F \in [G]^{<\kappa}$ ,  $x \in X$  a ball of radius  $F$  around  $x$ . Then we classify the subsets of a ballean by their

combinatorial size and, by analogy with topology, determine the ballean cardinal invariants, namely, density, cellularity, resolvability, spread etc. We conclude this section with a list of theorems which allow to calculate or evaluate the cardinal invariants for a large class of ballians including the ballians of  $G$ -spaces.

In Section 2 we characterize the large, small and thick subsets of an amenable group. In particular, we show that a subset  $S$  of an amenable group  $G$  is small if and only if  $\mu(KS) < 1$  for every Banach measure  $\mu$  on  $G$  and every finite subset  $K$ .

Section 3 is about the family  $\mathcal{S}_\kappa$  of all  $\kappa$ -small subsets of a group  $G$  and some its subfamilies, in particular, the family  $\mathcal{T}_\kappa$  of all  $\kappa$ -thin subsets and the family  $\mathcal{N}_\kappa$  of all non-generative subsets of  $G$ . The family  $\mathcal{S}_\kappa$  is an ideal in the Boolean algebra  $\mathcal{P}_G$  of all subsets of  $G$ . The main topic of this section is the group ideals. An ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called a group ideal if  $\mathcal{I}$  contains the ideal  $\mathcal{F}_G$  of all finite subsets and, for all  $A, B \in \mathcal{I}$ , we have  $AB \in \mathcal{I}$  and  $A^{-1} \in \mathcal{I}$ . We use a correspondence between the left invariant ideals in  $\mathcal{P}_G$  and closed left ideals in the semigroup  $\beta G$  (where  $\beta G$  is the Stone-Ćech compactification of  $G$ ) to describe the basic closed ideal in  $\beta G$ . We conclude this section with characterization (in the ballean context) of the ideal generated by the set of all thin subsets of a countable group  $G$ .

In Section 4, we describe the filtration method and apply it to partition of a group in large number of large and thick subsets, as well as, in small number of small subsets. A raisin of this section is in two new theorems on partitions of groups in thin subsets. In particular, one of them says that the statement “The group  $(\mathbb{R}, +)$  can be partitioned in countably many thin subsets” is equivalent to the Continuum Hypothesis. For the second part of this section we selected some 2-Ramsey type theorems. More precisely, these theorems are true for any partition of a group in two subsets, but fail to be true for partitions in  $> 2$  subsets.

Section 5 is a survey of results obtained by T. Banach and N. Lyaskovska on packing numbers. A packing number of a subset  $A$  of a group  $G$  is the cardinal  $pack(A) = \sup\{|F| : gA \cap g'A = \emptyset \text{ for all distinct } g, g' \in F\}$ .

In Section 6 we consider some modifications and refinements of thin subsets which appeared recently in [25]. The most interesting and delicate of them are sparse subsets coming from characterization of strongly prime ultrafilters in  $\beta G$ . Also, some results from this section show that every infinite group  $G$  can be generated by “very” thin subset.

The last Section 7 is much more “kaleidoscopic” than the previous. We survey briefly some recent cardinal invariants and configurations arising from the colorings of groups and  $G$ -spaces.

During the exposition we fix some open questions and problems. As

a rule, all necessary references can be found in comments at the end of each section.

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## 1. Ballean context

### 1.1. Ball structures and balleans

A *ball structure* is a triple  $\mathcal{B} = (X, P, B)$ , where  $X, P$  are non-empty sets and, for every  $x \in X$  and  $\alpha \in P$ ,  $B(x, \alpha)$  is a subset of  $X$  which is called a *ball of radius  $\alpha$  around  $x$* . It is supposed that  $x \in B(x, \alpha)$  for all  $x \in X$  and  $\alpha \in P$ . The set  $X$  is called the *support* of  $\mathcal{B}$ ,  $P$  is called the *set of radii*.

Given any  $x \in X$  and  $A \subseteq X$ , we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure  $\mathcal{B} = (X, P, B)$  is called a *ballean* if

- for any  $\alpha, \beta \in P$ , there exist  $\alpha', \beta' \in P$  such that, for every  $x \in X$ ,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- for any  $\alpha, \beta \in P$ , there exists  $\gamma \in P$  such that, for every  $x \in X$ ,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

A ballean  $\mathcal{B}$  is called *connected* if, for any  $x, y \in X$ , there exists  $\alpha \in P$  such that  $y \in B(x, \alpha)$ .

A subset  $A \subseteq X$  is called *bounded* if there exist  $x \in X$  and  $\alpha \in P$  such that  $A \subseteq B(x, \alpha)$ .

Given a ballean  $\mathcal{B} = (X, P, B)$ , we define a preordering  $\leq$  on  $P$  by the rule:  $\alpha \leq \beta$  if and only if  $B(x, \alpha) \subseteq B(x, \beta)$  for every  $x \in X$ . A subset  $P' \subseteq P$  is called *cofinal* if, for every  $\alpha \in P$ , there exists  $\alpha' \in P'$  such that  $\alpha \leq \alpha'$ . The *cofinality*  $cf\mathcal{B}$  is the minimal cardinality of cofinal subsets.

A connected unbounded ballean  $\mathcal{B}$  is called *ordinal* if there exists a well-ordered by  $\leq$  cofinal subset of  $P$ .

Let  $\mathcal{B}_1 = (X_1, P_1, B_1)$ ,  $\mathcal{B}_2 = (X_2, P_2, B_2)$  be balleans. A mapping  $f : X_1 \rightarrow X_2$  is called a  $\prec$ -*mapping* if, for every  $\alpha \in P_1$ , there exists  $\beta \in P_2$  such that  $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$  for every  $x \in X_1$ . If  $f$  is a bijection such that  $f$  and  $f^{-1}$  are  $\prec$ -mappings, we say that  $f$  is an

asymorphism. If  $X_1 = X_2$  and the identity mapping  $id : X_1 \rightarrow X_1$  is an asymorphism, we identify  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , and write  $\mathcal{B}_1 = \mathcal{B}_2$ .

Every subset  $Y \subseteq X$  determines the *subballean*  $\mathcal{B}_Y = (Y, P, B_Y)$  of  $\mathcal{B} = (X, P, B)$ , where  $B_Y(y, \alpha) = Y \cap B(y, \alpha)$  for all  $y \in Y, \alpha \in P$ .

Two balleans  $\mathcal{B} = (X, P, B)$  and  $\mathcal{B}' = (X', P', B')$  are called *coarsely equivalent* if there exist  $Y \subseteq X$  and  $Y' \subseteq X'$  such that  $\mathcal{B}_Y, \mathcal{B}_{Y'}$  are asymorphic and  $B(Y, \alpha) = X, B'(Y', \alpha') = X'$  for some  $\alpha \in P, \alpha' \in P'$ .

Every metric space  $(X, d)$  determines the metric ballean  $(X, \mathbb{R}^+, B_d)$ , where  $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$ . Clearly, every metric ballean is ordinal.

**Theorem 1.1.** *For a ballean  $\mathcal{B} = (X, P, B)$ , the following statements are equivalent*

- (i)  $\mathcal{B}$  is connected and  $cf\mathcal{B} \leq \aleph_0$ ;
- (ii)  $\mathcal{B}$  is asymorphic to some metric ballean;
- (iii)  $\mathcal{B}$  is coarsely equivalent to some metric ballean.

Let  $G$  be a group with the identity  $e$ . We recall that a  $G$ -space is a set  $X$  endowed with an action  $G \times X \rightarrow X, (g, x) \mapsto gx$  such that  $ex = x$  and  $g(hx) = (gh)x$  for all  $x \in X$  and  $g, h \in G$ . A  $G$ -space  $X$  is *transitive* if, for any  $x, y \in X$ , there exists  $g \in G$  such that  $gx = y$ .

Given an infinite  $G$ -space  $X$  and an infinite cardinal  $\kappa$  such that  $\kappa \leq |G|$ , we consider the ballean  $\mathcal{B}_\kappa(G, X) = (X, [G]^{<\kappa}, B)$ , where

$$B(x, F) = Fx \cup \{x\},$$

for all  $x \in X, F \in [G]^{<\kappa}$ . In the case  $\kappa = \aleph_0$ , we omit  $\kappa$  and write  $\mathcal{B}(G, X)$ . In the case  $G = X$  and the regular left action of  $G$  on  $X$ , we write  $\mathcal{B}_\kappa(G)$  instead of  $\mathcal{B}_\kappa(G, G)$ .

By the definition, a ballean  $\mathcal{B} = (X, P, B)$  has a *bounded geometry* if there exist  $\beta \in P$  and a function  $h : P \rightarrow \omega$  such that  $cap_\beta B(x, \alpha) \leq h(\alpha)$  for all  $x \in X, \alpha \in P$ , where  $\beta$ -*capacity* of  $B(x, \alpha)$  is

$$cap_\beta B(x, \alpha) = \sup\{|S| : S \text{ is a } \beta\text{-separated subset of } B(x, \alpha)\}$$

A subset  $S$  is called  $\beta$ -*separated* if  $B(x, \beta) \cap B(y, \beta) = \emptyset$  for all distinct  $x, y \in S$ .

Clearly,  $\mathcal{B}$  has a bounded geometry provided that, for every  $\alpha \in P$  there exists  $h(\alpha) \in \omega$  such that  $B(x, \alpha) \leq h(\alpha)$  for all  $x \in X, \alpha \in P$ . In particular, a ballean  $\mathcal{B}(G, X)$  of a  $G$ -space  $X$  is of bounded geometry.

**Theorem 1.2.** *Every ballean of bounded geometry is coarsely equivalent to some ballean  $\mathcal{B}(G, X)$  of  $G$ -space  $X$ .*

## 1.2. Subsets of ballean

Let  $\mathcal{B} = (X, P, B)$  be a ballean. We say that a subset  $A$  of  $X$  is

- *large* if there exists  $\alpha \in P$  such that  $X = B(A, \alpha)$ ;
- *small* if  $L \setminus A$  is large for every large subset  $L$ ;
- *thick* if  $\text{int}(A, \alpha) \neq \emptyset$  for every  $\alpha \in P$ , where  $\text{int}(A, \alpha) = \{x \in X : B(x, \alpha) \subseteq A\}$ ;
- *extralarge* if  $\text{int}(A, \alpha)$  is large for every  $\alpha \in P$ ;
- *piecewise large* if there exists  $\beta \in P$  such that  $\text{int}(B(A, \beta), \alpha) \neq \emptyset$  for every  $\alpha \in P$ ;
- *pseudodiscrete* if, for every  $\alpha \in P$ , there exists a bounded subset  $Y$  of  $X$  such that  $B(x, \alpha) \cap A = \{x\}$  for every  $x \in A \setminus Y$ .

In the following propositions we keep together some relationships between the introduced types of subsets.

**Proposition 1.1.** *For a ballean  $\mathcal{B} = (X, P, B)$  and a subset  $S \subseteq X$ , the following statements are equivalent*

- (i)  $S$  is small;
- (ii)  $S$  is not piecewise large;
- (iii)  $X \setminus S$  is extralarge;
- (iv)  $(X \setminus S) \cap L$  is large for every large subset  $L$  of  $X$ .

**Proposition 1.2.** *Let  $\mathcal{B} = (X, P, B)$  be a ballean. A subset  $T$  of  $X$  is thick if and only if  $T \cap L \neq \emptyset$  for every large subset  $L$  of  $X$ .*

We recall that a family  $\mathcal{I}$  of subsets of a set  $X$  is called an *ideal* in the Boolean algebra  $\mathcal{P}_X$  of all subsets of  $X$  if, for any  $A, B \in \mathcal{I}$  and  $A' \subseteq A$ , one has  $A \cup B \in \mathcal{I}$  and  $A' \in \mathcal{I}$ . We note that the family of all bounded subsets of a connected ballean with the support  $X$  forms an ideal in  $\mathcal{P}_X$ .

**Proposition 1.3.** *For every ballean  $\mathcal{B} = (X, P, B)$  the family  $\mathcal{S}$  of all small subsets of  $X$  is an ideal in  $\mathcal{P}_X$ . In particular,  $X$  cannot be partitioned in finitely many small subsets.*

Each of above defined types of subsets of a ballean has its counterparts in topology according to the following vocabulary:

large subset	dense subset
small subset	nowhere dense subset
thick subset	subset with nonempty interior
extralarge subset	subset with dense interior
piecewise large subset	somewhere dense subset
pseudodiscrete subset	discrete subset

Let  $G$  be an infinite group,  $X$  be a  $G$ -space,  $\kappa$  be an infinite cardinal with  $\kappa \leq |G|$ . A subset  $A \subseteq X$  is  $\kappa$ -large (resp.  $\kappa$ -small,  $\kappa$ -thick) if and only if  $A$  is large (resp. small, thick) in the ballean  $\mathcal{B}_\kappa(G, X)$ .

