

Weak turbulence of Kelvin waves in superfluid He

Victor S. L'vov^{1,3} and Sergey Nazarenko²

¹*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*

²*Mathematics Institute, Warwick University, Coventry, CV4 7AL, UK*

³*Theoretical Department, Institute for Magnetism, National Academy of Sciences of Ukraine, Kiev, Ukraine*

E-mail: victor.lvov@weizmann.ac.il

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Physics of small-scale quantum turbulence in superfluids is essentially based on the knowledge of the energy spectrum of Kelvin waves, E_k . In our paper, we derive a new type of kinetic equation for Kelvin waves on quantized vortex filaments with random large-scale curvature which describes a step-by-step energy cascade over scales caused by five-wave interactions. This approach replaces the previously used six-wave theory, which was recently shown to be inconsistent due to nonlocality. Solving the four-wave kinetic equation, we found a new *local* spectrum with a universal (curvature-independent) exponent, $E_k \propto k^{-5/3}$, which must replace the nonlocal spectrum of the six-wave theory, $E_k \propto k^{-7/5}$ in future theory, e.g., in finding the quantum turbulence decay rate, found by Kosik and Svistunov under wrong assumption of the locality of energy transfer in the six-wave interactions.

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1. Physical background

Turbulence in superfluids [1,2] is one of most fascinating natural phenomena where transition from the laws of classical physics to the quantum laws occurs gradually as energy passes from large to small scales along the turbulent cascade. Such a coexistence of the classical and quantum physics in the same system and their interplay is a fundamental consequence of absence of viscosity, the role of which in classical turbulence is to quench the energy cascades at scales which are still large enough to be classical. In superfluids, on the other hand, when temperature is close to the absolute zero, such quenching mechanism is absent, and the energy flux unavoidably reaches the scales where the quantization of the vortex circulation (discovered by Feynman [3]) is essential. Recently, there have been significant advances in experimental techniques allowing studies of turbulence in various systems such as ³He [4,5], ⁴He [6,7] and Bose–Einstein condensates of supercold atoms [8,9]. Often, experimental devices are not small enough to probe the transitional and quantum scales directly. For this reason, an impressive progress in numerical simulations [10,11] is very important because they give access to characteristics of turbulence yet unavailable ex-

perimentally. In zero-temperature limit, one of the most interesting questions is the nature of the energy dissipation, namely the mechanisms of transfer the energy down to the tiny (almost atomic) scales where vortices can radiate their energy away by emitting phonons.

A commonly accepted model of superfluid turbulence comprises a randomly moving tangle of quantized vortex lines which can be characterized by the mean intervortex distance ℓ and the vortex core radius $a \ll \ell$. The vortex core radius has an atomic size and the conventional description used for fluid media fails within such a core. There are two approaches to deal with the vortex core. First one is a «microscopic» model in which the core is resolved: it is based on the Gross–Pitaevski equation,

$$\frac{\partial \Psi}{\partial t} + \nabla^2 \Psi - \Psi |\Psi|^2 = 0, \quad (1)$$

where Ψ is so-called condensate wave function. This model is systematically derived for the Bose–Einstein condensates in super-cold atoms, but not for the liquid helium. Nevertheless, it is frequently used for describing superfluid flows in helium because it contains several essential features of such superfluids, i.e., vortex quantization, acoustic

waves (phonons) in presence of a condensate, and it describes a gradual (nonsingular) reconnection of vortex lines.

However, the Gross–Pitaevski equation can be costly to study, and one often resorts to using so-called Biot–Savart formulation in the Euler equations for the ideal classical fluids, exploiting the fact that far away from the vortex cores the Gross–Pitaevski dynamics is isomorphic to the ideal classical flow via the Madelung transformation. In the Biot–Savart model, the vortices are postulated via a cutoff in the equations for the vortex line elements. Namely, the equations used are

$$\dot{\mathbf{r}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3}, \quad (2)$$

with a cutoff at the core radius a , i.e., integrating over the range $|\mathbf{r} - \mathbf{s}| > a$. Here κ is the circulation quantum. In what follows, we will base on the Biot–Savart model.

Naturally, at scales $L \gg \ell$ the discreteness is unimportant and they can be described classically with the energy flux toward smaller scales by the celebrated Richardson–Kolmogorov cascade. Then the energy is transferred through the crossover scale ℓ by some complicated mechanisms [17–19], thereby exciting smaller scales $\ell < \lambda < a$ which propagate along the individual vortex filaments as waves. These were predicted by Lord Kelvin more than one century ago [12] and experimentally observed in superfluid ^4He about 50 years ago. It is believed that Kelvin waves (KWs) play a crucial role in superfluid dynamics, transferring energy from ℓ to a much smaller scale, where it can dissipate via emission of bulk phonons. In a wide range of scales KWs are weakly nonlinear and can be treated within the theory of weak-wave turbulence [16]. Such an approach for KWs was initiated in [13] where a six-wave kinetic equation (KE) was presented, and a KW spectrum was obtained from this equation based on a dimensional analysis, $E_{KS}(k) \propto k^{-7/5}$. Dimensional analysis of the KE is based on the *assumption* that all integrals in the collision term are converges. Physically it means that the energy transfer over scales can be considered as step-by-step cascade, in which energy to a given range of wavevectors k comes from the smaller k' of the same order of magnitude and is transferred toward larger k'' , again of the order of k . This assumption, firstly suggested in 1941 by Kolmogorov for hydrodynamical turbulence is often called «locality of the energy transfer». Spectrum $E_{KS}(k) \propto k^{-7/5}$ was subsequently used in theoretical constructions in superfluid turbulence, e.g., to describe the classical-quantum crossover range of scales and to explain the dissipation rate in the superfluid turbulence [17–20]. However, it was recently shown in [14] that this spectrum is nonlocal and, therefore, nonrealizable. This crucial locality check was only possible after a highly nontrivial calculation of the six-wave interaction coefficient done in Ref. 14 which took into account previously omitted impor-

