Fizika Nizkikh Temperatur, 2009, v. 35, No. 5, p. 510–513

The relation between fractional statistics and finite bosonic systems in one-dimensional case

Andrij Rovenchak

Department for Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov Str., Lviv UA-79005, Ukraine
E-mail: andrij.rovenchak@gmail.com

Received November 13, 2008, revised December 8, 2008

The equivalence is established between the one-dimensional (1D) Bose-system with a finite number of particles and the system obeying the fractional (intermediate) Gentile statistics, in which the maximum occupation of single-particle energy levels is limited. The system of 1D harmonic oscillators is considered providing the model of harmonically trapped Bose-gas. The results are generalized for the system with power energy spectrum.

PACS: 05.30.Ch Quantum ensemble theory;
05.30.Jp Boson systems;
05.30.Pr Fractional statistics systems (anyons, etc.).

Keywords: finite bosonic systems, traped Bose-gas, fractional statistics.

1. Introduction

The observation of Bose–Einstein condensation (BEC) in ultracold trapped alkali gases [1,2] gave new stimulus to the study of this quantum phenomenon. In particular, the effects of a finite number of particles on BEC of an ideal gas was discussed in theoretical works [3–8], where the corrections to the bulk properties were found.

The aim of the present study is to propose the description of a bosonic system with a finite number of particles by means of finding such a model system, for which the treatment might be mathematically simpler. To some extent, such an approach has common features with finding boson–fermion equivalence in ideal gases [9] or Tonks–Girardeau gas [10] achieved experimentally in 2004 [11,12]. The anyon–fermion mapping is also known in the application to ultracold gases [13].

Haldane's exclusion statistics [14] was considered by Bergère [15]. The connection of the exclusion (anyon) statistics parameter and the interaction in one-dimensional systems was studied in Refs. 16–18. Recently, a combinatorial interpretation of exclusion statistics was given by Comtet *et al.* [19].

Another approach is seen in a different type of the fractional statistics, which is formally understood as an intermediate one between Fermi and Bose statistics. Namely, the maximum occupation of a particular energy level is limited to M, with $M=1$ corresponding to the fermionic distribution and $M=\infty$ being the bosonic one, respectively. This statistics is known as the Gentile statistics [20–22]. If the relation between the number of particles N in the real system and the parameter M in the model one can be found, the stated problem is solved.

For simplicity, a one-dimensional system is considered. The paper is organized as follows. Microcanonical approach for harmonic oscillators with single-particle energy levels given by $\varepsilon_m = \hbar\omega m$ is considered in Sec. 2. The oscillators, unlike classical particles, are *indistinguishable* reproducing thus a quantum case. Physically, this corresponds to bosons trapped in a highly asymmetric harmonic trap. In Sec. 3, the same system is treated within canonical and grand-canonical approaches. Section 4 contains the generalization of obtained results for the system with power energy spectrum $\varepsilon_m \propto m^s$. Short discussion in Sec. 5 concludes the paper.

2. Microcanonical approach

The number of microstates $\Gamma(E)$ in the system of 1D oscillators is the number of ways to distribute the energy $E=\hbar\omega n$ over the (indistinguishable) particles. Such a problem reduces to the problem in number theory known as the partition of an integer [23–25]. An asymptotic expression for (unrestricted) partition is given by the well-known Hardy–Ramanujan formula [26]:

$$p(n) = \frac{1}{4\sqrt{3}\,n} \mathrm{e}^{\pi\sqrt{2/3}\sqrt{n}} . \quad (1)$$

Thus one obtains:

$$\Gamma(E) = \frac{1}{4\sqrt{3}\,E/\hbar\omega} \mathrm{e}^{\pi\sqrt{2/3}\sqrt{E/\hbar\omega}} . \quad (2)$$

Using the entropy $S = \ln\Gamma$ from the definition of the temperature $1/T = dS/dE$ the following equation of state is obtained:

$$E = \frac{\pi^2}{6}\hbar\omega\left(\frac{T}{\hbar\omega}\right)^2 . \quad (3)$$

As the energy E is an extensive quantity, $E \propto N$, where N is the number of particles, the thermodynamic limit $\omega N = \mathrm{const}$ follows immediately from the above equation. The same result also might be obtained from different considerations [27].