**Proposition 1.4.** *Let  $G$  be an infinite group,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ ,  $A \subseteq G$ . If  $A$  is pseudodiscrete in the ballean  $\mathcal{B}_\kappa(G)$  then  $A$  is  $\kappa$ -thin. If  $A$  is  $\kappa$ -thin and  $\kappa$  is regular then  $A$  is pseudodiscrete in the ballean  $\mathcal{B}_\kappa(G)$ .*

### 1.3. Cardinal invariants

Following the vocabulary, for every ballean  $\mathcal{B} = (X, P, B)$ , we introduce some cardinal invariants, namely, *density*, *crowdness*, *cellularity*, *resolvability*, *coresolvability*, *extraresolvability* and *spread* defined as

$$\text{den } \mathcal{B} = \min\{|L| : L \text{ is a large subset of } X\},$$

$$\text{cr } \mathcal{B} = \sup\{\kappa : \text{there exists } \alpha(\kappa) \in P \text{ such that } |B(x, \alpha(\kappa))| \geq \kappa \text{ for every } x \in X\},$$

$$\text{cell } \mathcal{B} = \sup\{|\mathcal{F}| : \mathcal{F} \text{ is a disjoint family of thick subsets of } X\},$$

$$\text{res } \mathcal{B} = \sup\{|\mathcal{F}| : \mathcal{F} \text{ is a disjoint family of large subsets of } X\},$$

$$\text{cores } \mathcal{B} = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a partition of } X \text{ into small subsets}\},$$

$$\text{exres } \mathcal{B} = \sup\{|\mathcal{F}| : \mathcal{F} \text{ is a family of large subsets of } X \text{ such that } F \cap F' \text{ is small for all distinct } F, F' \in \mathcal{F}\}.$$

$$\text{spread } \mathcal{B} = \sup\{|Y|_{\mathcal{B}} : Y \text{ is a pseudodiscrete subset of } X\}, \text{ where } |Y|_{\mathcal{B}} = \min\{|Y \setminus V| : V \text{ is a bounded subset of } X\}.$$

**Theorem 1.3.** *For every ordinal ballean  $\mathcal{B} = (X, P, B)$ ,*

$$\text{den } \mathcal{B} = \text{cell } \mathcal{B} = \text{spread } \mathcal{B}.$$

*Moreover, there exists a disjoint family  $\mathcal{F}$  of thick subsets of  $X$  and a subset  $Y$  of  $X$  such that  $\text{cell } \mathcal{B} = |\mathcal{F}|$  and  $|Y|_{\mathcal{B}} = |Y| = \text{spread } \mathcal{B}$ .*

**Theorem 1.4.** *Let  $\mathcal{B} = (X, P, B)$  be a ballean,  $|X| = \kappa$  and let  $|P| \leq \kappa$ . Then  $\text{cell } \mathcal{B} = \text{den } \mathcal{B} = \kappa$  and there exists a disjoint family  $\mathcal{F}$  of cardinality  $\kappa$  consisting of thick subsets of  $X$  provided that one of the following conditions is satisfied*

- (i) *there exists  $\kappa' < \kappa$  such that  $B(x, \alpha) \leq \kappa'$  for all  $x \in X$  and  $\alpha \in P$ ;*
- (ii)  *$|B(x, \alpha)| < \kappa$  for all  $x \in X$ ,  $\alpha \in P$  and  $\kappa$  is regular.*

**Theorem 1.5.** *For every ballean  $\mathcal{B}$ ,*

$$\text{cr } \mathcal{B} \leq \text{res } \mathcal{B} \leq \text{cr } \mathcal{B} \cdot \text{cf } \mathcal{B}.$$

**Theorem 1.6.** *Let  $(X, d)$  be a metric space,  $\mathcal{B} = \mathcal{B}(X, d)$ . Then  $\text{res } \mathcal{B} = \text{cr } \mathcal{B}$  and  $X$  can be partitioned in  $\text{cr } \mathcal{B}$  large subsets.*

**Theorem 1.7.** *Let  $\mathcal{B}$  be a ballean such that  $\text{cr } \mathcal{B} < \aleph_0$ . Then*

$$\text{cr } \mathcal{B} = \text{res } \mathcal{B} = \text{exres } \mathcal{B}$$

Let  $\mathcal{B} = (X, P, B)$  be a ballean. Given any  $\alpha \in P$  and a cardinal  $\kappa$ , we put

$$X(\alpha, \kappa) = \{x \in X : |B(x, \alpha)| \leq \kappa\}.$$

**Theorem 1.8.** *Let  $(X, d)$  be an unbounded metric space,  $\mathcal{B} = \mathcal{B}(X, d)$ . Assume that  $\text{cr } \mathcal{B} = \aleph_0$  and, for every  $m \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $X(m, n)$  is piecewise large. Then*

$$\text{res } \mathcal{B} = \text{exres } \mathcal{B} = \aleph_0.$$

In particular,  $\text{res } \mathcal{B} = \text{exres } \mathcal{B} = \aleph_0$  provided that for  $\text{cr } \mathcal{B} = \aleph_0$  and, for every  $m \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $|B(x, m)| \leq n$  for every  $x \in X$ . It follows that  $\text{exres } \mathcal{B}(G) = \aleph_0$  for every countable group  $G$ .

**Theorem 1.9.** *Let  $(X, d)$  be an unbounded countable metric space,  $\mathcal{B} = \mathcal{B}(X, d)$ . Assume that there exists  $n \in \mathbb{N}$  such that  $X(n, k)$  is small for each  $k \in \mathbb{N}$ . Then  $\text{exres } \mathcal{B} = 2^{\aleph_0}$ .*

It follows from Theorems 1.7, 1.8 and 1.9 that, for a countable metric space  $(X, d)$ ,  $\text{exres } \mathcal{B}(X, d)$  could be either finite, or  $\aleph_0$ , or  $2^{\aleph_0}$ . It is easy to construct an example for each case.

The cardinal invariants density and crowding are much more simple than others. For example, it is easy to see that  $\text{den } \mathcal{B}_\kappa(G) = |G|$  and  $\text{cr } \mathcal{B}_\kappa(G) = \kappa$  for every infinite group  $G$  and every  $\kappa$ . Thus, these theorems allow us to calculate or evaluate more complicated cardinal invariants cellularity, resolvability, coresolvability, extraresolvability and spread. For coresolvability of group ballians, see Subsection 3.1.



**Comments.** For ballean as the asymptotic counterparts of the uniform topological spaces see [44]. A ballean with the support  $X$  can also be defined in terms of the entourages of diagonal in  $X \times X$ . In this case, it is called a coarse structure [48]. The proofs of results of this section can be found in [44] and in the original papers [17, 35–37].

## 2. Amenability

We recall that a *Banach measure*  $\mu : \mathcal{P}_G \rightarrow [0, 1]$  is a function defined on the family  $\mathcal{P}_G$  of all subsets of a group  $G$  such that

- $\mu(G) = 1$ ;
- if  $A \cap B = \emptyset$  then  $\mu(A \cup B) = \mu(A) + \mu(B)$ ;
- $\mu(xA) = \mu(A)$  for all  $x \in G$ ,  $A \subseteq G$ .

A group  $G$  is called *amenable* if there exists a Banach measure on  $\mathcal{P}_G$ . All results of this section were proved by means of the Folner criterion [20, Theorem 3.61]: a group  $G$  is amenable if and only if, for every finite subset  $H$  of  $G$  and every  $\varepsilon > 0$ , there exists a finite subset  $F$  of  $G$  such that  $H \subseteq F$  and, for every  $x \in H$ ,

$$\frac{|(xF \setminus F) \cap (F \setminus xF)|}{|F|} < \varepsilon.$$

**Theorem 2.1.** *For a subset  $L$  of an amenable group  $G$ , the following statements are equivalent:*

- (i)  $L$  is large;
- (ii)  $\mu(L) > 0$  for every Banach measure  $\mu$  on  $\mathcal{P}_G$ ;
- (iii) there exists  $\varepsilon > 0$  such that  $\mu(L) > \varepsilon$  for every Banach measure  $\mu$  on  $\mathcal{P}_G$ .

**Theorem 2.2.** *A subset  $S$  of an amenable group  $G$  is small if and only if  $\mu(FS) < 1$  for every Banach measure  $\mu$  on  $\mathcal{P}_G$  and every finite subset  $F$  of  $G$ .*

**Theorem 2.3.** *A subset  $T$  of an amenable group is thick if and only if there exists a Banach measure  $\mu$  on  $\mathcal{P}_G$  such that  $\mu(T) = 1$ .*

A family  $\mathcal{I}$  of subsets of a group  $G$  is called an *ideal* if, for any  $A, B \in \mathcal{I}$  and  $A' \subseteq A$ , we have  $A \cup B \in \mathcal{I}$ ,  $A' \in \mathcal{I}$ . An ideal  $\mathcal{I}$  is called *translation invariant* if  $xA \in \mathcal{I}$  for all  $x \in G$ ,  $A \in \mathcal{I}$ .

Given a Banach measure  $\mu$  on  $\mathcal{P}_G$ , we put

$$\mathcal{N}_\mu = \{A \in \mathcal{I} : \mu(A) = 0\}, \quad \mathcal{N} = \cap \{\mathcal{N}_\mu : \mu \text{ is a Banach measure on } \mathcal{P}_G\}$$

Clearly,  $\mathcal{N}_\mu$  and  $\mathcal{N}$  are translation invariant ideals. By Theorem 2.2,  $\mathcal{N}$  is contained in the ideal  $\mathcal{S}_G$  of all small subsets of  $G$ . A subset  $A \in \mathcal{N}$  is called an *absolute null subset*. Clearly, every  $P$ -small subset is an absolute null subset.

**Theorem 2.4.** *Let  $G$  be a countable amenable group,  $\varepsilon > 0$ . Then there exists a small subset  $A \subset G$  such that  $\mu(A) \geq 1 - \varepsilon$  for some Banach measure  $\mu$  on  $\mathcal{P}_G$ .*

**Theorem 2.5.** *Let  $G$  be an amenable group,  $\mathcal{I}$  be a translation invariant ideal in  $\mathcal{P}_G$ ,  $\mathcal{I} \neq \mathcal{P}_G$ . Then there exists a Banach measure  $\mu$  on  $\mathcal{P}_G$  such that  $\mathcal{I} \subseteq \mathcal{N}_\mu$ .*

**Theorem 2.6.** *For every amenable group  $G$ , we have*

$$\mathcal{S}_G = \cap \{\mathcal{N}_\mu : \mathcal{S}_G \subseteq \mathcal{N}_\mu, \mu \text{ is a Banach measure on } \mathcal{P}_G\}$$

**Theorem 2.7.** *If a group  $G$  contains an isomorphic copy of the free group  $F_2$  of rank 2, then  $G$  can be partitioned  $G = A_1 \cup A_2$  such that  $A_1, A_2$  are large and  $P$ -small. In particular, there exists a  $P$ -small but not small subset of  $G$ .*

**Question 2.1.** Let  $G$  be a non-amenable group. Does there exist a  $P$ -small not small subset of  $G$ ?

**Question 2.2 (T. Banach).** Give a combinatorial characterization of the ideal  $\mathcal{N}$  of absolute null subsets of an amenable group.

We say that a transitive  $G$ -space  $X$  is amenable if there exists a finitely additive measure  $\mu : \mathcal{P}_G \rightarrow [0, 1]$  such that  $\mu(X) = 1$  and  $\mu(gA) = \mu(A)$  for all  $g \in G, A \subseteq X$ .

**Question 2.3.** Characterize the large, small and thick subsets of an amenable  $G$ -space.

A function  $\mu : \mathcal{P}(G) \rightarrow \mathbb{R}^+$  is called *subadditive* if  $\mu(A \cup B) \leq \mu(A) + \mu(B)$  for all  $A, B \in \mathcal{P}(G)$ . Let  $\mathcal{I}$  be a (translation invariant) ideal in  $\mathcal{P}(G)$ ,  $\mathcal{I} \neq \mathcal{P}(G)$ . Then there exists a (translation invariant) subadditive function  $\mu : \mathcal{P}(G) \rightarrow \{0, 1\}$  such that  $\mathcal{I} = \mathcal{N}_\mu$ . Indeed, we put  $\mu(A) = 0$  if and only if  $A \in \mathcal{I}$ .

Given a subset  $A \subseteq G$  and a finite subset  $F \subseteq G$ , we put  $\mu_F(A) = \left| \frac{A \cap F}{F} \right|$  and

$$u(A) = \inf \left\{ \sup_{g \in G} \mu_F(gA) : F \text{ is finite and non-empty} \right\}.$$

By [51], the function  $u$  is subadditive for every amenable group  $G$ . It is easy to verify (using the Folner criterion) that  $\mathcal{N} \subseteq \mathcal{N}_u$  for every amenable group  $G$ .

**Question 2.4.** Is  $\mathcal{N} = \mathcal{N}_u$  for every amenable group  $G$ ?

**Comments.** The results of this section from [2, 24].

### 3. Ideals

#### 3.1. $\kappa$ -Small subsets and duality

**Theorem 3.1.** *Let  $G$  be an infinite Abelian group,  $\kappa$  be a limit cardinal such that  $\kappa \leq |G|$ . Then every  $P_\kappa$ -small subset of  $G$  is  $\kappa$ -small.*

By Theorem 3.3, every  $P$ -small subset of an Abelian group  $G$  is small. This statement does not hold for non-Abelian groups.

**Example 3.1.** Let  $F$  be a free group in an infinite alphabet  $A$  of cardinality  $\kappa$ ,  $a \in A$ ,  $F_a$  be the set of all group words whose first letter is  $a$ . Since the subsets  $\{xF_a : x \in A \setminus \{a\}\}$  are disjoint,  $F_a$  is  $\kappa$ -small. On the other hand,  $F = F_a \cup a^{-1}F_a$  so  $F_a$  is large and  $F_a$  is not  $\kappa'$ -small for every infinite cardinal  $\kappa' \leq \kappa$ .