tant contributions and which yielded explicit relations for this coefficient in relevant asymptotical limits.

In this paper, we exploit the consequences of the nonlocality of the 6-wave theory, and replace the latter with a new local 5-wave theory of KW turbulence. Our 5-wave theory arises from the 6-wave theory (completed in [14]) in the strongly nonlocal case, when one of the waves in the sextet is much longer than the other five and corresponds to the outer scale — infra-red (IR) cutoff. We derive a new spectrum of the KW turbulence which is local, and which must be used in future for revising the parts of the superfluid turbulence where the nonlocal spectrum of the 6-wave theory has previously been used.

2. On statistical description of weak-wave turbulence

Weak-wave turbulence refers to a class of strongly non-equilibrium statistical systems consisting of a large number of excited weakly nonlinear waves in nondissipative (Hamiltonian) dispersive media [16]. Such systems comprise a unique example where strongly nonequilibrium statistics can be addressed systematically, and states analogous to Kolmogorov–Richardson cascades of classical turbulence can be obtained analytically. Let us briefly overview the theory of weak-wave turbulence with application to the five- and six-wave systems (three- and four-wave processes are absent for KWs) starting from a classical Hamiltonian equation for the complex canonical amplitude of waves $a_{\mathbf{k}} \equiv a(\mathbf{k}, t)$ and $a_{\mathbf{k}}^*$ (classical analogues of the Bose-operators of particle creations and annihilation) with a wavevector \mathbf{k} :

$$i \frac{\partial a_{\mathbf{k}}}{\partial t} = \frac{\delta \mathcal{H}}{\delta a_{\mathbf{k}}^*}. \quad (3)$$

Here \mathcal{H} is a Hamiltonian which for the wave systems is

$$\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}; \quad \mathcal{H}_{\text{free}} = \int \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* d\mathbf{k}, \quad (4)$$

where $\omega_{\mathbf{k}}$ is the wave frequency. For KWs $\omega_{\mathbf{k}} = \Lambda \kappa k^2 / 4\pi$ where κ is the circulation quantum. \mathcal{H}_{int} is an effective interaction Hamiltonian for KWs, propagating along straight vortex line, that is equal to

$$\mathcal{H}_{1 \leftrightarrow 3} = \frac{1}{6} \int d\mathbf{k}_1 \dots d\mathbf{k}_4 \delta_{\mathbf{k}_1}^{2,3,4} [V_{\mathbf{k}_1}^{2,3,4} a_1 a_2^* a_3^* a_4^* + \text{c.c.}] \quad (5)$$

for four-wave systems or

$$\mathcal{H}_{3 \leftrightarrow 3} = \frac{1}{36} \int d\mathbf{k}_1 \dots d\mathbf{k}_6 \delta_{\mathbf{k}_1}^{4,5,6} W_{\mathbf{k}_1}^{4,5,6} a_1 a_2 a_3 a_4^* a_5^* a_6^* \quad (6)$$

for six-wave systems. Here we use shorthand notations: $a_j \equiv a_{\mathbf{k}_j}$, $k_j = |\mathbf{k}_j|$, $\delta_{1,2}^{3,4,5} \equiv \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5)$ and $\delta_{1,2,3}^{4,5,6} \equiv \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6)$.

These equations effectively describe weakly nonlinear waves of any nature [16], using *only* relevant dynamical

information, that presents in the system Hamiltonian \mathcal{H} . The main technical problem is to find \mathcal{H} for a particular complicated physical system. Fortunately, in case of KWs this cumbersome job was done for us and for general reader in Ref. 14.

