

Influence of nonlinear dissipation and external perturbations on transition scenarios to the chaos in the Lorenz-Haken system

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We studied an influence of nonlinear dissipation and external perturbations on transition scenarios to the chaos in the Lorenz-Haken system. It was shown that variation in the values of external potential parameters leads to parameters domain formation which results in the chaos appearance. We have found that in the modified Lorenz-Haken system transitions from the regular to chaotic dynamics can be of Ruelle-Takens scenario, Feigenbaum scenario, or through intermittency.

Исследовано влияние нелинейной диссипации и внешних возмущений на сценарии перехода к хаосу в модели Лоренца-Хакена. Установлено, что области параметров системы, определяющих возникновение хаотического поведения, связаны с изменением параметров внешнего потенциала. Показано, что в модифицированной системе Лоренца-Хакена переход от регулярного поведения к хаотическому возможен по трем сценариям, а именно, по сценарию Рюэлля-Таккенса, по сценарию Фейгенбаума, или вследствие перемежаемости, в зависимости от значений параметров системы.

1. Introduction

One of the most actual task in the theory of nonlinear dynamical systems is a setting of chaotic regime generation conditions and defining possibilities of the chaos control [1, 2]. It is well known, that in physical applications a control of the transition from the chaotic to periodic mode in multicomponent systems can be performed by using different mechanisms. For example, in laser physics there are: negative feedback [3], angle between two crystals which are entered to the Fabry-Perot cavity [4], full feedback intensity [1], intensity of activating (see [5, 6] and citations therein). In this connection, from the theoretical point of view, an actual task is to establish the chaos control and to determine transition characters between the chaotic and regular dynamics.

The main goal of this work is to study an influence of two additional nonlinearities that arise up in the chaotic system as a result of different physical processes on transitions character between the regular and chaotic dynamics. We will consider the modified Lorenz-Haken model which self-consistently can describe, for example, optical bistable systems [7], systems of defects in a solid [8, 9], etc. Due to condition of commensurability for all three modes relaxation times we will set domains of the system parameters of the chaos realization with a help of maximal Lyapunov exponent approach. We will obtain two different strange chaotic attractors and set possible transitions to the chaotic dynamics.

The paper is organized in the following manner. In Section 2 we present a model of our system incorporating a nonlinear dissipation and external perturbation terms. Section 3 is devoted to consideration of the conditions for transition to the chaotic regime and to determination the main characteristics of the strange chaotic attractor. The main results and prospects for the future are presented in Conclusions.

2. A model of the chaotic system

The Lorenz-Haken model can be written in a form [7]:

$$\begin{aligned} \dot{\eta} &= -\eta/\tau_\eta + g_\eta h, \\ \dot{h} &= -h/\tau_h + g_h \eta S, \\ \dot{S} &= (r - S)/\tau_S - g_S \eta h. \end{aligned} \quad (1)$$

Here a point means a derivative in time; τ_η, τ_h, τ_S — relaxation times of the order parameter $\eta(t)$, a conjugating field $h(t)$ and control parameter $S(t)$, accordingly; g_η, g_h, g_S — positive feed-back constants; r — pump intensity, measures an influence of environment. First elements in the left hand of the system Eq.(1) take into account dissipation effects, peculiar to the synergetic systems. Connection between the order parameter and conjugating field is linear (first equation), evolution of both the conjugating field $h(t)$, and control parameter $S(t)$ are determined by nonlinear feed-back relations (second and third equations, respectively). It is principally important that positive feed-backs in Eq.(1) which are provided by constant g_η and g_h, result in increase in conjugating field. These positive feed-backs are compensated by negative one due to principle of Le-Shatel'e. As a result one has decreasing in the control parameter (see third equation in Eq.(1)).

