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Level statistics for quantum Hall systems

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Level statistics for two classes of disordered systems at criticality are analyzed in terms of different realizations of the Chalker–Coddington network model. These include: 1) Re-examination of the standard $U(1)$ model describing dynamics of electrons on the lowest Landau level in the quantum Hall effect, where it is shown that after proper local unfolding the nearest-neighbor spacing distribution (NNSD) at the critical energy follows the Wigner surmise for Gaussian unitary ensembles (GUE). 2) Quasi-particles in disordered superconductors with broken time reversal and spin rotation invariance (in the language of random matrix theory this system is a representative of symmetry class D in the classification scheme of Altland and Zirnbauer). Here again the NNSD obeys the Wigner surmise for GUE, reflecting therefore only «basic» discrete symmetries of the system (time reversal violation) and ignoring particle–hole symmetries and other finer details (criticality). In the localized regime level repulsion is suppressed.

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1. Introduction

The statistics of energy levels in a disordered system is an important tool in determining its transport properties as well as its critical behavior. A central quantity in this study is the nearest-neighbor spacing distribution (NNSD) denoted by $p(s)$. Here the random variable s is the (fluctuating) level spacing under the proviso that the local average of the density of states is energy independent (otherwise, a proper unfolding procedure is required). The distribution $p(s)$ involves all N point correlation functions of the pertinent Green function and hence, it is generally not available in a closed form.

In dealing with disordered systems, it is useful to distinguish between systems undergoing an Anderson type metal–insulator transition, and those characterized by a quantum Hall (QH)-like transition where, in the thermodynamic limit, critical state energies are isolated points occuring between continuous intervals of localized state energies. As for level statistics pertaining to disordered systems of the Anderson metal–insulater transition kind, there are a couple of important properties which are well established: 1) Under certain conditions it is expected to be represented (on the metallic side) by random matrix spectra [1]. To be more precise, it is well described by one of the corresponding Gaussian ensembles, Gaussian orthogonal (GOE), Gaussian unitary (GUE) and Gaussian simplectic (GSE), depending on the symmetry class to which the physical system belongs. The main condition is that the corresponding energy intervals are smaller than the Thouless energy. 2) It has been shown [2] that in the limit of an infinite system there are only three types of distributions $p(s)$. They are the Poisson law for the insulating regime, the Wigner surmise for the metallic domain, and a third one for the critical region. On the other hand, for systems in the second group (such as the quantum Hall effect) there is no similar analysis. The main difficulty is related to the fact that in the absence of a metallic regime, it is not possible to approach the critical point from the metallic regime using the powerful tool of expansion in the small parameter $1/g$ (here g is the dimensionless conductance). Common sense suggests that the distribution follows the Poisson law in the insulating part of the spectra while again, the distribution in the critical region is different, and related to the relevant universality class. For the quantum Hall transition this is supported by numerous numerical calculations [3].

Recently, it has been noticed that the second group contains, beside the systems belonging to the quantum Hall effect universality class, other disordered systems whose phase diagram is much richer. They are related to the physics of disordered superconductors [4] and comprise of four novel universality classes, determined according to the symmetry properties of the corresponding Bogoliubov–de Gennes Hamiltonian under spin rotation and time reversal. Some of these new phase diagrams have already been exposed, mainly in class C (where time reversal symmetry is broken while spin rotation invarinace is preserved). So far, the level statistics in the critical regions of the four new universality classes has not yet been studied. The goal of the present work is to fill this gap, starting by elucidating the level statistics of one of these new classes, namely class D, for which both symmetries (spin rotation and time reversal) are violated.

Some of the disordered systems in the second group can be mapped on a network model. The most studied one is the quantum Hall system which is mapped on the Chalker–Coddington network model (CCNM) [5]. The CCNM is designed to describe transition between plateaus in the QH system using transfer matrix algorithm in an infinite cylinder geometry (in the Landauer sense). It was later suggested that if, instead of studying transport properties, the system is closed up as a torus, then the eigenvalue problem can be addressed and the level statistics can be studied [6] (although no Hamiltonian is specified). In our previous works, a somewhat modified CCNM has been constructed which can describe noninteracting quasiparticles in disordered superconductors [7,8]. It appears that such a description can serve as an appropriate physical realization of the new random matrix universality classes [4].