Statistical description of weakly interacting waves can be reached [16] in terms of the KE

$$\frac{\partial n(\mathbf{k}, t)}{\partial t} = \text{St}(\mathbf{k}, t), \quad (7)$$

for the waveaction spectrum $n(\mathbf{k}, t)$, defined by

$$\langle a(\mathbf{k}, t) a^*(\mathbf{k}', t) \rangle = n(\mathbf{k}, t) \delta(\mathbf{k} - \mathbf{k}'),$$

where $\langle \dots \rangle$ stands for the ensemble averaging. The collision integral $\text{St}(\mathbf{k}, t)$ can be found in various ways [16], including the Golden Rule widely used in quantum mechanics. For the five- and six-wave processes we have, respectively,

$$\begin{aligned} \text{St}_{1\leftrightarrow 3} = & \frac{\pi}{12} \int d\mathbf{k}_1 \dots d\mathbf{k}_3 \left\{ |V_{\mathbf{k}}^{1,2,3}|^2 \delta_{\mathbf{k}}^{1,2,3} \mathcal{N}_{\mathbf{k}}^{1,2,3} \times \right. \\ & \times \delta(\omega_{\mathbf{k}} - \omega_1 - \omega_2 - \omega_3) + \\ & \left. + 3 |V_1^{\mathbf{k},2,3}|^2 \delta_1^{\mathbf{k},2,3} \mathcal{N}_1^{\mathbf{k},2,3} \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_2 - \omega_3) \right\}, \quad (8) \end{aligned}$$

$$\mathcal{N}_1^{2,3,4} \equiv n_1 n_2 n_3 n_4 (n_1^{-1} - n_2^{-1} - n_3^{-1} - n_4^{-1});$$

$$\begin{aligned} \text{St}_{3\leftrightarrow 3} = & \frac{\pi}{12} \int d\mathbf{k}_1 \dots d\mathbf{k}_5 |W_{\mathbf{k},1,3}^{4,5,6}|^2 \delta_{\mathbf{k},1,3}^{4,5,6} \times \\ & \times \delta(\omega_{\mathbf{k}} + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5) n_{\mathbf{k}} n_1 n_2 n_3 n_4 n_5 \times \\ & \times (n_1^{-1} + n_2^{-1} + n_3^{-1} - n_4^{-1} - n_5^{-1} - n_6^{-1}). \quad (9) \end{aligned}$$

Scaling solutions of these KE's can be found under two conditions satisfied for various wave systems, e.g., gravity and capillary waves on the fluid surface, Langmuir and ion-sound waves in plasma, etc. [16].

Scale-invariance of the wave system, when the frequency of waves and the interaction coefficients are homogeneous functions of wave vectors:

$$\begin{aligned} \omega(\lambda k) &= \lambda^{\alpha_2} \omega(k), \quad V(\lambda \mathbf{k}_1, \lambda \mathbf{k}_2; \lambda \mathbf{k}_3, \lambda \mathbf{k}_4, \lambda \mathbf{k}_5) \equiv \\ & \equiv \lambda^{\alpha_5} V(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5), \end{aligned}$$

and a similar relationship for $W_{1,2,3}^{4,5,6}$ with an index α_6 .

Interaction locality, in a sense that the main contribution to the energy balance of a given k -wave (with wavevector k) originates from its interaction with k' -waves with $k' \sim k$. Mathematically it means that all integrals over k_1, k_2 , etc. in the KE's (7)–(9) *converge*, and therefore in the scale-invariant case the leading contribution to the collision integral indeed originates from the regions $k_2 \sim k, k_3 \sim k$, etc. Note that nonlocal spectra are not

solutions of the KE's (7)–(9) and, therefore, physically irrelevant.

In the *scale-invariant* wave systems one seeks for the *scale-invariant* solutions of the KE's:

$$n(k) = A k^{-x}, \quad (10)$$

where A is a dimensional number. To find the scaling index x for turbulent spectra with a constant energy flux over scales, we note that all KE's (7)–(9) conserve the total energy of the wave system,

$$\frac{dE}{dt} = 0, \quad E \equiv \int E_{\mathbf{k}} d\mathbf{k}, \quad E_{\mathbf{k}} \equiv \omega_{\mathbf{k}} n_{\mathbf{k}}.$$

Therefore the \mathbf{k} -space energy density, $E_{\mathbf{k}}$, satisfies a continuity equation:

$$\frac{\partial E_{\mathbf{k}}}{\partial t} + \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k} = 0.$$

Here ε_k is the energy flux over scales, expressed via an integral over sphere of radius k :

$$\varepsilon_k = \int_{k' < k} d\mathbf{k}' \omega_{k'} \text{St}(k', t).$$

Under the assumption of the interaction locality, one estimates the d -dimensional integral $\int d\mathbf{k}$ as k^d , the interaction coefficients

$$\begin{aligned} V_1^{2,3,4} &\sim V_k^{k,k,k} \sim V k^{\alpha_4}, \\ W_{1,2,3}^{4,5,6} &\sim W_{k,k,k}^{k,k,k} \sim W k^{\alpha_6}, \quad (11) \end{aligned}$$

and $n_k = A_p k^{-x_p}$ (for the p -wave interactions). Therefore:

$$\varepsilon_k \sim k^{4d} (V k^{\alpha_5})^2 (A_5 k^{-x_5})^4, \quad 2 \leftrightarrow 3 \text{ scattering};$$

$$\varepsilon_k \sim k^{5d} (W k^{\alpha_6})^2 (A_6 k^{-x_6})^5, \quad 3 \leftrightarrow 3 \text{ scattering}. \quad (12)$$

For the spectra of turbulence with a constant energy flux $\varepsilon_k = \varepsilon = \text{const}$, i.e., $\varepsilon_k \propto k^0$. For the p -wave process this gives the scaling exponent of $n(k)$, x_p , and the energy scaling exponent y_p , $E(k) \propto k^{-y_p}$:

$$x_p = d + \frac{2\alpha_p}{p-1}, \quad y_p = x_p - \alpha_2. \quad (13)$$

In fact, these expressions are valid for any $p > 2$. For the three- and the four-wave processes (with $p = 3$ and $p = 4$) this gives the well-known results, see, e.g., Ref. 16. Note however, that the 4-wave $1 \leftrightarrow 3$ is considered here for the first time, and it is different from the previously considered standard $2 \leftrightarrow 2$ processes.