Let us start with the analysis of the system (1) with passing to dimensionless variables. Such transition is arrived due to measuring time t, order parameter η, conjugating field h, and control parameter S in the followings units:

$$t \propto \tau_\eta, \quad \eta_e \propto (g_h g_S)^{-1/2},$$
$$h_e \propto (g_\eta^2 g_h g_S)^{-1/2}, \quad S_e \propto (g_\eta g_h)^{-1}.$$

Hence, dropping indexes, the system (1) becomes the following

$$\begin{aligned} \dot{\eta} &= -\eta + h, \\ \sigma\dot{h} &= -h + \eta S, \\ \varepsilon\dot{S} &= (r - S) - \eta h, \end{aligned} \quad (2)$$

where $\sigma \equiv \tau_h/\tau_\eta$, $\varepsilon \equiv \tau_S/\tau_\eta$. The system Eq.(2) is written in supposition of linear dependence of the order parameter relaxation time, as $\tau_\eta(\eta) = const$. However, most real physical systems are characterized by the nonlinear relaxation processes. In this connection let us suppose that the order parameter relaxation time τ_η increases with an increase in the order parameter η by the relation [10]:

$$\tau_\eta(\eta) = 1 - \frac{\kappa}{1 + \kappa + \eta^2}, \quad (3)$$

where κ — positive constant which plays a role of an dissipation intensity. From relation (3) it is seen that relaxation time $\tau_\eta(\eta)$ is independent of order parameter sign. Except for that, relation (3) has practical application, namely, it designs the action of optical filter, entered into the Fabry-Perot cavity of optically bistable system (for example solid-state laser [11]). Such acting provides an establishment of the stable periodic radiation (or time dissipative structure appearance) [11]. Using dependence (3), first equation of (2) is generalized by an additional nonlinear term $f_\kappa = -(\kappa\eta)/(1 + \eta^2)$.

Considering the system in external field, one needs to take into consideration external perturbations. In this article we will model such perturbations by the external potential V_e. Due to the standard catastrophe theory such potential is given by three types of catastrophes [12]. In general case one has

$$V_e = A\eta + \frac{B}{2}\eta^2 + \frac{C}{3}\eta^3 + \frac{D}{4}\eta^4 + \frac{E}{5}\eta^5, \quad (4)$$

where A, B, C, D, E — parameters of the theory. For the catastrophe A_2 one has: $B = D = E = 0$, for the catastrophe A_3: $C = E = 0$ and for the catastrophe A_4: $D = 0$. The modified Lorenz-Haken system has the form

$$\begin{aligned} \dot{\eta} &= -\eta + h + f_\kappa(\eta) + f_e(\eta) \\ \dot{h} &= -h + \eta S, \\ \dot{S} &= (r - S) - \eta h, \end{aligned} \quad (5)$$

where we suppose $\sigma \approx \varepsilon \approx 1$ and $f_e(\eta) \equiv -dV_e/d\eta$. Variation in parameters of both $f_\kappa(\eta)$ and $f_e(\eta)$ can induce a change in the attractor topology in phase space.

3. Chaos in the modified Lorenz-Haken system

The system Eq.(5) with nonlinear dependence of order parameter relaxation time versus order parameter in a form (3) ($\kappa \neq 0$) but with an absence of additional perturbations ($V_e = 0$) was considered in [13]. It was shown that in such a case the semirestricted domain of system parameters (pump intensity r and dissipation intensity κ) for dissipative structure realization is formed. It was set that in a case of linear dependence for order parameter relaxation time versus its values ($\kappa = 0$) the chaotic regime does not realize. In addition, it was found the chaos domain, and it was determined conditions of the chaos control. Finally it was defined fractal and statistical properties of the corresponding chaotic strange attractor.

The main goal of this work is to study an influence of external perturbation on regimes of transition to the chaos in Lorenz-Haken system, generalized by nonlinear relaxation time of order parameter in a form (3). For external perturbations we will use a potential of the fold catastrophe A_2, i.e. $V_e \equiv A\eta + 1/3C\eta^3$. To indicate the chaotic dynamic we will use the method of Lyapunov exponents which is provided by the Benettin algorithm [14]. Due to this algorithm each of Lyaponov exponents (number is defined by the dimension of the corresponding phase space) determines a speed of convergence/divergence of any two initially nearby trajectories in a fixed direction in the corresponding phase space, starting from points $\nu(t)$ and $\nu(t')$. The divergence/convergence of such trajectories is given by the dependence $\delta\nu(t) = \delta\nu(t_0)e^{\Lambda_M t}$, where Λ_M is a maximal (global) Lyapunov exponent, which is defined by the relation [14]

$$\Lambda_M \equiv \Lambda(\delta\}(t_0)) = \overline{\lim_{T\to\infty}} \frac{1}{T}\ln\left\| \frac{\delta\nu(t)}{\delta\nu(t_0)} \right\|.$$

