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Toward the theory of the Dirichlet problem for the Beltrami equations

The Dirichlet problem for the degenerate Beltrami equations in arbitrary finitely connected domains is studied. In terms of the tangent dilatations, a series of criteria for the existence of regular solutions in arbitrary simply connected domains, as well as pseudoregular and multivalent solutions in arbitrary finitely connected domains without degenerate boundary components, are formulated.

Keywords: *Beltrami equations, Dirichlet problem, prime ends, regular solutions, simply connected domains, finitely connected domains, pseudoregular and multivalent solutions.*

The purpose of this paper is to give a brief presentation of results in our paper [1] on the Dirichlet problem for the degenerate Beltrami equations in arbitrary bounded finitely connected domains.

Let D be a domain in the complex plane $\mathbb{C}$, and let $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a. e. (almost everywhere) in D. We study the Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z, \tag{1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f with respect to x and y, correspondingly.

The classical Dirichlet problem in a Jordan domain D for the uniformly elliptic Beltrami equation, i. e., when $|\mu(z)| \leqslant k < 1$ a. e., is the problem of the existence of a continuous function $f\colon D \to \mathbb{C}$ such that

$$\begin{cases} f_{\overline{z}} = \mu(z) f_z & \text{for a.e. } z \in D, \\ \lim\limits_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta) & \forall \zeta \in \partial D, \end{cases} \tag{2}$$

for a continuous function $\varphi\colon \partial D \to \mathbb{R}$. It was studied long ago, see, e. g., [2, 3].

The degeneracy of the ellipticity of the Beltrami equation will be controlled by the dilatation coefficient

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}, \tag{3}$$

as well as by the more refined quantity, see, e. g., [4–6],

$$K_\mu^T(z, z_0) = \frac{\left|1 - \dfrac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}, \tag{4}$$

taking not only the modulus of the complex coefficient μ but also its argument into account. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \qquad \forall z \in D \qquad \forall z_0 \in \mathbb{C}. \tag{5}$$

Our research is based on new existence theorems of homeomorphic $W^{1,1}_{\rm loc}$ solutions for the degenerate Beltrami equations in [7] and on the theory of prime ends by Carathéodory, see, e. g., [8]. The boundary behavior of $W^{1,1}_{\rm loc}$ homeomorphic solutions, as well as the Dirichlet problem for the degenerate Beltrami equations in Jordan domains, has been already studied, see, e. g., [5] and references therein.

Let E_D denote the space of prime ends of a domain D in $\mathbb{C}$, and let $\overline{D}_P = D \bigcup E_D$ stand for the completion of the domain D by its prime ends with the topology described in [8], Section 9.5. From now on, the continuity of mappings $f\colon \overline{D}_P \to \overline{D'}_P$ and the boundary functions $\varphi\colon E_D \to \mathbb{R}$ as functions of the prime end P should be understood with respect to the given topology. Now, the boundary condition for the Dirichlet problem is written as

$$\lim_{n\to\infty} \operatorname{Re} f(z_n) = \varphi(P), \tag{6}$$

where the limit is taken over all sequences of points $z_n \in D$ converging to the prime end P.

It was established in [5] that every homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation (1) in a domain $D \subseteq \mathbb{C}$ is the so-called lower Q-homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = K_\mu^T(z, z_0)$, $z \in D$. We established in [1] that it is also the so-called ring Q-homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = K_\mu^T(z, z_0)$, $z \in D$, if $K_\mu \in L^1(D)$ or $K_\mu^T(z, z_0)$ is integrable along the circles $|z - z_0| = r$ for a. e. small enough r at every $z_0 \in \overline{D}$. In other words, the latter means that the homeomorphic $W^{1,1}_{\rm loc}$ solutions of the Beltrami equation (1) satisfy certain inequalities in terms of a conformal modulus for families of curves that is the main geometric tool in the mapping theory.

Then in [1], we developed the theory of the boundary behavior with respect to prime ends for ring Q-homeomorphisms that form a wider class than lower Q-homeomorphisms and, in particular, established far-reaching generalizations of the Carathéodory theorem on a homeomorphic extension of conformal mappings to the boundary in prime ends. This is a basis to develop, in [1], the theory of the boundary behavior with respect to prime ends for generalized homeomorphic solutions to the degenerate Beltrami equation (1). Finally, the latter makes possible to reduce the Dirichlet problem for the degenerate Beltrami equations (1) to the case of analytic and harmonic functions in circular domains.