3. Kelvin-wave turbulence with six-wave interaction

To consider the KW system, one has to start with the Biot–Savart equations (2), consider an equilibrium state corresponding to an infinitely long straight vortex line and perturb it with small angle disturbances. This will correspond to a setup of weakly nonlinear KWs which are dispersive, and for description of which the weak-wave turbulence theory can be used. For this, one has to parametrize the transverse displacement vector of the perturbed like by the distance along the unperturbed line, pass to Fourier space and expand in small perturbation angles and small parameter $1/\Lambda$, where $\Lambda = \ln(\ell/a) \gg 1$. Such expansion in two small parameters is not easy. This is because in the leading order in $1/\Lambda$ the model is integrable, i.e., noncascading, and to describe the leading order of the energy transfers one has to go to next order in $1/\Lambda$. Second difficulty is that the lowest order process, the four-wave resonances, are absent for such one dimensional systems with concave up dispersion relation. Thus one has to go to the next order in the small nonlinearity too. Combination of these two facts makes finding of the effective interaction Hamiltonian \mathcal{H}_{int} for KWs a hard task. For the six-wave process, which assumes that the underlying vortex is perfectly straight, this task was accomplished only recently [14]. Effective $3\leftrightarrow 3$ -interaction coefficient W was shown to have a form

$$W_{1,2,3}^{4,5,6} = -\frac{3}{4\pi\kappa} k_1 k_2 k_3 k_4 k_5 k_6 F_{1,2,3}^{4,5,6}, \quad (14)$$

where F is a nonsingular dimensionless function of k_1, \dots, k_6 , close to unity in the relevant region of its arguments. In particular, $F \rightarrow 1$ when one or several k 's are much less than the maximum wavenumber in the sextet.

Equations (3), (6) and (14) provide general reader with *all* necessary information about KWs required for further developments in this paper. Those interested in further details about the derivations of these equations can find them in [14].

Notice that the form of Eq. (14) could be expected because it demonstrates a very simple physical fact: long KWs (with small k 's) can contribute to the energy of a vortex line only when they produce curvature. The curvature, in turn, is proportional to wave amplitude a_k and, at fixed amplitude, is inversely proportional to their wave-length, i.e., $\propto k$. Therefore in the effective motion equation each a_j has to be accompanied by k_j , if $k_j \ll k$. Exactly this statement is reflected by Eq. (14). One can say, that cumbersome calculations [14] support these reasoning, and additionally provide with explicit numerical factor $-3/4\pi$ and give an explicit expression for F which can be important in further research, required for careful comparison with future experiments or numerics.

Equation (14) estimates $W_{1,2,3}^{4,5,6}$ as Wk^6 . Thus, Eq. (12) reproduces the Kozik–Svistunov (KS) scaling for the $3\leftrightarrow 3$

processes, which for further discussion is written with a dimensionless constant C_{KS} :

$$n_{KS} = \frac{C_{KS} \kappa^{2/5} \epsilon^{1/5}}{k^{17/5}}, \quad E_{KS} = \Lambda \frac{C_{KS} \kappa^{7/5} \epsilon^{1/5}}{k^{7/5}}. \quad (15)$$

We repeat that KS spectrum (15) would be valid only if it was local, i.e., if all integrals (9) converged and thus one could estimate $St_{3\leftrightarrow 3}(k)$ as in Eq. (12). However, detailed analysis (given in Ref. 14 and shortly reproduced below) shows the KS spectrum is nonlocal and therefore physically nonrealizable. In order to find the valid spectrum of turbulent KW we will briefly reproduce this analysis, using the Eq. (14).

4. Nonlocality of the energy transfer with the six-wave interactions

Let us now check if the KS spectrum (15) is local or not. For this, we consider the $3\leftrightarrow 3$ collision term (9) for KW with the interaction amplitude $\mathcal{W}_{1,2,3}^{4,5,6}$ as in (14) and $n(k)$ as in Eq. (10). In this case $\int d\mathbf{k}_j$ are one-dimensional integrals $\int_{-\infty}^{\infty} dk_j$. In the IR region $k_1 \ll k, k_j$, $j = 2, 3, 4, 5$, we have $F \approx 1$ and the integral over k_1 scales as

$$\Psi \equiv \frac{2}{\kappa} \int_{1/\ell}^{\infty} k_1^2 n(k_1) dk_1 = \frac{2A}{\kappa} \int_{1/\ell}^{\infty} k_1^{2-x} dk_1. \quad (16)$$

Lower limit 0 in Eq. (16) is replaced by $1/\ell$, where ℓ is the mean inter-vortex separation ℓ , at which approximation of noninteracting vortex lines fails and one expects a cutoff of the power like behavior (10). Prefactor 2 in Eq. (16) reflects the fact that the ranges of positive and negative \mathbf{k}_1 give equal contributions, and factor $1/\kappa$ is introduced to make parameter Ψ dimensionless. Ψ has a meaning of the mean-square angle of the deviation of the vortex lines from straight. Therefore $\Psi \lesssim 1$; for highly polarized vortex lines $\Psi \ll 1$.