Here one takes into account an upper limit and $\|\nu\|$ is a norm; $\nu = \eta, h, S$; T — full time. It can be concluded that in the case $\Lambda_i < 0$, $i = 1, 2, 3$, and accordingly $\Lambda_M < 0$, all of phase trajectories will coincide to the fixed point (stable node or stable focus). At $\Lambda_i < 0$, $\Lambda_j < 0$ and $\Lambda_M \equiv \Lambda_k = 0$, $i \neq j \neq k$, $i, j, k = 1, 2, 3$ phase trajectories will lie down on a stable limit circle (dissipative structure). If $\Lambda_i < 0$, $\Lambda_j = 0$ and $\Lambda_M \equiv \Lambda_k > 0$, a dynamics of the system is chaotic. Lyapunov map of the modified Lorenz-Haken system (5) at $\kappa = 25.0$ and $A = 0.1$ is shown in Fig. 1. Here by gradation of grey color the value of the maximal (global) Lyapunov exponent is shown versus pump intensity r and parameter of an external potential C. White color determines the domains of stable system behavior (phase space is characterized by a fixed point — stable node or a stable focus). Grey color marks the domains of time dissipative structure existence (more dark domains correspond to the larger number of oscillation periods). Domains of chaos are shown by a black color. From Fig. 1 it is seen, that in the case $C = 0$ an existence of the chaotic mode requires the large values of pump intensity. In the opposite case ($C \neq 0$) at $|C| \sim 0.1$ the chaos is realized at $r \sim \kappa$.

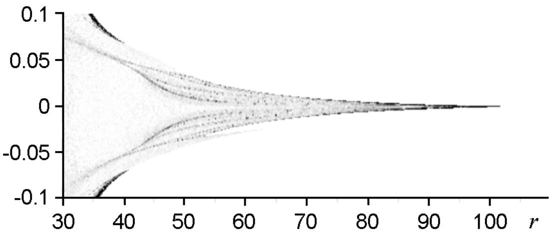

Fig. 1. Lyapunov map of the modified Lorenz-Haken system (5) at $\kappa = 25.0$ and $A = 0.1$.

Let us analyze a picture of dynamical regimes reconstruction of the system (5) with $C = 0$ and $A = 0.1$ in detail. Lyapunov map and the corresponding dependence of maximal Lyapunov exponent versus pump intensity at fixed value of dissipation intensity κ with characteristic phase portraits are shown in Fig. 2. Here due to earlier denoted scenario values of the maximal Lyapunov exponent are presented by gradation of grey color versus pump intensity and dissipation intensity. It is necessary to note that dark curves (between a) and b), b) and c) in the Lyapunov map) in the domain of dissipation structure existence (grey area) determine the parameter values corresponding to the doubling period bifurcation. Below the map there is a dependence of the maximal Lyapunov exponent versus pump intensity r at $A = 0.1$, $C = 0$ and $\kappa = 25.0$. The presence of two pronounced peaks in the domain of zero values of the maximal Lyapunov exponent (fluctuations around zero are connected with errors of numeral solution) determines the points of doubling period bifurcation. Corresponding phase portraits illustrating dissipative structure with one, n and $m > n$ periods are shown with the help of insertions a), b) and c), accordingly. It is principally, that both transitions from a) to b), and from b) to c) are characterized by appearance of a few additional harmonics. As it is seen from the dependence $\Lambda_M(r)$ in a point d) the maximal Lyapunov exponent has positive value and phase space is characterized by the irregular behavior of a trajectory. Increase in pump intensity leads to dissipative structure formation with one period (cf. phase portraits d) and e)). Further increase in r