It is then natural to attempt an investigation of level statistics of these new symmetry classes by using the CCNM. Our main results are summarized below. 1) As a starting reference point we revisit the familiar QH system by repeating calculations for the original $U(1)$ model [6]. The results of Ref. 6 are indeed reproduced ($p(s)$ deviates slightly from the Wigner surmise for GUE). Moreover, we show that after *proper local unfolding*, the NNSD of the $U(1)$ model at the critical energy is *exactly identical* with the Wigner surmise. In order to stress the necessity of unfolding we argue that the density of states (DOS) averaged over all samples is indeed uniform, whereas it has some structure for each sample. 2) We then present results for class D of disordered superconductors that have neither time-reversal nor spin-rotation invariance. Once again NNSD at the critical energy (after unfolding) coincides with the Wigner surmise. We find the DOS to have a periodic structure (period $\pi/2$), as one would expect from the form of the unitary operator. Beside the critical region, we also present results for the localized regime and show that level repulsion disappears. Thus, although the $U(1)$ and class D models considered here describe different systems and have different phase diagrams, yet the NNSD in the critical region is the same, depending only on the broken time-reversal symmetry. The fact that class D obeys particle–hole symmetry is not reflected in its NNSD. Our findings are in agreement with recent works [9] where it is argued that k-body embedded Gaussian ensembles of random matrices for sufficiently high rank k of the random interaction behave generically (i.e., in order to have exact RMT results it is not necessary for the Hamiltonian to be a full random matrix).

2. The $U(1)$ network model

In the original CCNM, electrons move along unidirectional links forming closed loops in analogy with semiclassical motion on equipotential contours. Scattering between links is allowed at nodes in order to map tunneling through saddle point potentials. Propagation along links yields a random phase φ, thus links are presented by diagonal matrices with elements in the form $\exp(i\varphi)$ (hence the notation $U(1)$ model). The transfer matrix for one node relates a pair of incoming and outgoing amplitudes on the left to a corresponding pair on the right; it has the form

$$\mathbf{T} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \qquad (1)$$

The node parameter θ is related to the electron energy in the following way

$$\varepsilon = -\frac{2}{\pi} \ln(\sinh\theta), \qquad (2)$$

where ε is a relative distance between the electron energy and the barrier height.

If the network forms a torus, then on every link the electron motion appears once as an outgoing one and once as an incoming one. The collection of relations between incoming and outgoing amplitudes defines the system's S matrix or rather, a discrete-time unitary evolution operator, $U(\varepsilon)$ [6]. The eigenphases of U serve as input for level statistics analysis. For a square network of $N \times N$ nodes, U is an $(2 \times N^2) \times (2 \times N^2)$ unitary matrix. The action of U on a vector Ψ of flux amplitudes defined on the start of each link maps the system onto itself, providing therefore an *implicit* eigenvalue equation $U(\varepsilon)\Psi = \Psi$. Since the dependence of the matrix elements of U on ε is comlicated, it is

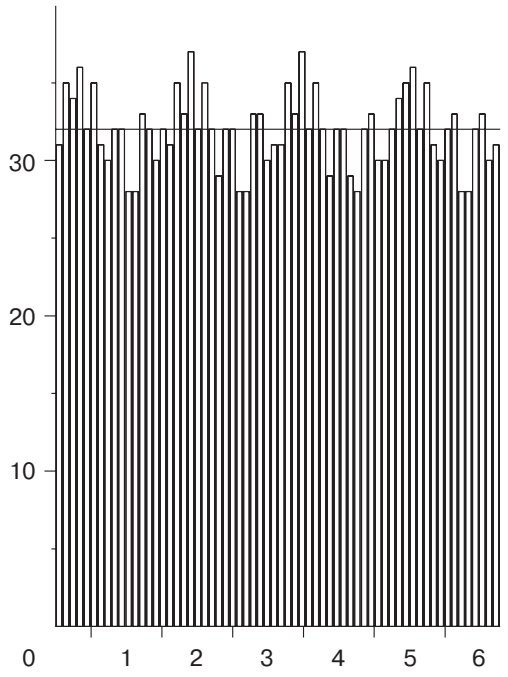

Fig. 1. Histogram for the DOS of a single sample at the critical point $\varepsilon = 0$ for $U(1)$ model without unfolding.