**Question 3.1.** Let  $G$  be an infinite Abelian group,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ . Is every  $P_\kappa$ -small subsets of  $G$  small? By Theorem 3.1, this is so if  $\kappa$  is a limit ordinal.

**Theorem 3.2.** *Let  $G$  be an infinite group,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ ,  $A$  be a  $\kappa$ -thin subset of  $G$ . Then the following statements hold*

- (i) *if  $\kappa$  is regular then  $A$  is  $\kappa$ -small;*
- (ii) *if  $\kappa'$  is a cardinal such that  $\kappa < \kappa' \leq |G|$ , then  $A$  is  $\kappa'$ -small;*
- (iii) *if  $\kappa = \omega$  then  $A$  is  $\kappa'$ -small for each  $\kappa'$  such that  $\omega \leq \kappa' \leq |G|$ .*

**Theorem 3.3.** *In every infinite group  $G$ , there exist a subset  $A$  such that  $A$  is  $\kappa$ -small for every  $\kappa$ ,  $\omega \leq \kappa \leq |G|$  but  $A$  is not  $P$ -small.*

If  $A$  is a  $P_\kappa$ -small subset of a group  $G$  then  $A$  is  $P_{\kappa'}$ -small for each  $\kappa'$  such that  $\omega \leq \kappa' \leq \kappa$ . Surprisingly, this is not true for  $\kappa$ -small subsets.

For a group  $G$  and a cardinal  $\kappa$ , we denote by  $\mathcal{S}_\kappa$  the ideal of all  $\kappa$ -small subsets of  $G$ .

**Theorem 3.4.** *Let  $G$  be an uncountable group,  $\kappa$  be a cardinal such that  $\omega < \kappa \leq |G|$ . Then the following statements hold*

(i) *if  $G$  is Abelian then  $\mathcal{S}_\omega \setminus \mathcal{S}_\kappa \neq \emptyset$ ;*

(ii) *if  $\kappa = |G|$  and  $\kappa$  is regular then  $\mathcal{S}_\kappa \setminus \mathcal{S}_\omega \neq \emptyset$ .*

Given a family  $\mathcal{F}$  of subsets of a group  $G$ , we put

$$\mathcal{F}^* = \{X \subset G : X^{-1}A \neq G \text{ for every } A \in \mathcal{F}\}.$$

In [49], W. Sređiński asked if there exists in (ZFC) a family  $\mathcal{F}$  of subsets of  $\mathbb{R}$  such that  $\mathcal{F}^* = \{\text{countable subset of } \mathbb{R}\}$ . In [50], Solecki answered this question in the affirmative proving that, for every infinite Abelian group  $G$  and every infinite cardinal  $\kappa$  with  $\kappa \leq |G|$ , there exists a translation invariant ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  such that  $\mathcal{I}^* = [G]^{<\kappa}$ . This was done as a byproduct of some very complicated construction.

Now we give much more elegant solution of the Sređinski problem using the ideal  $\mathcal{S}_\kappa$ .

**Theorem 3.5.** *Let  $G$  be an infinite group of regular cardinality  $\kappa$ . Then  $\mathcal{S}_\kappa^* = [G]^{<\kappa}$ .*

**Theorem 3.6.** *Let  $G$  be an infinite Abelian group and  $\kappa$  be a limit cardinal such that  $\kappa \leq |G|$ . Then  $\mathcal{S}_\kappa^* = [G]^{<\kappa}$ .*

**Theorem 3.7.** *Let  $G$  be an divisible Abelian group and  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ . Then  $\mathcal{S}_\kappa^* = [G]^{<\kappa}$ .*

**Question 3.2.** Is  $\mathcal{S}_\kappa^* = [G]^{<\kappa}$  for every infinite group  $G$  and every infinite cardinal  $\kappa$  such that  $\kappa \leq |G|$ ?

**Theorem 3.8.** *For every infinite group  $G$  of regular cardinality  $\kappa$ , there does not exist a family  $\mathcal{F}$  of subsets of  $G$  such that  $\mathcal{S}_\kappa = \mathcal{F}^*$ .*

**Comments.** This section is selection from [39].

### 3.2. Counting $\Omega$ -ideals

Let  $\mathbb{A}$  be a universal algebra of signature  $\Omega = \bigcup_{n \in \omega} \Omega_n$ ,  $\Omega_n$  is the set of  $n$ -ary functions in  $\Omega$ ,  $\mathcal{P}_{\mathbb{A}}$  be the Boolean algebra of all subsets of  $\mathbb{A}$ ,  $\mathcal{F}_{\mathbb{A}}$  be the family of all finite subsets of  $\mathbb{A}$ . A non-empty family  $\mathcal{I}$  of subsets  $\mathbb{A}$  is called an *ideal* in  $\mathcal{P}_{\mathbb{A}}$  if, given any  $A, B \in \mathcal{I}$  and  $A' \subseteq A$ , one has  $A \cup B \in \mathcal{I}$  and  $A' \in \mathcal{I}$ .

We say that an ideal  $\mathcal{I}$  in  $\mathcal{P}_{\mathbb{A}}$  is an  $\Omega$ -*ideal*, if  $\mathcal{F}_{\mathbb{A}} \subseteq \mathcal{I}$  and, for any  $A \in \mathcal{I}$  and  $f \in \Omega_n$ , we have  $f(A^n) \in \mathcal{I}$ . Given a subset  $A \subseteq \mathbb{A}$ , we denote by  $\mathcal{I}(A)$  the smallest by inclusion  $\Omega$ -ideal in  $\mathcal{P}_{\mathbb{A}}$  containing  $A$ , and say that  $\mathcal{I}(A)$  is the *monogenic  $\Omega$ -ideal* generated by  $A$ .

**Theorem 3.9.** *Let  $\mathbb{A}$  be a countably infinite universal algebra of countable signature  $\Omega$ . Then there exist  $2^{\aleph_0}$  monogenic  $\Omega$ -ideals and  $2^{2^{\aleph_0}}$   $\Omega$ -ideals in  $\mathcal{P}_{\mathbb{A}}$ .*

A proof of this statement is based on the following combinatorial lemma.

**Lemma 3.1.** *Let  $X, Y$  be infinite sets,  $Z$  be an infinite subsets of  $Y$ ,  $n \in \omega$ ,  $f : X^n \rightarrow Y$  be an arbitrary function. Given any  $V \in [X]^{\omega}$  and  $Z \in [Y]^{\omega}$  there exists an infinite subset  $U$  of  $V$  such that  $Z \setminus f(U^n)$  is infinite.*

Let  $\mathcal{B} = (X, P, B)$  be a ballean. Suppose that the support  $X$  of  $\mathcal{B}$  has a structure of an  $\Omega$ -algebra. We say that  $\mathcal{B}$  is an  $\Omega$ -*ballean* if, for every  $f \in \Omega_n$ , the mapping

$$X^n \rightarrow X, (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$$

is a  $\prec$ -mapping, where  $X^n$  is the support of the product of  $n$  copies of  $\mathcal{B}$ . Thus, an  $\Omega$ -ballean is a counterpart of a topological  $\Omega$ -algebra. If  $\mathcal{B}$  is a connected  $\Omega$ -ballean, then the family  $\mathcal{I}$  of all bounded subsets of  $X$  is an  $\Omega$ -ideal. In some cases, this ideal  $\mathcal{I}$  uniquely determines  $\mathcal{B}$ . For the case of groups see [44, Chapter 6] and the next section.

Let  $\Omega = \{+, \cdot\}$  and let an  $\Omega$ -algebra  $X$  be a ring. For every  $\Omega$ -ideal  $\mathcal{I}$  in  $\mathcal{P}_X$ , the ball structure  $\mathcal{B} = (X, P, B)$ , where  $P = \{A \in \mathcal{I} : 0 \in A\}$  and  $B(x, A) = x + A$ , is an  $\Omega$ -ballean. Since  $\mathcal{F}_X \subseteq \mathcal{I}$ ,  $\mathcal{B}$  is connected. Thus, we get a bijective correspondence between the connected  $\Omega$ -balleans on  $X$  and the  $\Omega$ -ideals in  $\mathcal{P}_X$ . Under this correspondence, our theorem states that every countably infinite ring admits  $2^{2^{\aleph_0}}$  ballean structures which respect with its algebraic structure. This is a counterpart of Theorem 1.3 from [14] stating that every countable ring admits  $2^{2^{\aleph_0}}$  Hausdorff ring topologies.

**Comments.** This section is a summary of [40].

### 3.3. Group ideals

We say that an ideal  $\mathcal{I}$  in the Boolean algebra  $\mathcal{P}_G$  is a *group ideal* if  $\mathcal{I}$  contains the ideal of all finite subsets of  $G$ , and

$$A, B \in \mathcal{I} \Rightarrow AB \in \mathcal{I}, A^{-1} \in \mathcal{I}.$$

Given a subset  $A$  of  $G$ , we denote by  $gp(A)$  the smallest group ideal in  $\mathcal{P}_G$  containing  $A$ , and say that  $gp(A)$  is *monogenic group ideal* generated by  $A$ .

We have at least two motivation to study group balleans. On one hand, every group ideal in  $\mathcal{P}_G$  determines a ball structure on  $G$  which respect with the algebraic structure of  $G$  (see [44, Chapter 6]). On the other hand, every group ideal induces a closed two-side ideal in the semigroup  $\beta G$  (see Section 3.4).

**Theorem 3.10.** *For every countable group  $G$ , there exists  $2^{\aleph_0}$  monogenic group ideals and  $2^{2^{\aleph_0}}$  group ideals in  $\mathcal{P}_G$ .*

**Theorem 3.11.** *Let  $G$  be an infinite Abelian group of cardinality  $\kappa$ . Then there exists  $2^\kappa$  monogenic group ideals and  $2^{2^\kappa}$  group ideals in  $\mathcal{P}_G$ .*

We say that a subset  $A$  of a group  $G$  is *non-generative* if, for any  $n \in \omega$  and  $F \in \mathcal{F}_G$ ,  $G \neq X(F, A, n)$ , where

$$X(F, A, n) = \underbrace{(FA)^{\pm 1} \dots (FA)^{\pm 1}}_n.$$

Clearly,  $A$  is non-generative if and only if  $gp(A) \neq \mathcal{P}_G$ . If  $\mathcal{I}$  is a group ideal and  $\mathcal{I} \neq \mathcal{P}_G$  then every subset  $A \in \mathcal{I}$  is non-generative.

Following [42], we say that a sequence  $(a_n)_{n \in \omega}$  of elements of a group  $G$  is a *T-sequence* if there exists a Hausdorff group topology on  $G$  in which  $(a_n)_{n \in \omega}$  converges to the identity  $e$  of  $G$ .

**Theorem 3.12.** *For every infinite group  $G$ , the following statements hold*

- (i) *if  $(a_n)_{n \in \omega}$  is an injective T-sequence then the subset  $\{a_n : n \in \omega\}$  is non-generative;*
- (ii) *there exists an injective sequence  $(b_n)_{n \in \omega}$  such that the subset  $\{b_n : n \in \omega\}$  is non-generative but  $(b_n)_{n \in \omega}$  is not a T-sequence.*

**Corollary 3.1.** *Let  $(a_n)_{n \in \omega}$  be an injective sequence in  $\mathbb{Z}$  such that either  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$  or  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$  and  $r$  is transcendental. Then the subset  $\{a_n : n \in \omega\}$  is non-generative in  $\mathbb{Z}$ .*

By [42, Theorems 2.2.4 and 2.2.6], for every algebraic number  $r > 1$ , there exist two injective sequences of positive integers such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = r$$

and  $(a_n)_{n \in \omega}$  is a  $T$ -sequence in  $\mathbb{Z}$  but  $(b_n)_{n \in \omega}$  is not a  $T$ -sequence. By Theorem 3.12 the subset  $\{a_n : n \in \omega\}$  is non-generative. We do not know whether, for every algebraic number  $r \geq 1$ , there is an injective sequence  $(a_n)_{n \in \omega}$  of integers such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$  and the subset  $\{a_n : n \in \omega\}$  is generative.

For a group  $G$ , we denote by  $\mathcal{N}_G$  the minimal ideal in  $\mathcal{P}_G$  containing all non-generative subsets of  $G$ . Thus,  $\mathcal{N}_G$  is the family of all finite unions of non-generative subsets.

**Question 3.3.** Given a group  $G$  and  $n \in \mathbb{N}$ , how to detect whether given subset  $A$  of  $G$  is a union on  $n$  non-generative subsets.

**Theorem 3.13.** *For every countable group  $G$ ,  $\mathcal{N}_G$  is not a group ideal.*

We recall that partially ordered set  $(L, \leq)$  is a *lattice* if any two elements  $a, b \in L$  have the exact lower bound (denoted by  $a \wedge b$ ) and the exact upper bound (denoted by  $a \vee b$ ). A lattice  $L$  is called *complete* if any subset  $S$  of  $L$  has the exact lower bound (denoted by  $\bigwedge S$ ) and the exact upper bound (denoted by  $\bigvee S$ ). In this case, by  $\min L$  and  $\max L$  we denote  $\bigwedge L$  and  $\bigvee L$ . An element  $a \in L$  is called an *atom* (resp. *coatom*) if the interval  $(\min L, a)$  (resp.  $(a, \max L)$ ) is empty.