Clearly, integral (16) IR-diverges if $x > 3$, which is the case for the KS spectrum (15) with $x_6 = 17/5$. Note that all the similar integrals over \mathbf{k}_2 , \mathbf{k}_3 , \mathbf{k}_4 , and \mathbf{k}_5 in Eq. (9) also diverge exactly in the same manner as integral (16). Moreover, when two of the wavenumbers belonging to the same side in the sextet tend to zero simultaneously then each of such wavenumbers will yield an integral as in (16), and the net result will be the product of these integrals, i.e., a stronger singularity than in the case of just one small wavenumber. On the other hand, small wavenumbers which are on the opposite sides of the resonant sextet do not lead to a stronger divergence because of an extra smallness arising in this case in Eq. (9) from $n_1^{-1} + n_2^{-1} + n_3^{-1} - n_4^{-1} - n_5^{-1} - n_6^{-1}$.

Divergence of the integrals in Eq. (9) means that KS spectrum (15) is not a solution of the KE (9) and thus non-realizable. One should find another, self-consistent solution of this KE.

5. Effective four-wave theory of KW turbulence

Nonlocality of the six-wave theory is a serious problem. It indicates that dominant sextets contributing to the $3 \leftrightarrow 3$ -scattering are those for which two of the wavenumbers from the same side of the six-wave resonance conditions

$$\begin{aligned} \omega_k + \omega_1 + \omega_2 &= \omega_3 + \omega_4 + \omega_5, \\ \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5, \end{aligned} \quad (17)$$

are very small, $k_j \lesssim 1/\ell$. Thus these equations effectively become

$$\begin{aligned} \mathbf{k} &= \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3, \quad \mathbf{k}_1 = \mathbf{k} + \mathbf{k}_2 + \mathbf{k}_3, \\ \mathbf{k}_2 &= \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_3, \quad \text{or} \quad \mathbf{k}_3 = \mathbf{k} + \mathbf{k}_2 + \mathbf{k}_1, \end{aligned} \quad (18)$$

and respective conditions for the frequencies, which implies a 4-wave process of the $1 \leftrightarrow 3$ type. In the other words, one can interpret such nonlocal sextets on straight vortex lines as quartets on curved vortices, with the slowest modes in the sextet responsible for the large-scale curvature R of the underlying vortex line in the 4-wave approach.

To derive an effective 4-wave KE, let us start with the 6-wave collision integral (9) and find the leading contributions to it when the spectrum n_k is steeper than k^{-3} in the IR region. There are four of them. The first one originates from the region where k_1 and k_2 are much smaller than the rest of k_j 's. The three other contributions originate from the other side of the sextet: regions where either k_3 and k_4 , or k_3 and k_5 , or k_4 and k_5 are small. These contributions are equal and we may find only one of them and multiply the result by three. Notably, the sum of the four contributions can be written exactly in the form of the $1 \leftrightarrow 3$ -collision term (8) with the effective $1 \leftrightarrow 3$ -interaction amplitude

$$V_1^{2,3,4} = -3\Psi \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 / (4\pi\sqrt{2}), \quad (19)$$

because, as shown in Ref. 14,

$$\lim_{k_1 \rightarrow 0} F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 | \mathbf{k}_4 \mathbf{k}_5, \mathbf{k}_6) = 1.$$

Deriving Eq. (8) with $V_1^{2,3,4}$, Eq. (19), we took only leading contributions in the respective IR regions, factorized the integrals over these wavevectors like in Eq. (16) and took only the zeroth order terms with respect to the small wavevectors (by putting these wavenumbers to zero) in the rest of the expression (9).

Equation (8) with $V_1^{2,3,4}$ as in Eq. (19) is an effective 4-wave KE, which we were aiming to obtain. This KE corresponds to interacting quartets of KWs propagating along a vortex line having a random large-scale curvature $R \lesssim \ell$. Equation (19) estimates $V_1^{2,3,4}$ as $\mathcal{V}k^4$ with $\mathcal{V} \sim \Psi$. Using this scaling in Eq. (12), we arrive at a spectrum for the $1 \leftrightarrow 3$ processes with scaling exponents $x_4 = 11/3$ and $y_5 = 5/3$,

$$n_{LN} = \frac{C_{LN} \varepsilon^{1/3}}{\Psi^{2/3} k^{11/3}} \Rightarrow E_{LN} = \frac{C_{LN} \Lambda \kappa \varepsilon^{1/3}}{\Psi^{2/3} k^{5/3}}, \quad (20)$$

Local ($1 \leftrightarrow 3$) L'vov–Nazarenko (LN) spectrum.