If one considers a finite system of bosons or a system of particles obeying fractional statistics the number of ways to distribute the energy $E = \hbar\omega n$ over N particles is the problem of restricted partitions of an integer number n [25]. For convenience, hereafter $\hbar\omega$ is the unit of both energy and temperature.

The expression for the finite system is given by the number of partitions of n into at most N summands and asymptotically equals [28]:

$$\Gamma_{\mathrm{fin}}(n) = \frac{1}{4\sqrt{3}n} \mathrm{e}^{\pi\sqrt{2/3}\sqrt{n}} \exp\left\{-\frac{\sqrt{6}}{\pi}\sqrt{n}\,\mathrm{e}^{-\frac{\pi}{\sqrt{6}}\frac{N}{\sqrt{n}}}\right\} . \quad (4)$$

The result reducing to the fractional statistics was considered by Srivatsan *et al*. [29], it corresponds to the number of partitions of n where every summand appears at most M times:

$$\Gamma_{\mathrm{frac}}(n) = \frac{1}{4\sqrt{3}\,n} \exp\left\{\pi\sqrt{2/3}\sqrt{n}\left(1-\frac{1}{\sqrt{M}}\right)^{1/2}\right\}\left(1-\frac{1}{\sqrt{M}}\right)^{1/2} . \quad (5)$$

As one can see, $\Gamma(E)$ from Eq. (2) constitutes the leading factor both in Eqs. (4) and (5). The respective entropies are

$$S_{\mathrm{fin}} = \ln\Gamma_{\mathrm{fin}} = \ln\Gamma + \Delta S_{\mathrm{fin}} \text{ and } S_{\mathrm{frac}} = \ln\Gamma_{\mathrm{frac}} = \ln\Gamma + \Delta S_{\mathrm{frac}} . \quad (6)$$

Comparing the corrections ΔS_{fin} and ΔS_{frac}, one finds the equivalence condition linking the maximum occupation parameter M and the number of particles N:

$$M \sim \exp\left(\frac{\pi}{\sqrt{6}}\frac{N}{\sqrt{n}}\right).$$

3. Canonical and grand-canonical approach

It is straightforward to show that in the case of the defined fractional statistics the occupation number of the energy level ε equals [20–22]

$$f_M(\varepsilon,\mu,T) = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}-1} - \frac{M+1}{\mathrm{e}^{(M+1)(\varepsilon-\mu)/T}-1} , \quad (7)$$

where μ is the chemical potential and T is the temperature.

The chemical potential is related to the number of particles $\mathcal{N}$ as follows:

$$\mathcal{N} = \sum_{j=0}^{\infty} f_M(\varepsilon_j,\mu,T) \quad (8)$$

and energy E equals

$$E = \sum_{j=0}^{\infty} \varepsilon_j f_M(\varepsilon_j,\mu,T) . \quad (9)$$

However, the case of a finite system is much easier to implement in the canonical approach. It is possible to show that the partition function of N indistinguishable 1D oscillators is given by (cf. [24])

$$Z_N = \prod_{j=1}^{N}\left(1-\mathrm{e}^{j/T}\right)^{-1} , \quad (10)$$

from which the energy E_N can be calculated. In the limit of large N the leading term is given by

$$E_N - E_{\mathrm{Bose}} \sim \frac{N\mathrm{e}^{-N/T}}{\mathrm{e}^{1/T}-1} , \quad (11)$$

where E_{Bose} is the energy of an infinite bosonic system.