A lattice  $L$  is called *modular* if  $(a \vee b) \wedge c \leq a \vee (b \wedge c)$  for all  $a, b, c \in L$  such that  $a \leq c$ .

For a group  $G$ , we denote by  $\mathbf{L}_G$  the set of all group ideals in  $\mathcal{P}_G$  partially ordered by inclusion. Given any subset  $\mathbf{S} \subset \mathbf{L}_G$ ,  $\bigcap \mathbf{S}$  is the exact lower bound of  $\mathbf{S}$  and

$$\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a group ideal, } \mathcal{I}' \subseteq \mathcal{I} \text{ for every } \mathcal{I}' \in \mathbf{S}\}$$

is the exact upper bound of  $\mathbf{S}$ , so  $\mathbf{L}_G$  is a complete lattice with  $\min \mathbf{L}_G = \mathcal{F}_G$  and  $\max \mathbf{L}_G = \mathcal{P}_G$ . It is easy to show that  $\mathbf{L}_G$  is modular for every Abelian group  $G$ .

**Example 3.2.** Let  $L$  be a free group with the free generators  $a, b$ . To show that the lattice  $\mathbf{L}_F$  is not modular, we put

$$A = \{a^n : n \in \mathbb{Z}\}, \quad B = \{b^n : n \in \mathbb{Z}\}, \quad C = \{a^n, b^n a^n b^n : n \in \mathbb{Z}\},$$

and  $\mathcal{I}_1 = gp(A)$ ,  $\mathcal{I}_2 = gp(B)$ ,  $\mathcal{I}_3 = gp(C)$ . Then  $\mathcal{I}_1 \subseteq \mathcal{I}_3$  and

$$(\mathcal{I}_1 \vee \mathcal{I}_2) \wedge \mathcal{I}_3 = \mathcal{I}_3, \quad \mathcal{I}_1 \vee (\mathcal{I}_2 \wedge \mathcal{I}_3) = \mathcal{I}_1,$$

but  $\mathcal{I}_1 \neq \mathcal{I}_3$ .

**Theorem 3.14.** *If an infinite group  $G$  is either countable or Abelian then  $\mathbf{L}_G$  has no atoms.*

In contrast to Theorem 3.14, by the Zorn Lemma, every group ideal  $\mathcal{I} \in \mathbf{L}_G$ ,  $\mathcal{I} \neq \mathcal{P}_G$  is contained in some coatom of  $\mathbf{L}_G$ .

**Theorem 3.15.** *Let  $G$  be a countable group,  $\mathbf{C}$  be a set of all coatoms in  $\mathbf{L}_G$ . Then  $\wedge \mathbf{C} = \mathcal{F}_G$ ,  $\vee \mathbf{C} = \mathcal{P}_G$ .*

**Theorem 3.16.** *For every countable group  $G$ , there exists a sublattice  $\mathbf{R}$  of  $\mathbf{L}_G$  isomorphic to the lattice  $(\mathbf{R}, \leq)$ .*

**Comments.** All results of this section from [16]. For vector ideals see [25].

### 3.4. Applications to $\beta G$

Let  $X$  be a discrete space,  $\beta X$  be the Stone-Ćech compactification of  $X$ ,  $X^* = \beta X \setminus X$ . We take the points of  $\beta X$  to be the ultrafilters on  $X$  identifying  $X$  with the principal ultrafilters. For every subset  $A \subseteq X$ , we put  $\bar{A} = \{q \in \beta X : A \in q\}$ . The topology of  $\beta X$  can be defined by stating that the family  $\{\bar{A} : A \subseteq X\}$  is a base for the open sets. For every filter  $\varphi$  on  $X$ , the subset  $\bar{\varphi} = \bigcap \{\bar{A} : A \in \varphi\}$  is closed in  $\beta X$ , and for every non-empty closed subset  $\delta \subseteq \beta X$ , there exists a filter  $\varphi$  on  $X$  such that  $\delta = \bar{\varphi}$ . To our purposes, it is more convenient to determine the closed subsets of  $\beta X$  by means of the ideals in the Boolean algebra  $\mathcal{P}_X$  of all subsets of  $X$ .

Given an ideal  $\mathcal{I}$  in  $\mathcal{P}_X$  and a closed subset  $\delta \subseteq \beta X$ , we put

$$\hat{\mathcal{I}} = \{q \in \beta X : X \setminus A \in q \text{ for every } A \in \mathcal{I}\}, \quad \check{\delta} = \{A \subseteq X : \delta \subseteq \overline{X \setminus A}\}.$$

Then  $\hat{\cdot}$  is a bijection between the set of all ideals in  $\mathcal{P}_X$  and the set of all non-empty closed subsets of  $\beta X$ , and  $\check{\cdot}$  is its inverse mapping.

The universal property of  $\beta X$  says that every mapping  $f : X \rightarrow Y$ , where  $Y$  is a compact Hausdorff space, can be extended to the continuous mapping  $f^\beta : \beta X \rightarrow Y$ .

We note (see [41, Corollary 3.6]) that, for every infinite discrete space  $X$  of cardinality  $\kappa$ ,  $|\beta X| = 2^{2^\kappa}$ .

Now let  $G$  be a discrete group with identity  $e$ . Using the universal property of the space  $\beta G$ , we extend the group multiplication from  $G$  to  $\beta G$  in two steps. Given  $g \in G$ , the mapping

$$x \mapsto gx : G \rightarrow \beta G$$



extends to the continuous mapping

$$q \mapsto gq : \beta G \rightarrow \beta G.$$

Then, for each  $q \in \beta G$ , we extend the mapping  $g \mapsto gq$ , defined from  $G$  into  $\beta G$ , to the continuous mapping

$$p \mapsto pq : \beta G \rightarrow \beta G.$$

The product  $pq$  of ultrafilters  $p, q$  can also be defined by the rule: given a subset  $A \subseteq G$ ,

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

Let  $P \in p$  and  $Q_x, x \in P$ . Then  $\bigcup\{xQ_x : x \in P\}$  is a neighbourhood of  $pq$  in  $\beta G$ , and the family of all subsets of this form is a base of the neighbourhoods of  $pq$ . It is easy to verify that the binary operation  $(p, q) \mapsto pq$  is associative, so  $\beta G$  is a semigroup, and  $G^*$  is a subsemigroup of  $\beta G$ . It follows from the second step of the extension that, for every  $q \in \beta G$ , the mapping  $p \mapsto pq$  is continuous, so the semigroup  $\beta G$  is right topological. For the structure of compact right topological semigroup  $\beta G$  and its combinatorial applications see [22, 41]. For application of  $\beta G$  to topologies on groups see [45] and [I. V. Protasov. Algebra in the Stone-Ćech compactification: applications to topologies on groups, Algebra Discrete Math, No 1, 2009, 83–110].

An ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called *left invariant* if  $gA \in \mathcal{I}$  for all  $g \in G, A \in \mathcal{I}$ .

We note that  $\hat{\cdot}$  is a bijection between the set of all left invariant ideals in  $\mathcal{P}_G$  and the set of closed left ideals in  $\beta G$ .

Let  $G$  be an infinite group,  $\mathcal{S}_G$  be the ideal of all small subset of  $G$ ,  $\mu_G$  the minimal closed (two-side) ideal of  $\beta G$ , so

$$\mu_G = \bigcap \{\mu : \mu \text{ is a closed ideal of } \beta G\}.$$

**Theorem 3.17.** *For every infinite group  $G$ , we have  $\hat{\mathcal{S}}_G = \mu_G$ .*

**Theorem 3.18.** *Let  $\mathcal{I}$  be a group ideal in  $\mathcal{P}_G$ . Then  $\hat{\mathcal{I}}$  is an ideal in the semigroup  $\beta G$ .*

For an infinite group  $G$ , we consider the ideal  $\mathcal{N}_G$  of all finite union of non-generative subsets of  $G$ , and put

$$\nu_G = \bigcap \{\hat{\mathcal{I}} : \mathcal{I} \text{ is a group ideal in } \mathcal{P}_G\}.$$

By Theorem 3.18,  $\nu_G$  is an ideal. If  $G$  is countable, by Theorem 3.13,  $\hat{\nu}_G$  is not a group ideal.

**Theorem 3.19.** *For every infinite group  $G$ , we have  $\widehat{N}_G = \nu_G$ .*

Let  $G$  be an amenable group,  $\mathcal{Z}_G$  be the ideal of all absolute zero-subsets of  $G$ . Then  $\mathcal{N}_G \subseteq \mathcal{Z}_G \subseteq \mathcal{S}_G$ . If  $G$  is countable then  $\mathcal{S}_G \setminus \mathcal{Z}_G \neq \emptyset$ . It follows that  $\nu_G \neq \mu_G$ .

**Question 3.4.** Is  $\nu_G \neq \mu_G$  for every (countable) group  $G$ ?

For any infinite group  $G$ , the subset  $G^*G^*$  and  $\overline{G^*G^*}$  are ideals of the semigroup  $\beta G$ . We say that an ultrafilter  $p \in G^*$  is *prime* if  $p \notin G^*G^*$ , and *strongly prime* if  $p \notin \overline{G^*G^*}$ .

We do not know a characterization of prime ultrafilter through its members. To give a characterization of strongly prime ultrafilters, we say that a subset  $P \subseteq G$  is *sparse* if, for any infinite subset  $R$  of  $G$ , there exists a finite non-empty subset  $F \subseteq R$  such that  $\bigcap_{x \in F} xP$  is finite.

**Theorem 3.20.** *An ultrafilter  $p \in G^*$  is strongly prime if and only if there exists a sparse subset  $P \in p$ .*

**Question 3.5.** Let  $G$  be an infinite amenable group,  $p \in G^*$  be a prime ultrafilter. Does there exists an absolute zero-subset  $P \in p$ ?

**Comments.** All results of this section from [16].

### 3.5. Ideals generated by thin subsets

Let  $G$  be an infinite group,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ ,  $\mathcal{T}_\kappa$  be a family of all  $\kappa$ -thin subsets of  $G$ . Then the smallest ideal  $\mathcal{T}_\kappa^*$  in the Boolean algebra  $\mathcal{P}_G$  is the family of all finite unions of  $\kappa$ -thin subsets. Thus, to characterize  $\mathcal{T}_\kappa^*$ , we need some test which, for given  $A \subseteq G$  and  $m \in \mathbb{N}$ , detect whether  $A$  can be represented as a union of  $\leq m$  thin subsets. We consider this question in the ballean context, taking in mind that, by Proposition 1.4, for every a regular cardinal  $\kappa$ , a subset  $A \subseteq G$  is  $\kappa$ -thin if and only if  $A$  is pseudodiscrete in the ballean  $\mathcal{B}_\kappa(G)$ .

For a ballean  $\mathcal{B} = (X, P, B)$ , we use the following notation:

$\mathcal{T}(\mathcal{B})$  is the family of all pseudodiscrete subsets of  $X$ ;

$\bigcup_m \mathcal{T}(\mathcal{B})$  is the family of all unions of  $\leq m$  pseudodiscrete subsets of  $X$ ;

$\mathcal{T}^*(\mathcal{B})$  is the ideal generated by  $\mathcal{T}(\mathcal{B})$ .

Clearly,  $\mathcal{T}^*(\mathcal{B}) = \bigcup_{m \in \mathbb{N}} (\bigcup_m \mathcal{T}(\mathcal{B}))$ . We use also the notation  $\mathcal{T}_m(\mathcal{B})$  for a family of subsets of  $X$  putting  $A \in \mathcal{T}_m(\mathcal{B})$  if and only if, for every  $\alpha \in P$ , there exists a bounded subset  $Y_\alpha$  of  $X$  such that  $|B(x, \alpha) \cap A| \leq m$  for every  $x \in A \setminus Y_\alpha$ .

**Theorem 3.21.** *For every ballean  $\mathcal{B}$ , we have  $\bigcup_m \mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}_m(\mathcal{B})$ .*

The following theorem gives a characterization of  $\mathcal{T}^*(\mathcal{B})$  in the case of an ordinal ballean.

**Theorem 3.22.** *For every ordinal ballean  $\mathcal{B}$  and  $m \in \mathbb{N}$ , we have  $\mathcal{T}_m(\mathcal{B}) = \bigcup_m \mathcal{T}(\mathcal{B})$ .*

**Theorem 3.23.** *Let  $G$  be an infinite group of regular cardinality  $\kappa$ ,  $m \in \mathbb{N}$ . A subset  $A \subseteq G$  can be partitioned in  $\leq m$   $\kappa$ -thin subsets if and only if, for every subset  $F \in [G]^{<\kappa}$ , there exists a subset  $Y \in [G]^{<\kappa}$  such that  $|A \cap Fx| \leq m$  for every  $x \in A \setminus Y$ .*

**Comments.** This section is a summary of [26].

## 4. Partitions

### 4.1. Filtrations

Let  $G$  be an infinite group with the identity  $e$ . A *filtration* of  $G$  is a family  $\{G_\alpha : \alpha < |G|\}$  of subgroups of  $G$  such that

- (i)  $G_0 = \{e\}$  and  $G = \bigcup\{G_\alpha : \alpha < |G|\}$ ;
- (ii)  $G_\alpha \subset G_\beta$  for all  $\alpha < \beta < |G|$ ;
- (iii)  $\bigcup\{G_\alpha : \alpha < \beta\} = G_\beta$  for every limit ordinal  $\beta$ ;
- (iv)  $|G_\alpha| < |G|$  for every  $\alpha < |G|$ .