6. Local step-by-step energy transfer with four-wave $1 \leftrightarrow 3$ interactions

Mathematically, locality of the energy transfer in the $1 \leftrightarrow 3$ -wave processes means convergence of the multi-dimensional integral in the corresponding collision term (8). Here we will show that proof of convergence in Eq. (8) is a delicate issue and cannot be done only on the basis of power counting because the latter would give a divergent answer.

6.1. Proof of the infrared convergence

Let us show that in the IR region, when at least one of the wave vectors, say k_2 , is much smaller than k , only a quadrupole cancelation of the largest, next to the largest and the two further sub-leading contributions appear to result in the final, convergent result for the collision term (8).

Three integrations in Eq. (8) are restricted by two conservation laws, namely by

$$1 \rightarrow 3: \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3, \quad k^2 = k_1^2 + k_2^2 + k_3^2 \quad (21)$$

in the first term, and by

$$3 \rightarrow 1: \quad \mathbf{k} + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_1, \quad k^2 + k_2^2 + k_3^2 = k_1^2 \quad (22)$$

in the second term. Therefore, only one integration, say with respect to \mathbf{k}_2 , remains in each term.

In the IR region $k_2 \ll k_1 \ll k$, we find from Eqs. (21), (22) for the $1 \rightarrow 3$ and the $3 \rightarrow 1$ terms:

$$\begin{aligned} 1 \rightarrow 3: \quad \mathbf{k}_1 &= \mathbf{k} - \frac{k_2^2}{\mathbf{k}_1 + \mathbf{k}_2} \approx \mathbf{k} - \frac{k_2^2}{\mathbf{k}}, \\ 1 \rightarrow 3: \quad \mathbf{k}_3 &= -\frac{\mathbf{k}_1 \mathbf{k}_2}{\mathbf{k}_1 + \mathbf{k}_2} \approx -\mathbf{k}_2, \end{aligned} \quad (23)$$

$$\begin{aligned} 3 \rightarrow 1: \quad \mathbf{k}_1 &= \mathbf{k} + \frac{k_2^2}{\mathbf{k} + \mathbf{k}_2} \approx \mathbf{k} + \frac{k_2^2}{\mathbf{k}}, \\ 3 \rightarrow 1: \quad \mathbf{k}_3 &= -\frac{\mathbf{k} \mathbf{k}_2}{\mathbf{k} + \mathbf{k}_2} \approx -\mathbf{k}_2. \end{aligned} \quad (24)$$

These equations demonstrate three important facts:

- 1) in both cases in the leading order $k_3 \simeq k_2$, i.e., when $k_2 \ll k$ then k_3 is small as well;
- 2) the difference between \mathbf{k}_1 and \mathbf{k} is of the second order in small k_2 : $|\mathbf{k}_1 - \mathbf{k}| \simeq k_2^2/k$;
- 3) these leading contributions to $\mathbf{k}_1 - \mathbf{k}$ have the same modulus and different sign in the $1 \leftrightarrow 3$ term and in the $3 \leftrightarrow 1$ term.

Therefore in the leading order the expressions for \mathcal{N} in Eq. (8) can be written as

$$\mathcal{N}_k^{1,2,3} \simeq -x(k_2/k)^2 n_k n_2 n_3 \simeq -\frac{x A^3}{k^{(x+2)}} k_2^{2(1-x)}, \quad (25)$$

$$\mathcal{N}_1^{k,2,3} \simeq +x(k_2/k)^2 n_k n_2 n_3 \simeq +\frac{x A^3}{k^{(x+2)}} k_2^{2(1-x)}, \quad (26)$$

where we substituted n_j from Eq. (10). Importantly, these estimates (in the leading order) have *the same magnitude and different signs*.

Next step is to compute integrals

$$I_{1 \rightarrow 3} \equiv \int d\mathbf{k}_1 d\mathbf{k}_3 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \\ \times \delta(k^2 - k_1^2 - k_2^2 - k_3^2) = \frac{|\mathbf{k} + \mathbf{k}_2|}{2|k^2 + 2\mathbf{k}\mathbf{k}_2 - k_2^2|} \rightarrow \frac{1}{2k}; \quad (27)$$

$$I_{3 \rightarrow 1} \equiv \int d\mathbf{k}_1 d\mathbf{k}_3 \delta(\mathbf{k} + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1) \times \\ \times \delta(k^2 + k_2^2 + k_3^2 - k_1^2) = \frac{1}{2|\mathbf{k} + \mathbf{k}_1|} \rightarrow \frac{1}{2k}, \quad (28)$$

i.e., in the leading order these results coincide and do not contain the smallness.

Now we can find the contributions to $St_{1 \leftrightarrow 3}$, given by Eq. (8), from the region $k_2 \ll k$. According to Eq. (19) we can write $V_k^{1,2,3} = V_1^{k,2,3} = V\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3$. Using our estimates (25), (26) for \mathcal{N} and Eqs. (27), (28) we have

$$1 \rightarrow 3: St_{1 \rightarrow 3}^{k_2 \ll k} \approx -\frac{x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2; \quad (29)$$

$$3 \rightarrow 1: St_{3 \rightarrow 1}^{k_2 \ll k} \approx +\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2. \quad (30)$$

One can see that, in spite of the deep cancelations in the estimates for \mathcal{N} , the integrals (29), (30) diverge if $x \geq 3.5$, which is satisfied for LN-scaling exponent $x = 11/3$.