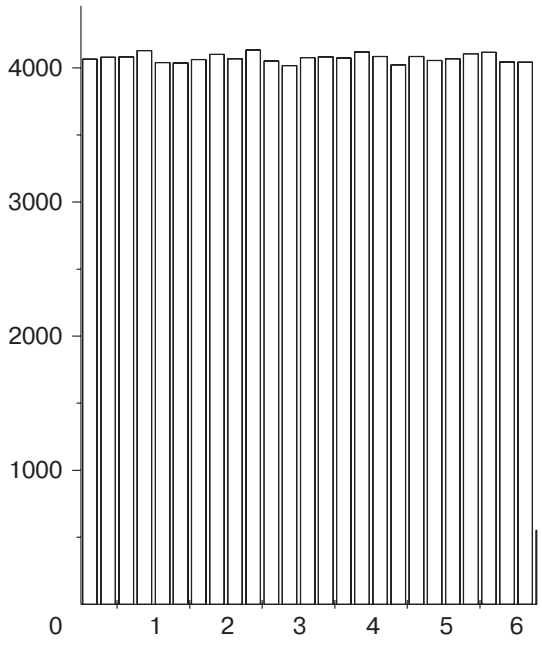

Fig. 2. Histogram for the DOS of 50 samples at the critical point $\varepsilon = 0$ for $U(1)$ model.

practically impossible to find solutions of that equation (even numerically). Instead, it has been suggested [6] to find the eigenvalues of the equation

$$U\Psi_n = \exp\,[i\omega_n(\varepsilon)]\Psi_n \qquad (3)$$

and to study statistics of ω_n for a given ε. The rationale behind it is twofold. First, there are sufficiently many states even in a narrow window near a particular energy ε to provide good statistics. Second, the behavior of the curves $\omega_n(\varepsilon)$ is rather smooth, and therefore the statistics of ω_n for a given ε is expected to be the same as the statistics of ε_n for $\omega = 0$ (which are the true energy eigenvalues). We argue that the second hypothesis is justified only after a proper unfolding procedure is executed. Indeed, from the RMT point of view the eigenvalue problem of Eq. (3) belongs to the circular unitary ensemble (CUE). In the standard CUE all the eigenvalues lie on the unit circle and are equally spaced, so there is no need for unfolding of the spectra. However, Eq. (3) represents a physical problem in which not all the elements of U are independent random variables. Therefore, the question of whether the $U(1)$ model is a *bona fide* CUE should be examined. Our calculations show that the averaged (over 50 samples) DOS is indeed uniform, whereas it has some pronounced structure for each particular sample. Since the level statistics should be manifested for each individual sample (as in the study of nuclear spectra), the spectrum of each sample should then be properly unfolded. Here we make use of the fact that the dimensionless unfolded distance between two levels is

$$\Delta s_n = 2k\frac{E_{n+1} - E_n}{E_{n+k} - E_{n-k}}, \qquad (4)$$

where k is a number of neighbors to be optimized by the requirement of having a constant DOS. This procedure encodes the important local fluctuations of level spacing. We have checked that for $k \geq 6$ the result is practically independent of k, provided of course that $k << 2 \times N^2$. To substantiate this point we plot in Fig. 1 the DOS of a single sample (at the crit-

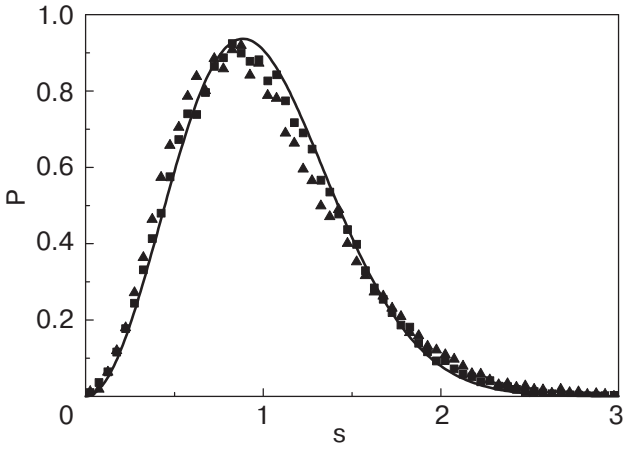

Fig. 3. Nearest neighbor spacing distribution $P(s)$ for $U(1)$ model at $\varepsilon = 0$ (critical regime). The curve is the Wigner surmise for the GUE; without unfolding Mettzer–Klesse (▲), ±3 neighbor local unfolding (■).