We note that every uncountable group admits a filtration  $\{G_\alpha : \alpha < |G|\}$  such that  $|G_\alpha| = |\alpha|$  for every infinite ordinal  $\alpha < |G|$ , where  $|\alpha| = |\{\beta : \beta < \alpha\}|$ .

For each  $\alpha < |G|$ , we decompose  $G_{\alpha+1} \setminus G_\alpha$  in right cosets by  $G_\alpha$  and fix some system  $X_\alpha$  of representatives so  $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$ . Take an arbitrary element  $g \in G \setminus \{e\}$  and choose the smallest subgroup  $G_\alpha$  with  $g \in G_\alpha$ . By (iii),  $\alpha = \alpha_1 + 1$  for some ordinal  $\alpha_1 < |G|$ . Hence,  $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$  and there exist  $g_1 \in G_{\alpha_1}$ ,  $x_{\alpha_1} \in X_{\alpha_1}$  such that  $g = g_1 x_{\alpha_1}$ . If  $g_1 \neq e$ , we choose the ordinal  $\alpha_2$  and the elements  $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$  and  $x_{\alpha_2} \in X_{\alpha_2}$  such that  $g_1 = g_2 x_{\alpha_2}$ . Since the set of ordinals  $< |G|$  is well-ordered, after finite number  $s(g)$  of steps, we get the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} \cdots x_{\alpha_2} x_{\alpha_1}, \quad x_{\alpha_i} \in X_{\alpha_i}.$$

We note that this representation is unique and put

$$\gamma_1(g) = \alpha_1, \quad \gamma_2(g) = \alpha_2, \dots, \gamma_{s(g)}(g) = \alpha_{s(g)},$$

$$\Gamma(g) = \{\gamma_1(g), \dots, \gamma_{s(g)}(g)\}$$

and, for every natural number  $n$ , put

$$D_n = \{g \in G : s(g) = n\},$$

so  $G = \bigcup_{n \in \omega} D_n$ , where  $D_0 = \{e\}$ .

**Theorem 4.1.** *Every infinite group  $G$  can be partitioned in  $\aleph_0$  subsets which are  $\kappa$ -small for every cardinal  $\kappa$  such that  $\omega \leq \kappa \leq cf|G|$ .*

*Proof.* If  $G$  is countable, the statement is trivial because every singleton is  $\kappa$ -small. We suppose that  $G$  is uncountable, use above filtration  $\{G_\alpha : \alpha < |G|\}$  of  $G$  and note  $G \setminus \{e\} = \bigcup_{n=1}^\infty D_n$ . We fix a natural number  $n$  and show that  $D_n$  is  $\kappa$ -small. Take an arbitrary  $F \in [G]^{<\kappa}$ . Since  $\kappa \leq cf|G|$ , there exists  $\beta < |G|$  such that  $F \subseteq G_\beta$  so  $FD_n \subseteq G_\beta D_n$ . Now it suffices to prove that  $G \setminus G_\beta D_n$  is  $\omega$ -large. We choose the elements  $a_1, a_2, \dots, a_{n+1}$  in  $G$  such that

$$a_1 \in G_{\beta+1} \setminus G_\beta, a_2 \in G_{\beta+2} \setminus G_{\beta+1}, \dots, a_{n+1} \in G_{\beta+n+1} \setminus G_{\beta+n}.$$

We take an arbitrary element  $g \in G_\beta D_n$  and put  $g = g_0$ . If  $\beta + n \in \Gamma(g)$ , we put  $\epsilon_0 = 0$ , otherwise  $\epsilon_0 = 1$ . Note that  $\beta + n \in \Gamma(a_{n+1}^{\epsilon_0} g_0)$  and put  $g_1 = a_{n+1}^{\epsilon_0} g_0$ . If  $\beta + n - 1 \in \Gamma(g_1)$ , we put  $\epsilon_1 = 0$ , otherwise  $\epsilon_1 = 1$ . Note that  $\{\beta + n - 1, \beta + n\} \in \Gamma(a_n^{\epsilon_1} g_1)$  and put  $g_2 = a_n^{\epsilon_1} g_1$ . After  $n + 1$  steps we get

$$\{\beta, \beta + 1, \dots, \beta + n\} \subseteq \Gamma(a_1^{\epsilon_n} a_2^{\epsilon_{n-1}} \dots a_{n+1}^{\epsilon_0} g).$$

It follows that  $a_1^{\epsilon_n} a_2^{\epsilon_{n-1}} \dots a_{n+1}^{\epsilon_0} g \notin G_\beta D_n$  and put  $A = \{e, a_1, a_2, \dots, a_{n+1}\}$ ,  $K = A^n$ . We have shown that  $G_\beta D_n \subseteq K^{-1}(G \setminus G_\beta D_n)$ . Hence,  $G \subseteq K^{-1}(G \setminus G_\beta D_n)$ . This shows that  $G \setminus G_\beta D_n$  is  $\omega$ -large.  $\square$

**Theorem 4.2.** *Let  $G$  be an infinite group,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ . Then  $G$  can be partitioned in  $\kappa$   $\kappa$ -large subsets.*

*Proof.* For  $\kappa = \aleph_0$ , this is Theorem 3.12 from [43]. For  $\kappa > \aleph_0$ , we choose a filtration  $\{G_\alpha : \alpha < |G|\}$  such that  $|G_\alpha| = |\alpha|$  for every infinite ordinal  $\alpha < |G|$ . For each  $\alpha < \kappa$ , we put

$$L_\alpha = \{g \in G : \gamma_{s(g)} = \alpha\}.$$

Then  $G_{\alpha+1} L_\alpha = G$  so  $L_\alpha$  is  $\kappa$ -large. Since the subsets  $\{L_\alpha < \kappa\}$  are pairwise disjoint, to get a desired partition of  $G$ , we join  $G \setminus \bigcup_{\alpha < \kappa} L_\alpha$  to some subset  $L_\alpha$ .  $\square$

**Theorem 4.3.** *Let  $G$  be an infinite group,  $\kappa$  be an infinite cardinal such that either  $\kappa < |G|$  or  $\kappa = |G|$  and  $\kappa$  is regular. Then  $G$  can be partitioned in  $|G|$   $\kappa$ -thick subsets.*

*Proof.* This is a partial case of Theorem 1.4. Here we use a filtration  $\{G_\alpha : \alpha < |G|\}$  to point out directly a desired partition in the case  $\kappa < cf|G|$ ,  $|G| > \aleph_0$ . We partition the cardinal  $|G|$  in  $|G|$  subsets of cardinality  $|G|$ :  $G = \bigcup_{\lambda < |G|} I_\lambda$ . We put  $T_I = \bigcup_{\lambda \in I} (G_{\lambda+1} \setminus G_\lambda)$  and show that  $T_I$  is  $\kappa$ -thick. Let  $X$  be a subset of  $G$  such that  $|X| < \kappa$ . Since  $\kappa \leq cf|G|$  and  $I$  is cofinal in  $|G|$ , there exists  $\lambda \in I$  such that  $X \subset G_\lambda$ . We take an arbitrary  $g \in G_{\lambda+1} \setminus G_\lambda$ . Then  $G_\lambda g \in T_I$  so  $Xg \subseteq T_I$ .  $\square$

**Question 4.1.** Can every infinite group  $G$  be partitioned in  $\aleph_0$  subsets which are  $\kappa$ -small for every infinite cardinal  $\kappa \leq |G|$ ? By the Theorem 4.1, this is so if  $|G|$  is regular.

**Question 4.2.** Can every infinite group  $G$  be partitioned in  $|G|$   $|G|$ -thick subsets? By the Theorem 4.3 this is so if  $|G|$  is regular.

In the following question, we formalize a problem of partitions of group into thin subsets.

**$(\kappa, \mu)$ -Question.** *Let  $G$  be an infinite group,  $\kappa, \mu$  be infinite cardinals such that  $\kappa \leq |G|$ ,  $\mu < |G|$ . How to detect whether  $G$  can be partitioned in  $\leq \mu$   $\kappa$ -thin subsets?*

The following two theorems essentially clarify this question.

**Theorem 4.4.** *Let  $G$  be an infinite group,  $\kappa, \mu$  be infinite cardinals such that  $\kappa \leq |G|$  and  $\mu^+ < |G|$ . If either  $\kappa < |G|$  or  $\kappa = |G|$  and  $\kappa$  is regular then  $G$  cannot be partitioned in  $\leq \mu$   $\kappa$ -thin subsets.*

**Theorem 4.5.** *Let  $\mu$  be an infinite cardinal,  $G$  be a subgroup of a direct product  $\bigotimes_{\alpha < \mu^+} G_\alpha$  of groups  $\{G_\alpha : \alpha < \mu^+\}$  of cardinality  $\leq \mu$ . Then  $G$  can be partitioned in  $\leq \mu$  thin subsets.*

The proofs of Theorem 4.4 and 4.5 are based on the following combinatorial lemmas.

**Lemma 4.1.** *Let  $\mu$  be an infinite cardinal,  $X$  be a set such that  $cf|X| > \mu^+$ ,  $\chi : [X]^2 \rightarrow \mu$  be an arbitrary mapping. Then there exist  $\lambda \in \mu$ , distinct elements  $x, y \in X$  and a subset  $Z$  of  $X$  such that  $x \notin Z$ ,  $y \notin Z$ ,  $|Z| = |X|$  and  $\chi(\{x, z\}) = \chi(\{y, z\}) = \lambda$  for every  $z \in Z$ .*

**Lemma 4.2.** *For any infinite cardinal  $\mu$  and  $m \in \mathbb{N}$ , there exists a mapping  $\chi : [\mu^+]^m \rightarrow \mu$  with the following property. For any  $\lambda \in \mu$ ,  $l \in \mathbb{N}$ ,  $l < m$  and any disjoint subsets  $A, B \in [\mu^+]^l$ , we have*

$$|\{C \in [\mu^+]^{m-l} : (A \cup B) \cap C = \emptyset, \chi(A \cup C) = \chi(B \cup C) = \lambda\}| \leq 1.$$

It is well known that every Abelian group can be embedded in a direct product of countable groups. Thus, Theorem 4.4 and Theorem 4.5 answer the  $(\kappa, \mu)$ -Question for every Abelian group  $G$  and all cardinals  $\kappa, \mu$  with exception  $\kappa = |G|$ ,  $\kappa$  is singular and  $\mu < cf\kappa$ . If  $\mu^+ < |G|$  and either  $\kappa < |G|$  or  $\kappa = |G|$  and  $\kappa$  is regular, we apply Theorem 4.4. If  $\mu^+ \geq |G|$ , we use Theorem 4.5. If  $\kappa = |G|$ ,  $\kappa$  is singular and  $\mu \geq cf\kappa$ , we partition  $G$  in  $cf\kappa$  subsets of cardinality  $< \kappa$  and note that each subset of this partition is  $|G|$ -thin. The exceptional case remains open.

**Question 4.3.** Let  $G$  be a group of singular cardinality  $\kappa$ ,  $\mu$  be an infinite cardinal such that  $\mu < cf\kappa$ . Can  $G$  be partitioned in  $\mu$   $\kappa$ -thin subsets?

For a natural number  $k$ , it is more convenient to call a subset  $A$  of a group  $G$  to be  $k$ -thin if  $|gA \cap A| \leq k$  for every  $g \in G$ ,  $g \neq e$ .

**Question 4.4.** Can  $\mathbb{R}$  be partitioned in  $\aleph_0$  subsets such that each subset of the partition is  $k$ -thin for some  $k \in \mathbb{N}$ ? Does there exist  $k \in \mathbb{N}$  such that  $\mathbb{R}$  can be partitioned in  $\aleph_0$   $k$ -thin subset? Can  $\mathbb{R}$  be partitioned in  $\aleph_0$  subsets linearly independent over  $\mathbb{Q}$  (each such subset is 1-thin)?

By Theorem 4.4, under  $\neg CH$  the answers to all subquestions of Question 4.4 are negative. In contrast to Theorem 4.5, stating that under CH  $\mathbb{R}$  can be partitioned in  $\aleph_0$  thin subsets, we expect the negative answer to Question 4.4 with no additional to ZFC set-theoretical assumptions.

We conclude this section with two propositions showing that above results (concerning partitions of groups) fail to be true for partitions of  $G$ -spaces.

A permutation  $f$  of a set  $X$  is called finitary if there exists a finite subset  $F$  of  $X$  such that  $f(x) = x$  for all  $x \in X \setminus F$ . A space  $X$  endowed with the action of the group  $G$  of all finitary permutations of  $X$  is called a *finitary  $G$ -space*.

A set  $X$  endowed with the action of the group of all permutations of  $X$  is called a *universal  $G$ -space*.

**Proposition 4.1.** *Let  $X$  be an infinite finitary  $G$ -space,  $A \subseteq X$ . Then the following statements hold*

- (i)  *$A$  is large if and only if  $G \setminus A$  is finite;*

- (ii)  $A$  is small if and only if  $A$  is finite;
- (iii)  $A$  is thick if and only if  $A$  is infinite;
- (iv)  $A$  is thin if and only if  $A$  is finite.