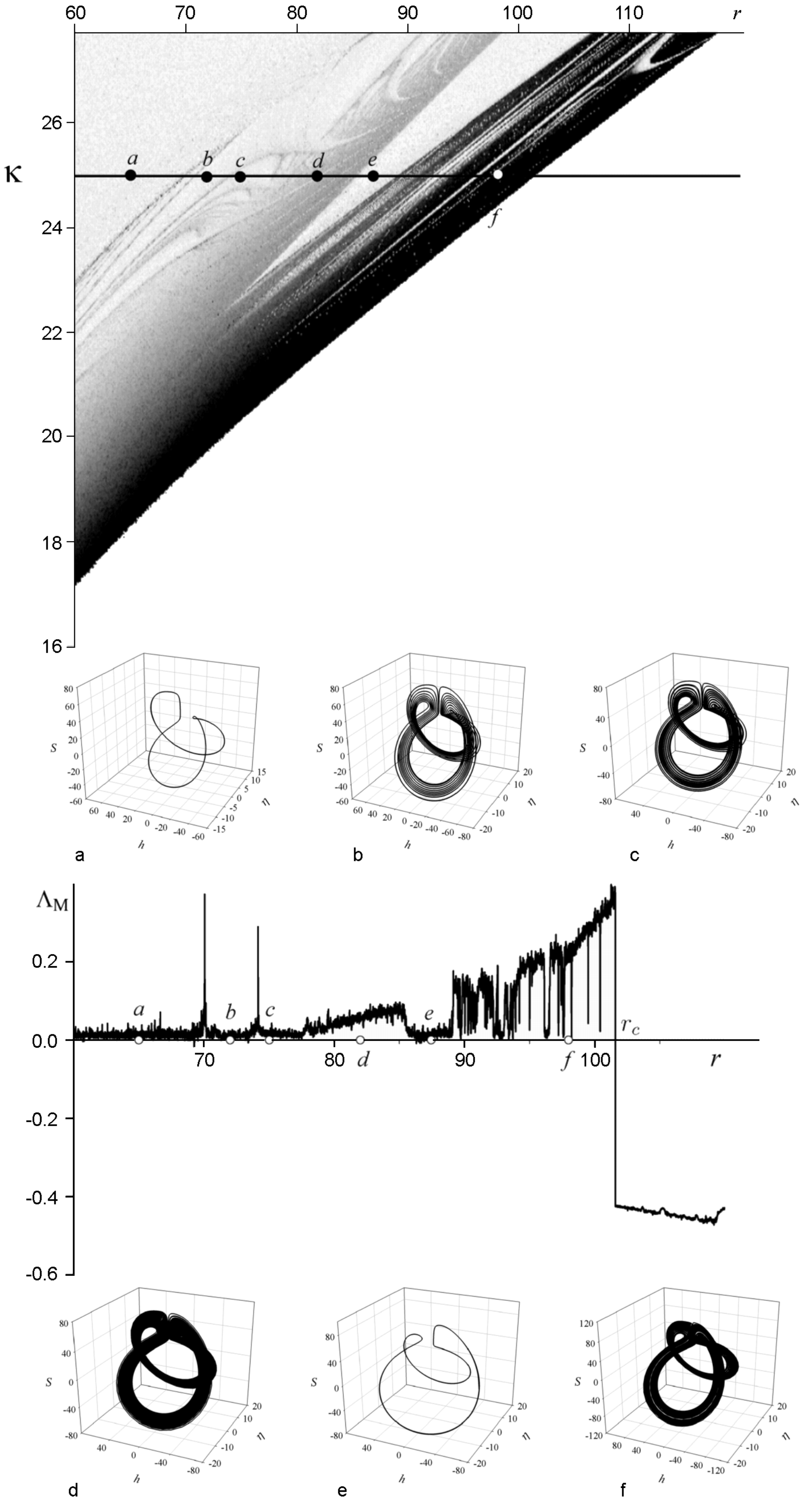

Fig. 2. Lyapunov map of the modified Lorenz-Haken system (5) at $A = 0.1$ and $C = 0$; dependence of maximal Lyapunov exponent at $A = 0.1$, $C = 0$ and $\kappa = 25.0$ and corresponding phase portraits.