For the fractional-statistics system the grand-canonical approach is used. The fugacity $z = \mathrm{e}^{\mu/T}$ is represented as $z = z_{\mathrm{Bose}} + \Delta z$ with z_{Bose} satisfying

$$\mathcal{N} = \sum_i \frac{1}{z_{\mathrm{Bose}}^{-1}\,\mathrm{e}^{\varepsilon_i/T}-1} . \quad (12)$$

It is found that $\Delta z \sim 1/M$ in the limit of large M, from which the correction to the energy given by Eq. (9) follows:

$$E_M - E_{\mathrm{Bose}} \sim \frac{1}{M} . \quad (13)$$

Comparing Eqs. (11) and (13) one obtains the following relation between the parameters M and N:

$$M \sim \frac{1}{N}\mathrm{e}^{N/T} . \quad (14)$$

In the exponent, the temperature T is related to the energy level n of (6) via Eq. (3) (with $E = n$). Result (14) thus reproduces the microcanonical one (6) up to the negligible factor of $1/N$ — it must be taken into account that only leading terms were preserved in the logarithms of (4) and (5).

4. Power energy spectrum

In this section, a general power energy spectrum $\varepsilon = am^s$ $(s > 0)$ is considered. By choosing appropriate energy units, one can set the constant $a = 1$. In fact, only $s = 1$ and $s = 2$ cases are realized in real physical systems [29], but other values can effectively occur in some exotic model systems or in the density of states of a system confined by an external potential within a WKB approach.

To obtain $\Gamma_{\text{fin}}(n)$ for arbitrary s it is worth to recall briefly the derivation of the expression for restricted partitions from [25].

Partition function $Z(\beta)$ and the number of microstates $\Gamma(E)$ are related via the Laplace transform:

$$Z(\beta) = \int_0^\infty \Gamma(E)\, \mathrm{e}^{-\beta E}\, dE\,, \quad \Gamma(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} Z(\beta)\, \mathrm{e}^{\beta E}\, d\beta\,. \tag{15}$$

The entropy $S(\beta)$ equals

$$S(\beta) = \beta E + \ln Z(\beta)\,. \tag{16}$$

For energy spectrum $\varepsilon_m = m^s$ the partition function is

$$Z(\beta) = \prod_{m=1}^{\infty} (1 - \mathrm{e}^{-\beta m^s})^{-1}\,. \tag{17}$$

Using the saddle-point method, one can evaluate $\Gamma(E)$ (15) as follows:

$$\Gamma(E) = \frac{\exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}}\,. \tag{18}$$

The entropy $S(\beta)$, after applying the Euler–Maclaurin summation formula, can be expressed in such a form

$$S(\beta) = \beta E - \sum_{m=1}^{\infty} \ln\left(1 - \mathrm{e}^{-\beta m^s}\right) = \beta E + \frac{C(s)}{\beta^{1/s}} + \frac{1}{2}\ln\beta + \ldots, \tag{19}$$

where

$$C(s) = \Gamma\left(1 + \frac{1}{s}\right)\zeta\left(1 + \frac{1}{s}\right), \tag{20}$$

$\Gamma(z)$ and $\zeta(z)$ being Euler's gamma-function and Riemann's zeta-function, respectively.

The stationary point β_0 is

$$\beta_0 = \left(\frac{C(s)}{sE}\right)^{s/(s+1)} = \lambda_s E^{-s/(s+1)}. \tag{21}$$

Thus, the number of microstates is

$$\Gamma(E) = \frac{\lambda_s}{(2\pi)^{(s+1)/2}}\sqrt{\frac{s}{s+1}}\, E^{-\frac{3s+1}{2(s+1)}} \exp\left[\lambda_s (s+1) E^{\frac{1}{s+1}}\right]. \tag{22}$$

Substituting E with n one can obtain the well-known Hardy–Ramanujan formula [26] for the number of partitions of an integer n into the sum of sth powers.