**Proposition 4.2.** *Let  $X$  be an infinite universal  $G$ -space,  $A \subseteq X$ . Then the following statements hold*

- (i)  $A$  is large if and only if  $|A| = |X|$ ;
- (ii)  $A$  is small if and only if  $|A| < |X|$ ;
- (iii)  $A$  is thick if and only if  $|G \setminus A| < |X|$ ;
- (iv)  $A$  is thin if and only if  $A$  is finite.

**Comments.** For usage of filtration in *Homological Algebra and Theory of Abelian group* see [15]. Some applications of filtrations to *Topological Algebra* can be find in [45] (one of them: every non-meager left topological group can be partitioned in  $\aleph_0$  dense subsets). Theorem 4.1 from [39], Theorem 4.2 from [35], Theorems 4.4 and 4.5 are new. For applications of Theorem 4.3 to topologies on group see [36], some generalization of Theorem 4.3 to semigroups are in [10].

## 4.2. Partitions of groups in two subsets

Given any finite partition of an infinite group, what can we say about the cells of the partition? This is the main question in the Ramsey Group Theory (see [22, Chapters 5,14,16,18]). Most results of this theory state that at least one cell of partition has rich algebraic structure or is “large” by the combinatorial size, so following Taras Banakh, we may say that a full chaos is impossible.

By [43, Theorem 12.7], for every finite partition of a group  $G = A_1 \cup \dots \cup A_n$ , there exist  $i \in \{1, \dots, n\}$  and a finite subset  $K$  of  $G$  such that  $G = KA_iA_i^{-1}$ . It is still unknown whether we can choose  $K$  such that  $|K| \leq n$  [43, Question 12.1]. This is so if either  $G$  is amenable [43, Theorem 12.8] or  $n = 2$ .

**Theorem 4.6.** *For a group  $G$ , the following statements are equivalent*

- (i)  $G$  is a torsion group with no elements of order 2;
- (ii) for any partition  $G = A \cup B$ , either  $G = AA^{-1}$  or  $G = BB^{-1}$ .

**Theorem 4.7.** *Let  $\mathbb{Z} = Z_1 \cup Z_2$  be a partition such that  $\mathbb{Z} \neq Z_i - Z_i$ ,  $i \in \{1, 2\}$ . Then the following statements hold*

- (i) there exists  $k \in \mathbb{N}$  such that  $Z_i - Z_i + k\mathbb{Z} = Z_i - Z_i$ ,  $i \in \{1, 2\}$ ;
- (ii) either  $2\mathbb{Z} = Z_i - Z_i$ ,  $i \in \{1, 2\}$  or there exists  $a > 1$  such that  $\mathbb{Z} \setminus a\mathbb{Z} \subseteq Z_i - Z_i$  and  $\{-a + 1, \dots, a - 1\} \subseteq Z_i - Z_i$ ,  $i \in \{1, 2\}$ .

For every  $k \in \mathbb{N}$ ,  $\mathbb{Z}$  can be partitioned  $\mathbb{Z} = Z_1 \cup Z_2$  such that  $(k + 2k\mathbb{Z}) \cap (Z_i - Z_i) = \emptyset$ ,  $i \in \{1, 2\}$ .

It would be interesting to extend Theorem 4.7 to the groups  $\mathbb{Z}^n$ ,  $n \in \mathbb{N}$ .

By van der Waerden theorem (see [19]), for every finite partition of  $\mathbb{Z}$ , one cell of the partition contains an arbitrarily long finite arithmetic progression.

**Theorem 4.8.** *For every 2-coloring of  $\mathbb{N}$ , there are infinitely many monochrome 3-term arithmetic progressions with common first term.*

**Theorem 4.9.** *There exists a 3-coloring of  $\mathbb{N}$  such that every set of monochrome 3-term arithmetic progressions with common first term is finite.*

**Theorem 4.10.** *There exists a 2-coloring of  $\mathbb{N}$  such that every set of monochrome 4-term arithmetic progressions with common first term is finite.*

For a subset  $A$  of  $\mathbb{N}$ , we put

$$PS(A) = \{a + b : a, b \in A, a \neq b\},$$

$$FS(A) = \{a_1 + \dots + a_n : n \in \mathbb{N}, a_1, \dots, a_n \in A, a_i \neq a_j \text{ for all } i < j \leq n\},$$

and note that  $A + A = PS(A) + 2A$ .

Let  $\mathbb{N} = A_1 \cup \dots \cup A_n$  be an arbitrary partition. We define a coloring  $\chi : [\mathbb{N}^2] \rightarrow \{1, \dots, n\}$  by the rule:  $\chi(\{a, b\}) = i$  if and only if  $a + b \in A_i$ . By the Ramsey theorem (see [19, Chapter 2]), there exist an infinite subset  $A$  of  $\mathbb{N}$  and  $i \in \{1, \dots, n\}$  such that  $\chi(\{a, b\}) = i$  for all distinct  $a, b \in A$ . It means that  $PS(A) \subseteq A_i$ . Much more delicate statements follow from the Hindman theorem (see [22, Chapter 5]): there exist  $i \in \{1, \dots, n\}$  and an infinite subset  $A \subseteq \mathbb{N}$  such that  $FS(A) \subseteq A_i$ .

One of the most intriguing open question in Combinatorics of Numbers, the Owings problem (see Amer. Math. Monthly, **81** (1974), 902) is: *Given any partition  $\mathbb{N} = A_1 + A_2$ , do there exist  $i \in \{1, 2\}$  and infinite subset  $A$  of  $\mathbb{N}$  such that  $A + A \not\subseteq A_i$ ?* Equivalently, does there exist an increasing sequence of even naturals  $(a_i)_{i \in \omega}$  such that the set  $\{\frac{a_i + a_j}{2} : i, j \in \omega\}$  is monochrome?



**Theorem 4.11.** *For every 2-coloring of  $\mathbb{N}$ , there exists an increasing sequence of even numbers  $a_1, a_2, \dots$  such that the sequence*

$$a_1, \frac{a_1 + a_2}{2}, a_2, \frac{a_2 + a_3}{2}, a_3, \dots, a_n, \frac{a_n + a_{n+1}}{2}, a_{n+1}, \dots$$

*is monochrome.*

**Comments.** Theorem 4.6 from [18], Theorem 4.7 is new, all other theorems from [53].

## 5. Packings

### 5.1. Packing numbers

For a subset  $A$  of a group  $G$ , we put

$$\text{pack}(A) = \sup\{|S| : S \subseteq G \text{ is such that} \\ \text{the family } \{gA : g \in S\} \text{ is disjoint}\}.$$

Answering a question from [13], T. Banach and N. Lyaskovska constructed, in any infinite group  $G$ , a subset  $A$  such that  $\text{pack}(A) = \aleph_0$  but  $A$  is not  $P$ -small, so the supremum in the definition of  $\text{pack}(A)$  is not accessible. In this connection, they introduced the following packing number

$$\text{Pack}(A) = \min\{\kappa : \forall S \subseteq G, |S| = \kappa, \\ \text{the family } \{gA : g \in S\} \text{ is not disjoint}\}.$$

If the supremum in the definition of  $\text{pack}(A)$  is accessible, then  $\text{Pack}(A) = (\text{pack}(A))^+$ , otherwise  $\text{Pack}(A) = \text{pack}(A)$ .

**Theorem 5.1.** *An infinite Abelian group  $G$  contains a subset  $A \subseteq G$  with  $\text{Pack}(A) = \kappa$  if and only if one of the following conditions holds*

- (i)  $2 \leq \kappa \leq |G|^+$  and  $\kappa \notin \{3, 4\}$ ;
- (ii)  $\kappa = 3$  and  $G$  is not isomorphic to  $\bigoplus_{i \in I} \mathbb{Z}_3$ ;
- (iii)  $\kappa = 4$  and  $G$  is not isomorphic to  $\bigoplus_{i \in I} \mathbb{Z}_2$  or to  $\mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2)$ .

**Theorem 5.2.** *An infinite Abelian group  $G$  contains a subset  $A$  with  $\text{pack}(A) = \kappa$  if and only if one of the following conditions holds*

- (i)  $1 \leq \kappa \leq |G|$  and  $\kappa \notin \{2, 3\}$ ;

- (ii)  $\kappa = 2$  and  $G$  is not isomorphic to  $\bigoplus_{i \in I} \mathbb{Z}_3$ ;
- (iii)  $\kappa = 3$  and  $G$  is not isomorphic to  $\bigoplus_{i \in I} \mathbb{Z}_2$  or to  $\mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2)$ .

Given a translation invariant ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  and a subset  $A \subseteq G$ , we define the relative packing numbers

$$\mathcal{I}\text{-pack}(A) = \sup\{|B| : B \subseteq G \text{ is such that } \{bA : b \in B\} \text{ is disjoint modulo } \mathcal{I}\},$$

$$\mathcal{I}\text{-Pack}(A) = \sup\{|B|^+ : B \subseteq G \text{ is such that } \{bA : b \in B\} \text{ is not disjoint modulo } \mathcal{I}\}.$$

A family  $\mathcal{F}$  of subsets of  $G$  is called *disjoint modulo  $\mathcal{I}$*  if  $F \cap H \in \mathcal{I}$  for all distinct  $H, F \in \mathcal{F}$ . It is clear that  $\{\emptyset\}\text{-pack}(A) = \text{pack}(A)$ ,  $\{\emptyset\}\text{-Pack}(A) = \text{Pack}(A)$ ,  $\mathcal{I}\text{-pack}(A) \leq \mathcal{I}\text{-Pack}(A)$  and

$$\mathcal{I}\text{-pack}(A) = \sup\{\kappa : \kappa < \mathcal{I}\text{-Pack}A\},$$

so the value of  $\mathcal{I}\text{-pack}(A)$  can be recovered from that of  $\mathcal{I}\text{-Pack}(A)$ .

**Theorem 5.3.** *Let  $G$  be an infinite group of cardinality  $\kappa$ ,  $A$  be a subset of  $G$  such that  $[G]^{<\omega}\text{-Pack}(A) = \kappa^+$ . Then  $A$  can be partitioned  $A = A_1 \cup A_2$  such that  $\text{Pack}(A_i) \leq (\text{cf}\kappa)^+$ ,  $i \in \{1, 2\}$ .*

**Theorem 5.4.** *Let  $G$  be an infinite Abelian group,  $\kappa$  be a cardinal. Then the following statements are equivalent*

- (i) *there exists a subset  $A \subseteq G$  such that  $[G]^{<\omega}\text{-Pack}(A) = \kappa$ ;*
- (ii) *there exists a subset  $A \subseteq G$  such that  $\text{Pack}(A) = \kappa$ .*

In fact, the packing numbers  $\mathcal{I}\text{-pack}(A)$  and  $\mathcal{I}\text{-Pack}(A)$  are the partial cases of the packing numbers  $\mathcal{I}\text{-pack}_n(A)$  and  $\mathcal{I}\text{-Pack}_n(A)$  defined for every natural number  $n \geq 2$  by the formulas:

$$\mathcal{I}\text{-pack}_n(A) = \sup\left\{|B| : B \subseteq G \text{ is such that } \left\{\bigcap_{c \in C} cA : C \in [B]^n\right\} \in \mathcal{I}\right\},$$

$$\mathcal{I}\text{-Pack}_n(A) = \sup\left\{|B|^+ : B \subseteq G \text{ is such that } \left\{\bigcap_{c \in C} cA : C \in [B]^n\right\} \in \mathcal{I}\right\}.$$

It is clear that  $\mathcal{I}\text{-pack}(A) = \mathcal{I}\text{-pack}_2(A)$  and  $\mathcal{I}\text{-Pack}(A) = \mathcal{I}\text{-Pack}_2(A)$ . Also,  $\mathcal{I}\text{-pack}_n(A) \leq \mathcal{I}\text{-pack}_{n+1}(A)$  and  $\mathcal{I}\text{-Pack}_n(A) \leq \mathcal{I}\text{-Pack}_{n+1}(A)$  for every  $n \geq 2$ .

If  $\mathcal{I} = \{\emptyset\}$ , then we shall write  $\text{pack}_n(A)$  and  $\text{Pack}_n(A)$  instead of  $\{\emptyset\}\text{-pack}_n(A)$  and  $\{\emptyset\}\text{-Pack}_n(A)$ .

**5.2. Packing-complete ideals**

It is clear that, for each subset  $A \in \mathcal{I}$  and  $n \geq 2$ , we get  $\mathcal{I}\text{-pack}_n(A) = |G|$  and  $\mathcal{I}\text{-Pack}_n(A) = |G|^+$ . We shall be interested in ideals for which the (partial) converse implications hold.

A translation invariant ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called  $\text{Pack}_n\text{-complete}$  (resp.  $\text{pack}_n\text{-complete}$ ) if  $\mathcal{I}$  contains each subset  $A \subseteq G$  with  $\mathcal{I}\text{-Pack}_n(A) \geq \aleph_0$  (resp.  $\mathcal{I}\text{-pack}_n(A) \geq \aleph_0$ ). Clearly, each  $\text{Pack}_{n+1}\text{-complete}$  ideal is  $\text{Pack}_n(A)\text{-complete}$ .