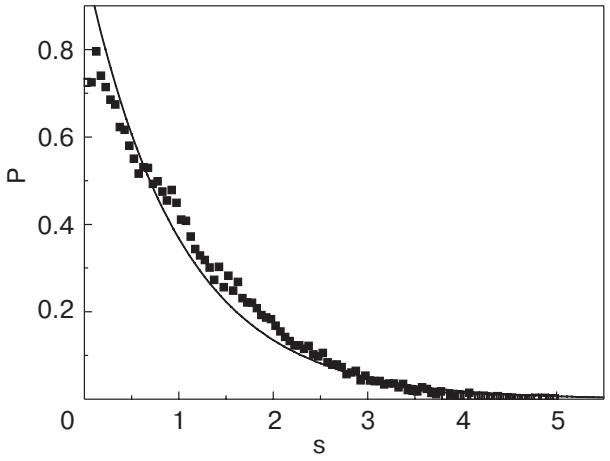

Fig. 4. Nearest neighbor spacing distribution $P(s)$ for $U(1)$ model at $\varepsilon = 1$ (localized regime). The curve is the Poissonian statistics, data after unfolding (■).

ical point $\varepsilon = 0$) without unfolding. It shows indeed that the averaged DOS is not constant in energy. On the other hand, after averaging over 50 samples, Fig. 2 indicates that the average DOS is constant, as expected for random matrices belonging to CUE. The corresponding NNSD are displayed in Fig. 3. Raw data without unfolding reproduce the results obtained in [6], whereas after local unfolding the NNSD nearly coincides with the Wigner surmise for GUE (expected to be true also for CUE at large N). Finally, we assert in Fig. 4 that in the localized regime ($\varepsilon = 1$) the NNSD follows the Poisson statistics $p(s) = e^{-s}$.

3. Disordered superconductors: Class D

The properties of quasiparticles in disordered superconductors have been the subject of much recent interest. The Hamiltonians of such systems are representatives of a set of symmetry classes different from the three classes which are familiar both in normal disordered conductors and in the Wigner–Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes, has been established [4]. Below we present numerical results on statistics of energy levels for a certain two-dimensional system with a particularly rich phase diagram. In the numenclature of Ref. 4 the corresponding symmetry is denoted as class D. It can be realized in superconductors with broken time-reversal invariance, and either broken spin-rotation invariance (as in d-wave superconductors with spin–orbit scattering) or spinless or spin-polarized fermions (as in certain p-wave states). A particular realization of class D (which will be adopted here) is the Cho–Fisher (CF) model [10] which has a rich phase diagram. Each realization has two parameters: a disorder strength $W(0 \le W \le 1)$, and a tunneling amplitude ε, which controls the value of the thermal Hall conductance at short distances. The phases on the links on the two sides of the same node (which are either π with probability W or 0 with probability $1 - W$) are correlated: the same random phase appears on both sides. It is equivalent to attributing random sign to the off-diagonal elements of the transfer matrix. The phase diagram in the (ε, W) plane contains a region of metallic states, and two distinct localized domains, which can be identified as regions with different quantized thermal Hall conductance. There is a critical state at $\varepsilon = 0$ for any W. The detailed structure of the phase diagram has been presented elsewhere [8].

In our numerical simulations we have studied 50 different network systems, of size $(2 \times 32^2)(2 \times 32^2)$ on the critical line $\varepsilon = 0$ and disorder strength parameter $W = 0.1$. The raw DOS appears to be a periodic function of ω with period $\pi/2$ reflecting the cubic symmetry of the CF model (Fig. 5). The NNSD is presented in Fig. 6 and are compared with the Wigner surmise for GUE. The agreement is rather evident, and it could not be achieved without unfolding. We are thus convinced that the critical form of the NNSD at the critical line of the CF model (in fact of class D in general) coicides with the GUE. Next, we move away from the critical line and put $\varepsilon = 1$, keeping the same value $W = 0.1$ which, according to our phase diagram, is well within a localized domain. The results are

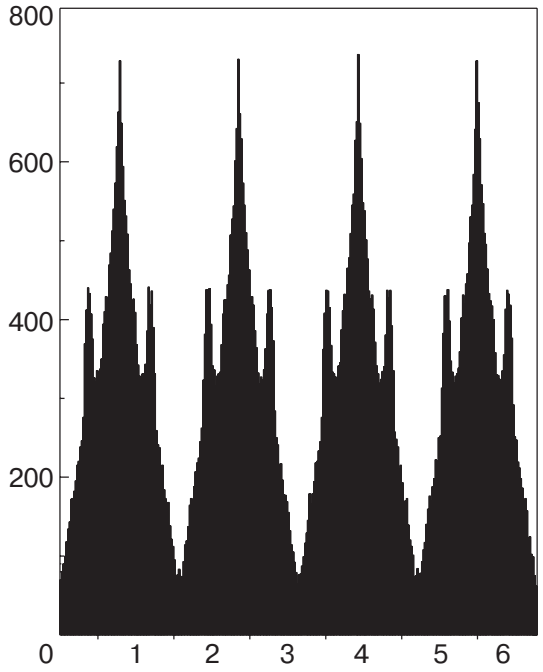

Fig. 5. Histogram for the DOS of 50 samples for CF model at the critical point $\varepsilon = 0$ and $W = 0.1$.