When the number of particles N in the system is finite, the correction to the above formula must be found. In this case, the partition function equals

$$\ln Z_N(\beta) = -\sum_{m=1}^{N} \ln\left(1 - \mathrm{e}^{-\beta m^s}\right) \tag{23}$$

and for the entropy one has

$$S_{\text{fin}}(\beta) = \beta E - \sum_{m=1}^{N} \ln\left(1 - \mathrm{e}^{-\beta m^s}\right). \tag{24}$$

After simple transformations it is easy to obtain the following:

$$S_{\text{fin}}(\beta) = S(\beta) - \frac{1}{s\beta^{1/s}}\,\Gamma\left(\frac{1}{s}, \beta N^s\right), \tag{25}$$

where $\Gamma(a, x)$ is incomplete Γ-function. Thus,

$$\Gamma_{\text{fin}}(E) = \Gamma(E)\exp\left[-\frac{1}{s\beta_0^{1/s}}\,\Gamma\left(\frac{1}{s}, \beta_0 N^s\right)\right]. \tag{26}$$

Applying the asymptotic expansion for $\Gamma(a, x)$ [30, Eq. (6.5.32)], we finally arrive at the following:

$$\Gamma_{\text{fin}}(E) = \Gamma(E)\exp\left[-\frac{1}{s\beta_0} N^{1-s} \mathrm{e}^{-\beta_0 N^s}\right]. \tag{27}$$

Substituting E with n one obtains the result for restricted partitions

$$\Gamma_{\text{fin}}(n) = \Gamma(n)\exp\left[-\frac{1}{\lambda_s s} n^{s/(s+1)} N^{1-s} \mathrm{e}^{-\lambda_s N^s n^{-s/(s+1)}}\right], \tag{28}$$

cf. also Eq. (17) from [31]. For this function, the notation $p_N^s(n)$ is traditionally used, note, however, that in the problem of integer partitions s must be integer. For $s = 1$ the obtained expression reduces to that of Erdös and Lehner [28], see Eq. (4).

The fractional-statistics result can be directly taken from [29]

$$\Gamma_{\text{frac}}(n) = \frac{\lambda_s}{(2\pi)^{(s+1)/2}}\sqrt{\frac{s}{s+1}}\left(1 - \frac{1}{(M+1)^{1/s}}\right)^{s/(s+1)} \times$$
$$\times\, n^{-\frac{3s+1}{2(s+1)}} \exp\left[\lambda_s\left(1 - \frac{1}{(M+1)^{1/s}}\right)^{s/(s+1)} (s+1) n^{\frac{1}{s+1}}\right]. \tag{29}$$

To obtain the relation between the parameters M and N, one can again consider the entropies $S_{\text{frac}} = \ln \Gamma_{\text{frac}}$ and $S_{\text{fin}} = \ln \Gamma_{\text{fin}}$:

$$S_{\text{frac}} - S = -\frac{1}{s\beta_0} N^{1-s} \mathrm{e}^{-\beta_0 N^s},$$
$$S_{\text{fin}} - S = -\frac{s\lambda_s}{1+s} M^{-1/s}, \tag{30}$$

where $S = \ln \Gamma$.

Dropping the constants, the following result is obtained:

$$M^{1/s} \sim n^{\frac{1-s}{1+s}} N^{s-1} \exp\left\{ \lambda_s n^{-\frac{s}{1+s}} N^s \right\}. \tag{31}$$

It is interesting to find in this general case the connection between energy E and temperature T from the definition $1/T = dS/dE$:

$$\frac{1}{T} = \lambda_s E^{-\frac{s}{1+s}} \Rightarrow E^{\frac{s}{1+s}} = \lambda_s T. \tag{32}$$

Thus, the leading contribution in the relation of M and N (31) is

$$M \sim \exp\left(\frac{sN^s}{T} \right), \tag{33}$$

which is compatible with (14).