A translation invariant ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called  $\text{Pack}_{<\omega}\text{-complete}$  if  $\mathcal{I}$  is  $\text{Pack}_n\text{-complete}$  for each  $n \geq 2$ .

**Theorem 5.5.** *For every amenable group  $G$ , the following ideals are  $\text{Pack}_{<\omega}\text{-complete}$ :*

- (i)  $\mathcal{N}_\mu = \{A \subseteq G : \mu(A) = 0\}$ ,  $\mu$  is a Banach measure on  $\mathcal{P}_G$ ;
- (ii)  $\mathcal{N} = \bigcap \{\mathcal{N}_\mu : \mu \text{ is a Banach measure on } \mathcal{P}_G\}$ ;
- (iii) the ideal  $\mathcal{S}_G$  of all small subsets of  $G$ .

For a translation invariant ideal  $\mathcal{I}$  of  $\mathcal{P}_G$  and  $n \geq 2$ , we put

$$\text{Pack}_n(\mathcal{I}) = \bigcap \{\mathcal{J} : \mathcal{I} \subset \mathcal{J}, \mathcal{J} \text{ is } \text{Pack}_n\text{-complete}\},$$

$$\text{Pack}_{<\omega}(\mathcal{I}) = \bigcap \{\mathcal{J} : \mathcal{I} \subset \mathcal{J}, \mathcal{J} \text{ is } \text{Pack}_{<\omega}\text{-complete}\},$$

and say that  $\text{Pack}_n(\mathcal{I})$  and  $\text{Pack}_{<\omega}(\mathcal{I})$  are the  $\text{Pack}_n\text{-completion}$  and  $\text{Pack}_{<\omega}\text{-completion}$  of  $\mathcal{I}$ .

**Proposition 5.1.** *The  $\text{Pack}_n\text{-completion}$  of  $\mathcal{I}$  is equal to the union*

$$\mathcal{I}_{<\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{I}_\alpha,$$

where  $\mathcal{I}_0 = \mathcal{I}$  and  $\mathcal{I}_\alpha$  is the smallest ideal in  $\mathcal{P}_G$  containing all subsets  $A \subseteq G$  with infinite  $\mathcal{I}_\beta\text{-Pack}_n(A)$  for some  $\beta < \alpha$ .

**Proposition 5.2.** *The  $\text{Pack}_{<\omega}$ -completion of  $\mathcal{I}$  is equal to the union*

$$\mathcal{I}_{<\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{I}_\alpha,$$

where  $\mathcal{I}_0 = \mathcal{I}$  and  $\mathcal{I}_\alpha$  is the smallest ideal in  $\mathcal{P}_G$  containing all subsets  $A \subseteq G$  with infinite  $\mathcal{I}_\beta$ - $\text{Pack}_n(A)$  for some  $\beta < \alpha$  and  $n < \omega$ .

**Question 5.1.** Are the ideals  $\text{Pack}_n(\{\emptyset\})$ ,  $n \geq 2$  in  $\mathcal{P}_{\mathbb{Z}}$  pairwise distinct? Do they differ from  $\text{Pack}_{<\omega}(\{\emptyset\})$ ?

**Question 5.2.** Characterize the smallest ideal in  $\mathcal{P}_{\mathbb{Z}}$  containing all  $P$ -small subsets.

### 5.3. Packing in topological group

For a topological group  $G$ , it is naturally to ask about the possible values of the packing numbers of subsets with good topological or descriptive properties. The authors of [3] addressed this general question to K. Omiljanowski.

We recall that a *Polish group* is a metrizable, separable and complete topological group.

**Theorem 5.6.** *If  $A$  is a  $\sigma$ -compact subset of a Polish group  $G$ , then  $\text{pack}(A) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ . Moreover, if  $A$  is compact, then*

(i)  $\text{pack}(A) = 2^{\aleph_0}$  if  $G$  is not locally compact;

(ii)  $\text{pack}(A) \in \{\aleph_0, 2^{\aleph_0}\}$  if  $G$  is locally compact but not compact;

(iii)  $\text{pack}(A) \in \mathbb{N} \cup \{2^{\aleph_0}\}$  if  $G$  is compact.

**Question 5.3.** Is there a compact group  $G$  and a  $\sigma$ -compact subset  $A$  of  $G$  with  $\text{pack}(A) = \aleph_0$ ?

**Question 5.4.** Is there a Polish group  $G$  and a Borel subset  $A \subset G$  with  $\aleph_0 < \text{pack}(A) < 2^{\aleph_0}$ ?

For the possible values of the packing numbers of the Borel and analytic subsets of a Polish group see [3, Section 3].

**Theorem 5.7.** *Every non-discrete Polish Abelian group  $G$  contains two closed subsets  $A, B$  such that  $\text{pack}(A) = \text{pack}(B) = 2^{\aleph_0}$  and  $|g(A \cup B) \cap (A \cup B)| = |A \cup B|$  for every  $g \in G$ .*

We recall that a subset  $A$  of a topological group  $G$  is *Haar null* if there is a Borel probability measure  $\mu$  on  $G$  such that  $\mu(xAy) = 0$  for all  $x, y \in G$ .

**Theorem 5.8.** *Every non-discrete Polish Abelian group  $G$  contains a closed nowhere dense Haar null subset  $C$  such that  $G = CC^{-1}$  and  $|gC \cap C| = |C|$  for every  $g \in G$ .*

**Comments.** This section with open questions is a poppuree from the papers [1–3,27]. Theorem 5.3 is a generalization of Theorem 3.5 from [24].

## 6. Around thin subsets

Let  $G$  be a group with the identity  $e$ ,  $X$  be a  $G$ -space,  $k \in \mathbb{N}$ . A subset  $A$  of  $X$  is said to be

- *k-thin* if  $|gA \cap A| \leq k$  for each  $g \in G$ ,  $g \neq e$ ;
- *almost thin* if the set  $\Delta(A)$  is finite, where  $\Delta(A) = \{g \in A : gA \cap A\}$  is infinite;
- *sparse* if, for every infinite subset  $S$  of  $G$ , there exists a non-empty finite subset  $F \subset S$  such that  $\bigcap_{g \in F} gA$  is finite;
- *k-sparse* if, for every infinite subset  $S$  of  $G$ , there exists a finite subset  $F \subset S$  such that  $|F| \leq k$  and  $\bigcap_{g \in F} gA$  is finite.

### 6.1. Sparse subsets

For motivation to study the sparse subsets see Theorem 3.21. We say that a subset  $A$  of  $G$  is *finitely sparse* if  $A$  is  $k$ -sparse for some  $k \in \mathbb{N}$ . The following theorem shows that the family of all sparse subsets of  $X$  and the family of all finitely sparse subsets form the ideals in the Boolean algebra  $\mathcal{P}_X$ . The key part in the proof of Theorem 6.1 (ii) plays the following observation: a subset  $A$  of a group  $G$  is  $k$ -sparse if and only if, for every infinite subset  $S$  of  $G$ , there exists an infinite subset  $S' \subseteq S$  such that, for each  $F \in [S']^k$ , the subset  $\bigcap_{g \in F} gA$  is finite.

**Theorem 6.1.** *Let  $X$  be a  $G$ -space,  $k, m \in \mathbb{N}$ ,  $A$  and  $B$  be subsets of  $X$ . Then*

- (i) *if  $A, B$  are sparse then  $A \cup B$  is sparse;*
- (ii) *if  $A$  is  $k$ -sparse and  $B$  is  $m$ -sparse then  $A \cup B$  is  $(k+m-1)$ -sparse.*

The following two theorems clarify the relationships between  $k$ -sparse and sparse subsets of groups.

**Theorem 6.2.** *For every infinite group  $G$  and every  $k \in \mathbb{N}$ , there exists  $(k+1)$ -sparse but not  $k$ -sparse subset of  $G$ .*

**Theorem 6.3.** *For every infinite group  $G$ , there exists a sparse subset which is not  $k$ -sparse for every  $k \in \mathbb{N}$ .*

The relationships between sparse, small and  $P$ -small subsets are reflected in the following theorems.

**Theorem 6.4.** *Every sparse subset  $A$  of an infinite group  $G$  is small. If  $G$  is amenable then  $A$  is an absolute zero-subset.*

**Theorem 6.5.** *For every infinite group  $G$ , there exists a sparse subset which is not  $P$ -small.*

**Theorem 6.6.** *For every infinite group  $G$ , there exists a small and  $P$ -small subset  $A$  which is not sparse.*

In construction of the subset  $A$  from Theorem 6.5, we used the following claim: if  $A$  is a subset of  $G$  such that  $AA^{-1}$  is not large then  $A$  is small and  $P$ -small.

## 6.2. Thin, almost thin and 2-sparse subsets

**Theorem 6.7.** *For every infinite group  $G$ , there exists a thin subset which is not  $k$ -thin for each  $k \in \mathbb{N}$ .*

**Theorem 6.8.** *For every infinite finitely generated Abelian group  $G$ , there exists a thin subset  $A$  such that whenever  $A$  is finitely partitioned  $A = A_1 \cup \dots \cup A_n$ , there exist  $j \in \{1, \dots, m\}$  and a finite subset  $K$  of  $G$  such that*

$$G = K + (A_j - A_j + A_j - A_j).$$

*In particular,  $A_j$  generates a subgroup of finite index.*

More on thin generating subsets of a group can be found in the subsection 6.3.

**Theorem 6.9.** *Every almost thin subset  $A$  of a group  $G$  can be partitioned in  $3^{|\Delta(A)|-1}$  thin subsets. If  $G$  has no elements of odd order then  $A$  can be partitioned in  $2^{|\Delta(A)|-1}$  thin subsets.*

**Theorem 6.10.** *A subset  $A$  of a group  $G$  is 2-sparse if and only if  $X^{-1}X \not\subseteq \Delta(A)$  for every infinite subset  $X$  of  $G$ . In particular, every almost thin subset of  $G$  is 2-sparse.*

**Theorem 6.11.** *For every countable thin subset  $A$  of a group  $G$ , there exists a subset  $B$  such that  $A \cup B$  is 2-sparse but not almost thin.*

**Theorem 6.12.** *For every infinite thin subset  $A$  of a group  $G$ , there exists a 2-thin subset  $B$  such that  $A \cup B$  is not 2-sparse.*

**Theorem 6.13.** *Suppose that a group  $G$  is either non-torsion, or for every  $n \in \mathbb{N}$ , there exists a finite subgroup  $H_n$  such that  $|H_n| > n$ . Then there exists a 2-sparse subset of  $G$  which cannot be partitioned in finitely many thin subsets.*

**Theorem 6.14.** *Every 2-sparse subset of a group  $G$  can be partitioned in two  $P$ -small subsets.*

### 6.3. Thin generating subsets of groups

By [43, Theorem 13.1], every infinite group  $G$  can be generated by some small subset. This statement can be essentially strengthened if we use the thin subsets instead of small.

In the following theorems "k-thin subset  $A$  of a group  $G$ " means that  $A$  is both left and right  $k$ -thin, i.e.  $|gA \cap A| \leq k$ ,  $|Ag \cap A| \leq k$  for every  $g \in G$ ,  $g \neq e$ .

**Theorem 6.15.** *Every infinite group  $G$  has a 2-thin system of generators. Moreover, if  $G$  has no elements of order 2 and  $G$  is either Abelian or a torsion group, then there exists a 1-thin system of generators of  $G$ .*

**Theorem 6.16.** *For every infinite group  $G$ , there exists a 2-thin subset  $X$  such that  $G = XX^{-1} \cup X^{-1}X$ .*

**Theorem 6.17.** *For every infinite group  $G$ , there exists a 4-thin subset  $X$  such that  $G = XX^{-1}$ .*

**Comments.** Theorems 6.14, 6.15, 6.17 are from [23], all other results from [24].

## 7. Colorings

### 7.1. Chromatic numbers

Let  $X$  be a set and  $\mathcal{F}$  be a family of subsets of  $X$  (the pair  $(X, \mathcal{F})$  can be thought as a hypergraph with the set of vertices  $X$  and the set of edges  $\mathcal{F}$ ). Following [43], a coloring  $\chi : X \rightarrow \kappa$  of  $X$  (i.e., a surjective mapping of  $X$  onto a cardinal  $\kappa$ ) is defined to be

- $\mathcal{F}$ -surjective if the restriction  $\chi|_F$  is surjective for all  $F \in \mathcal{F}$ ;
- $\mathcal{F}$ -injective if  $\chi|_F$  is injective for all  $F \in \mathcal{F}$ ;

- $\mathcal{F}$ -nonconstant if  $\chi|_F$  is not constant for all  $F \in \mathcal{F}$ ;
- $\mathcal{F}$ -bijective or  $\mathcal{F}$ -kaleidoscopic if  $\chi|_F$  is bijective for all  $F \in \mathcal{F}$ .