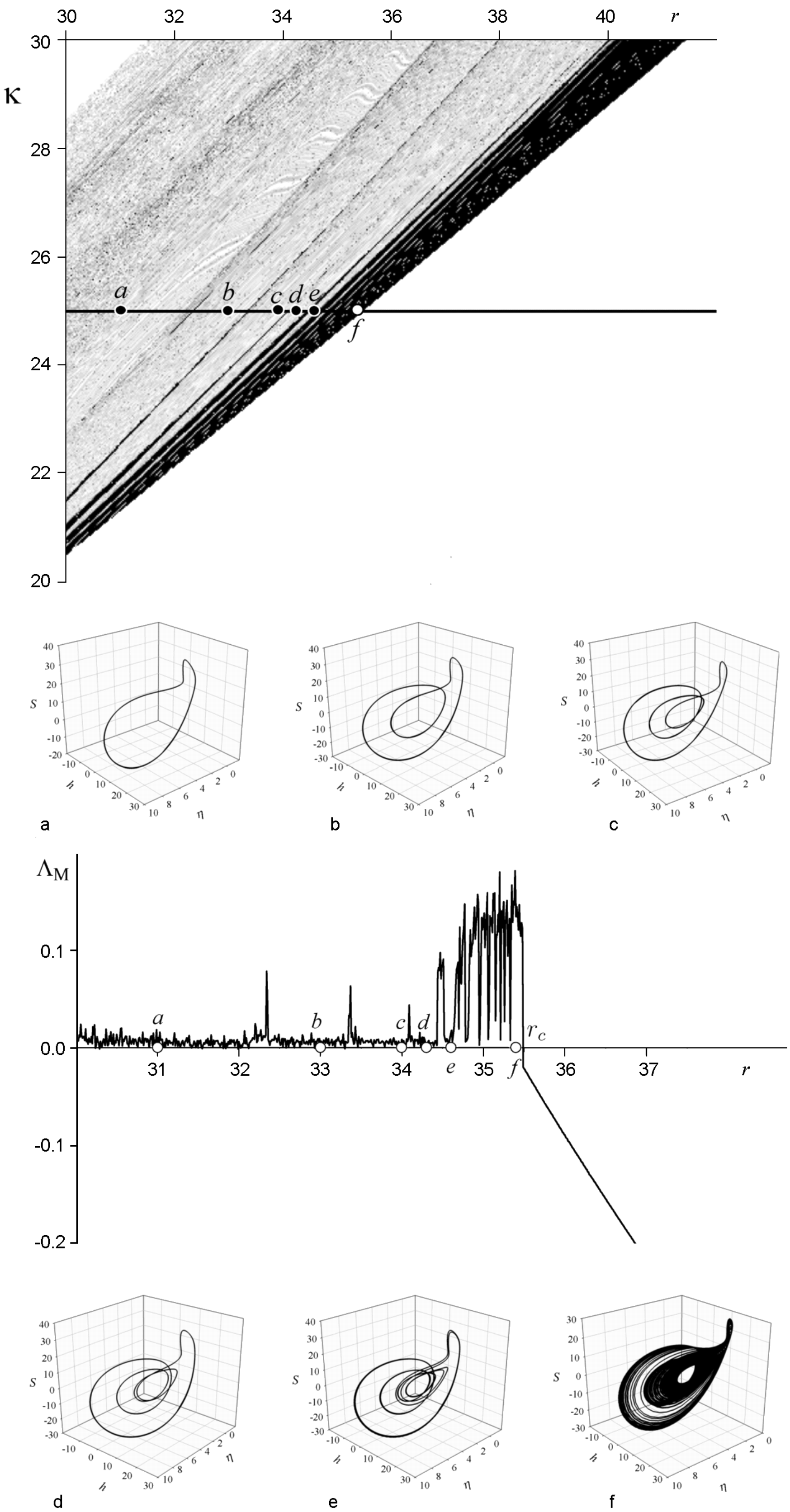

Fig. 3. Lyapunov map of the modified Lorenz-Haken system (5) at $A = 0.1$ and $C = 0.1$; dependence of maximal Lyapunov exponent at $A = 0.1$, $C = 0.1$ and $\kappa = 25.0$ and corresponding phase portraits.

leads to positive values of Λ_M and phase space is characterized by the chaos existence (corresponding phase portrait is shown with the help of insertion *f*)). So, one can conclude that at $A = 0.1$, $C = 0$ and $\kappa = 25.0$ with an increase in pump intensity r a transition to the chaotic regime occurs due to Ruelle-Takens scenario, when only negligible number of doubling period bifurcation leads to the chaos [15]. With a decrease in r (from $r > r_c$ to $r < r_c$) the maximal Lyapunov exponent takes positive value at $r = r_c$ in spontaneous manner. Corresponding transition to the chaos takes place through an intermittency [16].