5. Discussion

To summarize, the equivalence is established between the finite bosonic system and the system obeying fractional (intermediate) Gentile statistics in the case of one-dimensional harmonic trap. This approach is extended to a general power energy spectrum. While the expressions for two-dimensional (2D) partitions are also known [23], the application to asymmetric (elliptical) traps as well as the generalization for arbitrary 2D systems needs additional study.

Interacting systems are of special interest now. Weak interactions are known not to change the properties of a Bose-system drastically. Thus, one can use, e.g., a slightly modified excitation spectrum [32] and, upon calculating the properties of a model fractional-statistics system, obtain the results for a finite one from the established equivalence.

I am grateful to my colleagues Prof. Volodymyr Tkachuk, Dr. Taras Fityo, and Yuri Krynytskyi for useful discussions and comments regarding the presented material.

1. M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *Science* **269**, 198 (1995).
2. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).
3. W. Ketterle and N.J. van Druten, *Phys. Rev.* **A54**, 656 (1996).
4. W. Deng and P.M. Hui, *Solid State Commun.* **104**, 729 (1997).
5. R. Napolitano, J. DeLuca, V.S. Bagnato, and G.C. Marques, *Phys. Rev.* **A55**, 3954 (1997).
6. R.K. Pathria, *Phys. Rev.* **A58**, 1490 (1998).
7. M. Li, L. Chen, J. Chen, Z. Yan, and C. Chen, *Phys. Rev.* **A60**, 4168 (1999).
8. I.O. Vakarchuk and A.A. Rovenchak, *Condens. Matter Phys.* **4**, 431 (2001).
9. K.R. Patton, M.R. Geller, and M.P. Blencowe, *Physica* **A357**, 427 (2005).
10. M. Girardeau, *J. Math. Phys.* **1**, 516 (1960).
11. B. Peredes, A. Widera, V. Murg, O. Mandel, S. Folling, I. Cirac, G.V. Shlyapnikov, T.W. Hansch, and I. Bloch, *Nature* **429**, 277 (2004).
12. T. Kinoshita, T. Wenger, and D.S. Weiss, *Science* **305**, 1125 (2004).
13. M.D. Girardeau, *Phys. Rev. Lett.* **97**, 100402 (2006).
14. F.D.M. Haldane, *Phys. Rev. Lett.* **67**, 937 (1991).
15. M.C. Bergère, *J. Math. Phys.* **41**, 7252 (2000).
16. S.B. Isakov, *Phys. Rev. Lett.* **73**, 2150 (1994).
17. M.V.N. Murthy and R. Shankar, *Phys. Rev. Lett.* **73**, 3331 (1994).
18. Z.N.C. Ha, *Nucl. Phys.* **B435**, 604 (1995).
19. A. Comtet, S.N. Majumdar, and S. Ouvry, *J. Phys.* **A40**, 11255 (2007).
20. G. Gentile, *Nuovo Cim.* **17**, 493 (1940).
21. A. Isihara, *Statistical Physics*, Academic Press, New York–London (1971).
22. A. Khare, *Fractional Statistics and Quantum Theory*, World Scientific, Singapore (1997).
23. G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Mass. (1976).
24. S. Grossmann and M. Holthaus, *Phys. Rev. Lett.* **79**, 3557 (1997).
25. M.N. Tran, M.V.N. Murthy, and R.J. Bhaduri, *Ann. Phys.* **311**, 204 (2004).
26. G.H. Hardy and S. Ramanujan, *Proc. London Math. Soc.* **17**, 75 (1918).
27. F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
28. P. Erdös and J. Lehner, *Duke Math. J.* **8**, 345 (1941).
29. C.S. Srivatsan, M.V. N. Murthy, and R.K. Bhaduri, *Pramana — J. Phys.* **66**, 485 (2006).
30. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions, Tenth Printing*, National Bureau of Standards (1972).
31. A. Comtet, P. Leboeuf, and S.N. Majumdar, *Phys. Rev. Lett.* **98**, 070404 (2007).
32. A.A. Rovenchak, *J. Low Temp. Phys.* **148**, 411 (2007).