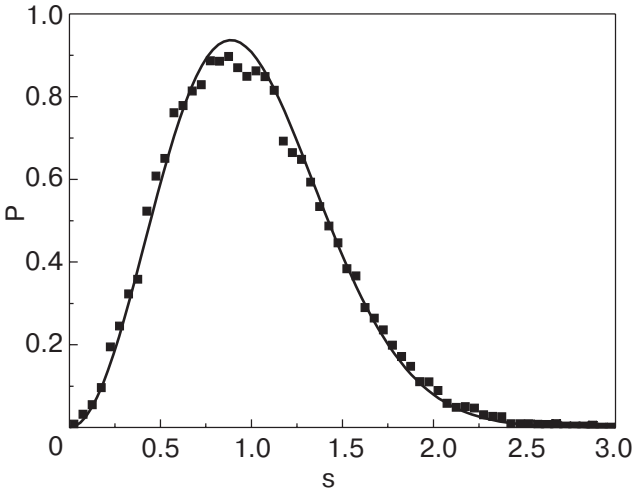

Fig. 6. Nearest-neighbor spacing distribution $P(s)$ for CF model at $\varepsilon = 0$ and $W = 0.1$ (metallic regime). The curve is the Wigner surmise for the GUE.

shown in Fig. 7. They are fitted by the Berry–Robnick approximation [11]. Usually, the large s behavior is more sensitive to localization than the small s behavior. In other words, even deep inside the localized regime one still finds level repulsion $p(0) = 0$. Remarkably, for the CF model we find $p(0.025) \approx 0.53$, which cannot be just attributed to statistical error in view of the fact that we study almost 10^6 energy levels. We have also calculated the compressibility η of the spectrum and have found an extremely small value ≈ 0.01, which is in agreement with the classical result for the GUE, where $\eta \to 0$ for large system sizes.

In conclusion, although mapping of a physical problem on a network model results in correlated and sparse matrices of unitary evolution operators, the results for NNSD seem to agree with the predictions of

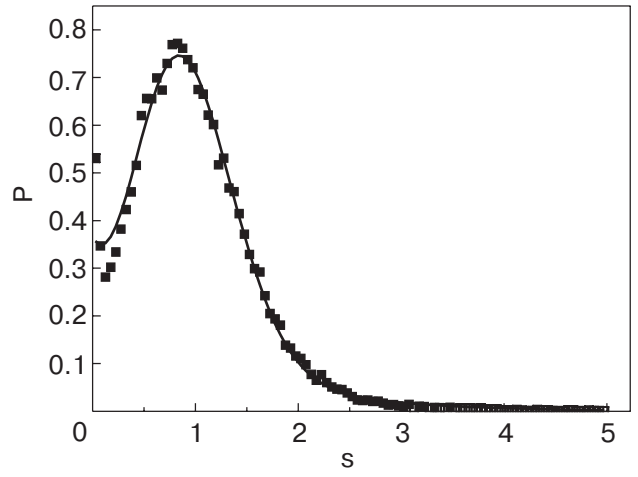

Fig. 7. Nearest-neighbor spacing distribution $P(s)$ for CF model at $\varepsilon = 1$ and $W = 0.1$ (localized regime). Curve is the Berry–Robnick fit for the transition from GUE to Poissonian statistics.

RMT which assume non-sparse matrices with uncorrelated matrix elements [9]. In the cases studied here this agreement is achieved after a proper local unfolding of the spectra is executed.

The main physical result is the following: Despite the occurrence of ten different random matrix symmetry classes according to time-reversal, spin-rotation, and particle–hole symmetries, with many different physical properties, some basic characteristics remain intact, depending only on time reversibility and spin rotation invariance. There have been numerous attempts to check whether the form of $p(s)$ in QH like systems deviates from that of GUE [12]. Our results indicate that as far the network model realization is concerned, $p(s)$ is satisfactorily accounted for by the Wigner surmise for the unitary ensemble. The violation of time reversal invariance either by a magnetic field (in QH systems) or spontaneously in unconventional superconductors is the dominant factor, which masks finer details such as quantum criticality.

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