Next, let us consider the case of $C \neq 0$. Lyapunov map at $A = 0.1$ and $C = 0.1$ is shown in Fig. 3. Below the map as well as in previous case a dependence of maximal Lyapunov exponent versus pump intensity at $A = 0.1$, $C = 0.1$ and $\kappa = 25.0$ and corresponding phase portraits are shown. Unlike to the previous case here with an increase in pump intensity r a successive complication of the attractor due to doubling period bifurcation is observed (see corresponding phase portraits in points a), b), c), d) and e)). Thus, in such a case ($A = 0.1$, $C = 0.1$ and $\kappa = 25.0$) an increase in r results in transition to the chaos by Feigenbaum scenario [15]. The chaotic attractor is shown with the help of insertion *f*). As well as in the previous case a decrease in r leads to transition to the chaos at $r = r_c$ through intermittency [16].

It is well known that dynamical systems can realize four types of attractors in the phase space, namely: non chaotic non strange attractor, chaotic non strange attractor, strange non chaotic attractor and chaotic strange attractor. The attractor is chaotic if the condition $\Lambda_M > 0$ is satisfied; strange attractor is characterized by the fractional value of the fractal dimension D. In [17] it was shown that the fractal dimension of the attractor which is realized in the dynamical system, is determined with the help of Lyapunov exponents by the relation

$$D \approx \frac{3}{2} + \frac{1}{2}\sqrt{1 - 8\frac{\Lambda_M}{\Lambda_{\min}}},$$

where Λ_{min} is the minimal Lyapunov exponent. Previously in the absent of external force, $f_e = 0$, the full analysis of the fractal dimension of strange chaotic attractor was made in [18]. It was shown, that strange chaotic attractor is characterized by the Lyapunov fractal dimension $D > 2$. For the considered attractor in a point f) at $\kappa = 25.0$, $r = 98.0$, $A = 0.1$ and $C = 0$ (see Fig. 2) one has: $\Lambda_M = 0.2774$, $\Lambda_{min} = -16.948$ and, accordingly, $D \approx 2.032$. For the attractor at $\kappa = 25.0$, $r = 34.5$, $A = 0.1$ and $C = 0.1$ (see Fig. 3) one has $\Lambda_M = 0.16545$, $\Lambda_{min} = -3.631$ and, accordingly, $D \approx 2.084$. Thus, the attractors in Fig. 2f and Fig. 3f are strange and chaotic.

4. Conclusions

We have studied the influence of nonlinear dissipation and external perturbations on transition scenarios to the chaos in the Lorenz-Haken system. Dissipation processes are defined by the nonlinear dependence of order parameter relaxation time versus its values. External perturbations are modeled by a potential of fold catastrophe. From a physical viewpoint we have considered the absorptive optical bistability system [7]. At that time used nonlinear dissipation relation related to the possibility of the additional medium in the Fabry-Perot cavity (phthalocyanine fluid, gases SF_6, $BaCl_3$, and CO_2 [7]) to absorb signals with weak intensities. Meanwhile, external perturbations model the influence of optical modulator which sets additional interphotons interaction processes in the Fabry-Perot cavity.

We have considered a case of commensurability of relaxation times for order parameter, conjugated field and control parameter. It was shown that variation in values of external potential parameters leads to formation of domain with $r \sim \kappa$ and with r by order of magnitude greater than κ of the chaos realization. The corresponding attractors are chaotic and strange. In the system considered transitions from the regular to chaotic dynamics can be of Ruelle-Takens scenario, Feigenbaum scenario, or through intermittency.

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Вплив нелінійної дисипації та зовнішніх збурень на сценарії переходу до хаосу в моделі Лоренця-Хакена

А.В.Дворниченко

Досліджено вплив нелінійної дисипації та зовнішніх збурень на сценарії переходу до хаосу в моделі Лоренця-Хакена. Встановлено, що області параметрів системи, які визначають виникнення хаотичної поведінки, пов'язані зі зміною параметрів зовнішнього потенціалу. Показано, що у модифікованій системі Лоренця-Хакена перехід від регулярної поведінки до хаотичної можливий за трьома сценаріями, а саме, за сценарієм Рюелля-Таккенса, за сценарієм Фейгенбаума або унаслідок переміжності, в залежності від значень параметрів системи.