Nevertheless one has to take into account the following: the $1 \leftrightarrow 3$ contribution to the collision integral has three identical divergent regions: $k_2 \sim k_3 \ll k_1 \approx k$, $k_1 \sim k_3 \ll k_2 \approx k$ and $k_2 \sim k_1 \ll k_3 \approx k$, and Eq. (29) estimates only the first one. Therefore the total contribution is

$$St_{1 \rightarrow 3}^{IR} = 3St_{1 \rightarrow 3}^{k_2 \ll k} \approx -\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2, \quad (31)$$

while the $3 \leftrightarrow 1$ contribution has only one divergent region $\mathbf{k}_1 \approx \mathbf{k}$. Therefore,

$$St_{1 \rightarrow 3}^{IR} = St_{1 \rightarrow 3}^{k_2 \ll k} \approx +\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2, \quad (32)$$

i.e., exactly the same result as in Eq. (29), but with the different sign. Therefore the divergent contributions (29), (30) cancel each other and one has to take into account the next order.

Notice that next order terms in the expansion over $k_2 \ll k$ results in the already convergent integral

$$St_{1 \leftrightarrow 3}^{IR} \propto \int_0^{k_{IR} \ll k} \mathbf{k}_2 k_2^{(6-2x)} d\mathbf{k}_2, \quad (33)$$

with the LN exponent $x = 11/3$. Moreover, typically excitation of KWs is symmetrical in $\mathbf{k} \leftrightarrow -\mathbf{k}$. In this case, this integral has an odd integrand and, therefore, it is equal to zero. Then the leading contribution to the $1 \leftrightarrow 3$ -collision term in the IR region can be summarized as follows:

$$St_{1 \leftrightarrow 3}^{IR} \sim \frac{V^2 A^3}{k^{x+1}} \int_0^{k_{IR} \ll k} k_2^{2(4-x)} d\mathbf{k}_2 \propto k_{IR}^{9-2x}. \quad (34)$$

The IR convergence require: $x < 9/2$.

With LN exponent $x = 9/3$ this gives

$$St_{1 \rightarrow 3}^{IR} \propto k_{IR}^{5/3} \equiv k_{IR} \delta_{IR}.$$

Here we introduce an «IR convergence reserve»: $\delta_{IR} = 5/3$.

6.2. Proof of the ultraviolet convergence

Convergence of the integral (8) in the UV region, when one of the wavevectors, say $k_2 \gg k$, can be established in a similar manner.

Notice first of all that in the $1 \rightarrow 3$ term in Eq. (8), there is no UV region, because by the 2nd of Eq. (21) we have $k_j \leq k$. In the $3 \rightarrow 1$ term to satisfy Eq. (22) in the leading order we can take $\mathbf{k}_2 \simeq \mathbf{k}_1$; $k_2 \geq k_{UV} \gg k$ (case $\mathbf{k}_3 \simeq \mathbf{k}_1$; $k_3 \geq k_{UV}$ gives an identical result). Using parametrization $\mathbf{k}_1 = \mathbf{k} + k_2^2/(\mathbf{k} + \mathbf{k}_2)$, $\mathbf{k}_3 = -\mathbf{k}\mathbf{k}_2/(\mathbf{k} + \mathbf{k}_2)$ (cf. (26)) we get some cancelations in $\mathcal{N}_1^{k,2,3}$ and the leading order result is

$$\mathcal{N}_1^{k,2,3} \propto x(x+1) \left(\frac{k_2}{k} \right)^{-2-x} - \left(\frac{k_2}{k} \right)^{-2x}. \quad (35)$$

Further, similarly to Eqs. (27), (28), one gets $I_{3 \rightarrow 1} \simeq 1/k_2$. As before, the interaction coefficient $V \propto k_2^2$ or $V^2 \propto k_2^4$. Counting the powers of k_2 one gets:

$$\text{St}_{1 \leftrightarrow 3}^{UV} \propto k_{UV}^y, \quad y = \max(-2x+4, -x+2). \quad (36)$$

The UV convergence require: $y < 0 \Rightarrow x > 2$.

One concludes that in the case $x = 11/3$, $\text{St}_{1 \rightarrow 3}^{UV} \propto k_{UV}^{-5/3} \equiv k_{UV}^{-\delta_{UV}}$, where we introduce an «UV convergence reserve» $\delta_{UV} = 5/3$.