The family  $\mathcal{F}$  is called *kaleidoscopic* if it admits an  $\mathcal{F}$ -kaleidoscopic coloring  $\chi : X \rightarrow \kappa$ . Other three notions lead us to the following cardinal characteristics of the family  $\mathcal{F}$ :

$$\varkappa_{sur}(\mathcal{F}) = \sup\{\kappa : \text{there is an } \mathcal{F}\text{-surjective coloring } \chi : X \rightarrow \kappa\};$$

$$\varkappa_{inj}(\mathcal{F}) = \min\{\kappa : \text{there is an } \mathcal{F}\text{-injective coloring } \chi : X \rightarrow \kappa\};$$

$$\varkappa_{non}(\mathcal{F}) = \min\{\kappa : \text{there is an } \mathcal{F}\text{-nonconstant coloring } \chi : X \rightarrow \kappa\}.$$

If  $X$  admits no  $\mathcal{F}$ -nonconstant colorings, we put  $\varkappa_{non}(\mathcal{F}) = 1$ . Due to historical traditions, we call the cardinals  $\varkappa_{sur}(\mathcal{F})$ ,  $\varkappa_{non}(\mathcal{F})$  and  $\varkappa_{inj}(\mathcal{F})$  the resolvability, coresolvability and achromatic number of  $\mathcal{F}$ , respectively.

These numbers have been studied in many different contexts. The concept of resolvability came from topology. In 1943 E. Hewitt [21] introduced the notion of an irresolvable topological space, i.e., a space  $X$  with  $\varkappa_{sur}(\mathcal{F}) = 1$ , where  $\mathcal{F}$  is the family of non-empty open subsets of  $X$ . The information about the status of the resolvable spaces and its algebraic aspects can be found in the surveys [7, 12, 32].

On the other hand, the study of coresolvability numbers is the central topic of Ramsey Theory, see [19, 22, 41]. For example, the Wan der Waerden Theorem is nothing else but the equality  $\varkappa_{non}(\mathcal{F}) = \aleph_0$  where  $\mathcal{F}$  is the family of all arithmetic progressions in  $\mathbb{N}$ .

Another very important field where the coresolvability and achromatic numbers appear naturally is Graph Theory. For example, the *chromatic number*  $\varkappa(Gr)$  of a graph  $Gr$  is nothing else but  $\varkappa_{non}(\mathcal{F}) = \varkappa_{inj}(\mathcal{F})$  where  $\mathcal{F}$  is the family of edges of  $Gr$ . In fact, many questions concerning coloring of a hypergraph  $(X, \mathcal{F})$  can be reduced to studying the graph  $Gr_{\mathcal{F}}$  whose set of vertices is  $X$  and two vertices  $v_1, v_2 \in X$  are connected by an edge in  $Gr_{\mathcal{F}}$  if and only if  $v_1, v_2 \in F$  for some  $F \in \mathcal{F}$ . Using this observation, we get (see [43, Theorem 5.7]) the following estimates for the numbers  $\varkappa_{sur}(\mathcal{F})$ ,  $\varkappa_{inj}(\mathcal{F})$  and  $\varkappa_{non}(\mathcal{F})$ :

$$\varkappa_{sur}(\mathcal{F}) \leq \min_{F \in \mathcal{F}} |F| \leq \sup_{F \in \mathcal{F}} |F| \leq \varkappa_{inj}(\mathcal{F}) \leq \sup_{x \in X} |St(x, \mathcal{F})|,$$

$$\varkappa_{non}(\mathcal{F}) \leq 1 + ord(\mathcal{F}) \quad \text{and} \quad \varkappa_{non}(\mathcal{F}) = 2 \Leftrightarrow \varkappa_{sur}(\mathcal{F}) \geq 2,$$

where  $ord(x, \mathcal{F}) = |\{F \in \mathcal{F} : x \in F\}|$ ,  $ord(\mathcal{F}) = \sup_{x \in X} ord(x, \mathcal{F})$ , and  $St(x, \mathcal{F}) = \cup\{F \in \mathcal{F} : x \in F\}$  for  $x \in X$ .



Given a subset  $F \subseteq X$  of a  $G$ -space  $X$ , we get the hypergraph  $(X, \{gF : g \in G\})$ . For the chromatic characteristics of this hypergraph and a couple of open questions see [43, Chapters 6,7]. Only one typical result [43, Theorem 7.1].

**Theorem 7.1.** *Let  $G$  be a group considered as the left regular  $G$ -space,  $F$  be a finite subset of  $G$  with  $|F| > 1$ ,  $\mathcal{F} = \{gF : g \in G\}$ . Then the following statements hold*

- (i)  $\chi_{inj}(\mathcal{F}) \leq |F|^2 - |F| + 1$  and  $\chi_{non}(\mathcal{F}) \leq 3$ ;
- (ii)  $\chi_{inj}(\mathcal{F}) = |F|^2 - |F| + 1$  if and only if either  $F = \{a, b\}$  is two-element subset with  $ab^{-1}$  being of odd order, or else  $F^{-1}F$  is a subgroup of size  $|F^{-1}F| = |F|^2 - |F| + 1$ ;
- (iii)  $\chi_{inj}(\mathcal{F}) = 2$  if  $F^{-1}F$  contains an element of infinite or even order.

A new chromatic characteristic of a hypergraph  $(X, \mathcal{F})$ , the color detector, appeared in [46] in connection with [5]. Let  $\lambda$  be a cardinal,  $\lambda < |X|$ . A coloring  $\chi : X \rightarrow \kappa$  is called  $\lambda$ -admissible if  $|\chi(F)| \leq \lambda$  for every  $F \in \mathcal{F}$ . Then  $\lambda$  is a color detector if

$$\lambda = \sup\{|\chi(X)| : \chi \text{ is an } \lambda\text{-admissible coloring of } X\}.$$

**Theorem 7.2.** *Let  $G$  be a group with the identity  $e$ ,  $F \subseteq G$  and  $e \in F$ . Then 1 is a color detector of the hypergraph  $(G, \{gF : g \in G\})$  if and only if  $G = \langle F \rangle$ .*

One of the most interesting branch of the Subset Combinatorics of Groups is the Symmetry and Colorings (see the survey [7] and [8]). A subset  $A$  of a group  $G$  is said to be *symmetric* if  $A = gA^{-1}g$  for some  $g \in G$ , the center of symmetry. A good part of efforts was concentrated around calculations of  $\nu(G, \kappa)$ : *the minimal number of colors necessary to color the group  $G$  so that  $G$  contains no monochrome symmetric subsets of size  $\kappa$* . We restrict ourself with only one result in this direction.

**Theorem 7.3.** *For an infinite Abelian group  $G$ ,*

$$\nu(G, \aleph_0) = \begin{cases} r_0(G) + 1, & \text{if } G \text{ is finitely generated;} \\ r_0(G) + 2, & \text{if } G \text{ is countable, not} \\ & \text{finitely generated and } |B(G)| < \aleph_0; \\ \max\{|B(G)|, \log |G|\}, & \text{if } G \text{ is uncountable or } |B(G)| \geq \aleph_0 \end{cases}$$

where  $r_0(G)$  is a free rank of  $G$ ,  $B(G) = \{g \in G : 2g = 0\}$ ,  $\log \kappa = \min\{\alpha : 2^\alpha \geq \kappa\}$ .

In particular,  $\nu(\mathbb{Z}^n, \aleph_0) = n+1$ ,  $\nu(\mathbb{Q}^n, \aleph_0) = n+2$  and  $\nu(\mathbb{R}^n, \aleph_0) = \aleph_0$  for  $n \in \mathbb{N}$ .

## 7.2. Kaleidoscopic configurations

Let  $X$  be a  $G$ -space. We say that a subset  $F$  of  $X$  is a *kaleidoscopic configuration* if there exists a coloring  $\chi : X \rightarrow \kappa$  such that the restriction  $\chi|_{gF} : gF \rightarrow \kappa$  is a bijection for every  $g \in G$ . Clearly,  $F$  is a kaleidoscopic configuration if and only if the hypergraph  $(X, \{gF : g \in G\})$  is kaleidoscopic.

To characterize the kaleidoscopic configurations in groups (considered as the left regular  $G$ -space), we use the following definitions.

A subset  $A$  of a group  $G$  is called

- *complemented* if there is a subset  $B \subseteq G$  such that the multiplication mapping  $\mu : A \times B \rightarrow G$ ,  $\mu(a, b) = ab$ , is bijective;
- *doubly complemented* if there is a complemented subset  $B \subseteq G$  such that the multiplication mapping  $\mu : A \times B \rightarrow G$ ,  $\mu(a, b) = ab$ , is bijective.

**Theorem 7.4.** *Let  $G$  be a group. Each kaleidoscopic subset of  $G$  is complemented and each doubly complemented subset of  $G$  is kaleidoscopic.*

It should be mentioned that Theorem 7.4 reduces the kaleidoscopic problem in Abelian groups to the well known factorization problem. For the current state of the latter problem see [Sándor Szabó, Arthur D. Sands, Factoring groups into subsets, CRC Press, 2009].

**Corollary 7.1.** *The cardinality  $|F|$  of a kaleidoscopic subset  $F$  of a finite group  $G$  divides  $|G|$ .*

**Corollary 7.2.** *A subset  $F$  of an Abelian group  $G$  is kaleidoscopic if and only if  $F$  is complemented.*

**Question 7.1.** Is each kaleidoscopic subset of a group double complemented?

Given a subset  $A$  of a group  $G$ , how to detect whether  $A$  is a kaleidoscopic configuration? For a finite Abelian group  $G$ , the following theorem reduces this question to the case in which  $|A|$  and  $|G|$  have the same set of prime divisors.

**Theorem 7.5.** *Let  $A$  be a kaleidoscopic configuration in a finite Abelian group  $G$ ,  $m$  be a natural number such that  $(m, |A|) = 1$ . Then  $mA$  is a kaleidoscopic configuration in  $mG$ .*

The following theorem reduces the detecting problem for  $\mathbb{Z}$  to the case of finite cyclic groups.

**Theorem 7.6.** *Let  $F$  be a kaleidoscopic configuration in  $\mathbb{Z}$ ,  $\chi : \mathbb{Z} \rightarrow \{1, \dots, n\}$  be corresponding kaleidoscopic coloring. Then there exists a natural number  $t$  such that  $\chi(z + t) = \chi(z)$  for every  $z \in \mathbb{Z}$ .*

Now we describe some general construction of kaleidoscopic configurations in a transitive  $G$ -space  $X$ . To this end we use the standard identification of  $X$  with the (left) coset space  $G/H$  where  $H = \{g \in G : gx_0 = x_0\}$ ,  $x_0$  is a fixed element of  $X$ .

Let  $G$  be a group,  $H$  be a subgroup of  $G$ , and let

$$G = G_0 \supset G_1 \supset \dots \supset G_n = H$$

be a chain of subgroups of  $G$ . The *coset tree*  $T = T(G_0, G_1, \dots, G_n)$  is a tree with the root  $G_0$ , whose vertices are all left cosets of  $G$  by  $G_i$ ,  $i \in \{1, \dots, n\}$ , and two vertices  $A \in G/G_i$  and  $B \in G/G_j$  are incident if and only if  $j = i + 1$  and  $B \subset A$ . Thus,  $G/H$  is the set of endvertices of  $T$ .

We say that a coloring of  $T$  in two colors (black and white) is *coherent* if

- (i) the root  $G_0$  is black;
- (ii) if a vertex is white then all its successors are white;
- (iii) if a vertex is black then either one of its successors is black and others are white (type I) or all its successors are black (type II);
- (iv) any two vertices on the same level are of the same type.

**Theorem 7.7.** *The set of all black endvertices in a coherent coloring of  $T$  forms a kaleidoscopic configuration in  $G/H$ .*

The above construction can also be used as the *coherent test* giving a sufficient condition for a subset  $F \subset G/H$  to be a kaleidoscopic configuration.

Let  $G_0 = G$ ,  $G_n = H$  and  $T(G_0, G_1, \dots, G_n)$  be a coset tree. Let  $F$  be a subset of  $G/H$ . We define a coloring of  $T$  in black and white by the following rule: a vertex  $gG_i$  is black if and only if  $gG_i \cap F \neq \emptyset$ . We say that  $F$  *satisfies the coherent test* for  $T(G_0, G_1, \dots, G_n)$  if this coloring is coherent. If this is so, by Theorem 7.7,  $F$  is a kaleidoscopic configuration. If  $F$  does not satisfy this test, we have nothing to say about the kaleidoscopicality of  $F$ .

**Question 7.2.** Let  $F$  be a kaleidoscopic configuration in a finite cyclic group  $G$ . Does there exist a decreasing system  $G = G_0, G_1, \dots, G_n = \{0\}$  of subgroups of  $G$  such that  $F$  satisfies the coherent test for  $T(G_0, G_1, \dots, G_n)$ ?

**Theorem 7.8.** Let  $p$  be a prime number,  $n \in \mathbb{N}$ . A subset  $F \subseteq \mathbb{Z}_{p^n}$  is a kaleidoscopic configuration if and only if  $F$  satisfies the coherent test for the coset tree  $T(\mathbb{Z}_{p^n}, p\mathbb{Z}_{p^n}, \dots, p^n\mathbb{Z}_{p^n})$ .

**Theorem 7.9.** Let  $p_1, \dots, p_n$  be distinct primes,  $N = p_1 \dots p_n$ . A subset  $F \subseteq \mathbb{Z}_N$  is a kaleidoscopic configuration if and only if  $|F|$  divides  $N$  and  $F$  satisfies the coherent test for the coset tree  $T(\mathbb{Z}_N, m\mathbb{Z}_N, \{0\})$  where  $m = \frac{N}{|F|}$ .

**Question 7.3.** Is there a polynomial test detecting whether a given subset  $A$  of a finite Abelian group is a kaleidoscopic?

By Theorems 7.8 and 7.9, such test exists in corresponding cases.

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