6.3. Counterbalanced interaction locality

Notably, $\delta_{IR} = \delta_{UV}$. This equality is not occasional. Observed «counterbalanced» IR-UV locality is a consequence of the scale-invariance of the problem. Indeed, for a given values of $k_{IR} \ll \tilde{k} \ll k_{UV}$ the IR-energy flux $k_{IR} \Rightarrow \tilde{k}$ (from the IR region $k \leq k_{IR}$ toward the region $\sim \tilde{k}$) should scale with (k_{IR}/\tilde{k}) exactly in the same manner as the UV-energy flux $\tilde{k} \Rightarrow k_{UV}$ (from the \tilde{k} -region toward the UV-region $k \leq k_{UV}$) scales with \tilde{k}/k_{UV} . This is because the UV-flux $\tilde{k} \Rightarrow k_{UV}$ from \tilde{k} -region can be considered as the IR flux toward k_{UV} -region. Remembering that the IR-energy flux $k_{IR} \Rightarrow \tilde{k}$ scales like $(k_{IR}/\tilde{k})^{\delta_{IR}}$, while the UV-flux $\tilde{k} \Rightarrow k_{UV}$ is proportional to $(\tilde{k}/k_{UV})^{\delta_{UV}}$, one immediately concludes that δ_{IR} should be equal to δ_{UV} .

The overall conclusion is that the collision term $\text{St}_{1 \leftrightarrow 3}$ is convergent in both the IR and the UV regions for $x = 11/3$ and the energy transfer in the $1 \leftrightarrow 3$ kinetic equation is local.

7. Discussion

— In this paper we have revised the theory of superfluid turbulence in the quantum range of scales where the turbulent cascade is due to nonlinear interaction of weak Kelvin waves on quantized vortex lines. In particular, we have addressed the problem that the previously used KS spectrum is nonlocal, i.e., an invalid mathematically and irrelevant physically solution.

— We have presented a new effective theory of Kelvin wave turbulence consisting of wave quintets interacting on vortex lines with random large-scale curvature. This four-wave theory replaces the nonlocal six-wave theory. We derived an effective four-wave kinetic Eqs. (8), (19), and solved it to obtain a new wave spectrum (20). We proved that this new spectrum is local, and therefore it is a valid solution of the kinetic equation, which should replace the nonlocal (and therefore invalid) Kosik–Svistunov spectrum (15) in the theory of quantum turbulence. In particular, it is now necessary to revise the theory of the classical-quantum crossover scales and its predictions for the turbulence dissipation rate Refs. 17–20. Further, a similar revision is needed for the analysis of laboratory experiments

and numerical simulations of superfluid turbulence, which have been done over the last five years with reliance on the un-physical KS spectrum (15).

— The difference between the LN-exponent $-5/3$ (see (20)) from the KS-exponent $-7/5$ (see (15)) is $4/15$ which is rather small. This may explain why the previous numerical experiments seem to agree with the KS spectrum, obtained numerically in Ref. 15. However, by inspection one can also see that these results also agree with the LN slope.

The different physics results in different expressions for the dimensional pre-factors in the KS and LN spectra, in particular the different dependence on the energy flux ε , as well as an extra dependence on the large-scale behavior (through Ψ) in (20). Careful examination of such pre-factors is necessary in future numerical simulations in order to test the predicted dependencies. Such numerical simulations can be done efficiently with the local nonlinear equation (LNE) suggested in Ref. 14 based on the detailed analysis of the nonlinear KW interactions:

$$i \frac{\partial \tilde{w}}{\partial t} + \frac{\kappa}{4\pi} \frac{\partial}{\partial z} \left\{ \left[\Lambda - \frac{1}{4} \left| \frac{\partial \tilde{w}}{\partial z} \right|^4 \right] \frac{\partial \tilde{w}}{\partial z} \right\} = 0. \quad (37)$$

The LNE model is similar but not identical to the Truncated LIA model of [15] (these models become asymptotically identical for weak KWs).

— Both pre-factors in the KS spectrum (15) and in the LN spectrum (20) contain very different numerical constants C : an order one constant in LN ($C_{LN} \sim 1$, yet to be found) and a zero constant in KS ($C_{KS} \equiv 0$ as a formal consequence of its nonlocality). Also we should note a mysterious very small numerical factor 10^{-5} in formula (16) for the energy flux in Ref. 13, that has no physical justification. Actually, nonlocality of the energy transfer over scales means that this number should be very large, rather than very small. This emphasizes the confusion, and highlights the need for numerical re-evaluation of the spectrum's prefactor.

To conclude comparison between KS and LN approach notice that the drastic difference in the numerical pre-factors constitutes an important difference between the KS and the LN spectra for a practical analysis of experimental data, while the difference between the underlying physics of the local and nonlocal energy cascades, that results in the difference between spectral indices, is important from fundamental, theoretical viewpoint.

— In this work, the effective local five-wave kinetic equation was derived from the six-wave kinetic equation by exploiting nonlocality of the latter. Strictly speaking, this derivation is valid only when the six-wave kinetic equation is valid, i.e., when all the scales are weakly nonlinear, including the ones at the infra-red cutoff. However, the resulting five-wave kinetic equation is likely to be applicable more widely, when only the small scales, and not the large scales, are weak. A similar picture was previously

observed for the nonlocal turbulence of Rossby/drift waves in Ref. 22 and for nonlocal MHD turbulence in Ref. 23. In future we plan to attempt derivation of the five-wave kinetic equation directly from the dynamical equations for the Kelvin waves, which would allow us to extend its applicability to the case with strong large scales.

— Finally we note that the suggested here theory can potentially be useful for other one-dimensional physical systems, including optical fibers, where nonlinear interactions of one-dimensional wave packages becomes important with increase in network capacity.

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