In what follows, we use the notations $B(z_0, r) := \{z \in \mathbb{C}\colon |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$, $S(z_0, r) := \{z \in \mathbb{C}\colon |z - z_0| = r\}$, $\mathbb{D} := \mathbb{B}(0, 1)$, and $\overline{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$.

1. On BMO and FMO functions. Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of bounded mean oscillation in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\rm loc}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B|\, dxdy < \infty, \tag{7}$$

where the supremum is taken over all disks B in D, and

$$u_B = \frac{1}{|B|} \int\limits_B u(z)\, dxdy.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in paper [9] and soon became an important concept in harmonic analysis, partial differential equations, and related areas.

Following [10], we say that a function $u\colon D \to \mathbb{R}$ has finite mean oscillation at a point $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} |u(z) - \widetilde{u}_\varepsilon(z_0)|\, dxdy < \infty, \tag{8}$$

where

$$\widetilde{u}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} u(z)\, dxdy$$

is the mean value of the function $u(z)$ over the disk $B(z_0, \varepsilon)$ with small $\varepsilon > 0$. We also say that a function $u\colon D \to \mathbb{R}$ is of finite mean oscillation in D, abbr. $u \in$ FMO(D) or simply $u \in$ **FMO**, if (8) holds at every point $z_0 \in D$.

Remark 1. Clearly, BMO $\subset$ FMO. There exist examples showing that FMO is not BMO $_{\mathrm{loc}}$, see, e. g., [7]. By definition, FMO $\subset L^1_{\mathrm{loc}}$, but FMO is not a subset of L^p_{loc} for any $p > 1$ in comparison with BMO $_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$.

Proposition 1. *If, for some collection of numbers $u_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} |u(z) - u_\varepsilon|\, dxdy < \infty, \tag{9}$$

then u is of finite mean oscillation at z_0.

Corollary 1. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} |u(z)|\, dxdy < \infty, \tag{10}$$

then u has finite mean oscillation at z_0.

Remark 2. Note that the function $u(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\mathbb{B}$ and hence also to FMO. However, $\widetilde{u}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (10) is only sufficient but not necessary for a function u to be of finite mean oscillation at z_0.

2. The Dirichlet problem in simply connected domains. Given a continuous function $\varphi(P) \not\equiv \mathrm{const}$, $P \in E_D$, we will say that f is a regular solution of the Dirichlet problem (6) for the Beltrami equation (1) if f is a continuous discrete open mapping $f\colon D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\mathrm{loc}}$ with the Jacobian

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0 \qquad \text{a. e.}, \tag{11}$$

satisfying (1) a. e. and the boundary condition (6) for all prime ends of the domain D.

Recall that a mapping $f\colon D \to \mathbb{C}$ is called discrete if $f^{-1}(y)$ for every point $y \in \mathbb{C}$ consists of isolated points and open if the image of every open set $U \subseteq D$ is open in $\mathbb{C}$.

For $\varphi(P) \equiv c \in \mathbb{R}$, $P \in E_D$, a regular solution of the problem is any constant function $f(z) = c + ic'$, $c' \in \mathbb{R}$.

Theorem 1. *Let D be a bounded simply connected domain in $\mathbb{C}$, and let $\mu\colon D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\mathrm{loc}}$ and such that*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|(z_0,r)} = \infty \qquad \forall z_0 \in \overline{D} \tag{12}$$

for some $0 < \delta(z_0) < d(z_0) = \sup\limits_{z\in D} |z - z_0|$, where

$$\|K_\mu^T\|(z_0,r) := \int\limits_{S(z_0,r)} K_\mu^T(z,z_0)\, ds.$$

Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Here and below, we set that K_μ^T is equal to zero outside of the domain D.

Corollary 2. *Let D be a bounded simply connected domain in $\mathbb{C}$, and let $\mu\colon D \to \mathbb{D}$ be a measurable function such that*

$$k_{z_0}^T(\varepsilon) = O\biggl(\log\frac{1}{\varepsilon}\biggr) \qquad as \qquad \varepsilon \to 0 \qquad \forall z_0 \in \overline{D}, \tag{13}$$

where $k_{z_0}^T(\varepsilon)$ is the average of the function $K_\mu^T(z,z_0)$ over the circle $S(z_0,\varepsilon)$.

Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Remark 3. In particular, the conclusion of Theorem 1 holds if

$$K_\mu^T(z,z_0) = O\biggl(\log\frac{1}{|z-z_0|}\biggr) \qquad \text{as} \qquad z \to z_0 \qquad \forall z_0 \in \overline{D}. \tag{14}$$

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{C}$, let $\mu\colon D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\mathrm{loc}}$, and let*

$$K_\mu^T(z,z_0) \leqslant Q_{z_0}(z) \in \mathrm{FMO}(z_0) \qquad \forall z_0 \in \overline{D}. \tag{15}$$

Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Corollary 3. *Let D be a bounded simply connected domain in $\mathbb{C}$, and let $\mu\colon D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\mathrm{loc}}$ such that*

$$\limsup_{\varepsilon\to 0} \fint\limits_{B(z_0,\varepsilon)} K_\mu^T(z,z_0)\, dm(z) < \infty \qquad \forall z_0 \in \overline{D}. \tag{16}$$

Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Remark 4. In particular, by (5), the conclusion of Theorem 2 holds if

$$K_\mu(z) \leqslant Q(z) \in \mathrm{BMO}(\overline{D}). \tag{17}$$

Theorem 3. *Let D be a bounded simply connected domain in $\mathbb{C}$, and let $\mu\colon D \to \mathbb{D}$ be a measurable function such that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall z_0 \in \overline{D}. \tag{18}$$

Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Remark 5. Here, we are able to give a number of other conditions of logarithmic type. In particular, condition (18) can be replaced by the condition

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\mu^T(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall z_0 \in \overline{D}, \tag{19}$$

and condition (13) can be replaced by the weaker condition

$$k_{z_0}^T(r) = O\left(\log\frac{1}{r}\log\log\frac{1}{r}\right). \tag{20}$$

Theorem 4. *Let D be a bounded simply connected domain in $\mathbb{C}$, let $\mu\colon D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{\mathrm{loc}}$, and let*

$$\int\limits_{D\cap B(z_0,\varepsilon_0)} \Phi_{z_0}(K_\mu^T(z,z_0))\,dm(z) < \infty \qquad \forall z_0 \in \overline{D} \tag{21}$$

for $\varepsilon_0 = \varepsilon(z_0) > 0$ and a nondecreasing convex function $\Phi_{z_0}\colon [0,\infty) \to [0,\infty)$ with

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi_{z_0}^{-1}(\tau)} = \infty \tag{22}$$

for $\delta_0 = \delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

Remark 6. Moreover, it was shown by us that condition (22) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (6) for every Beltrami equation (1) with the integral restrictions (21) and every continuous function $\varphi\colon E_D \to \mathbb{R}$.

Corollary 4. *Let D be a bounded simply connected domain in $\mathbb{C}$, let $\mu\colon D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1_{loc}$, and let*

$$\int\limits_{D\cap B(z_0,\varepsilon_0)} e^{\alpha_0 K_\mu^T(z,z_0)}dm(z) < \infty \qquad \forall z_0 \in \overline{D} \tag{23}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and $\alpha_0 = \alpha(z_0) > 0$. Then the Beltrami equation (1) *has a regular solution f of the Dirichlet problem* (6) *for every continuous function $\varphi\colon E_D \to \mathbb{R}$.*

3. The Dirichlet problem in multiply connected domains. As was probably first noted by B. Bojarski, see, e. g., Sect. 6 of Chapter 4 in [3], the Dirichlet problem for the

Beltrami equations, generally speaking, has no regular solution in the class of functions continuous (single-valued) in $\mathbb{C}$ with generalized derivatives in the case of multiply connected domains D. Hence, the natural question arose: Do solutions exist in wider classes of functions in this case? It turned out that the solutions of this problem can be found in the class of functions admitting a certain number (related to the connectedness of D) of poles at prescribed points. This number should involve the multiplicity of these poles from the Stoilow representation.

A discrete open mapping $f\colon D \to \overline{\mathbb{C}}$ of the Sobolev class $W^{1,1}_{\rm loc}$ (outside of poles) satisfying (1) a. e. and the boundary condition (6) are called a pseudoregular solution of the Dirichlet problem if the Jacobian $J_f(z) \neq 0$ a. e.

It was demonstrated in [1] that, under the same conditions on the complex coefficient μ as in Section 2, in bounded m — connected domains with nondegenerate boundary components, for every prescribed integer $k \geqslant m-1$, the Beltrami equation (1) has a pseudoregular solution f of the Dirichlet problem (6) with k poles at prescribed points in D for every continuous function $\varphi\colon E_D \to \mathbb{R}$.

It was also shown in [1] that, under the same conditions in finitely connected domains, the Dirichlet problem (6) for the Beltrami equations (1) admits multivalent solutions in the spirit of the theory of multivalent analytic functions in addition to pseudoregular solutions.

We say that the discrete open mapping $f\colon B(z_0,\varepsilon_0) \to \mathbb{C}$, where $B(z_0,\varepsilon_0) \subseteq D$, is a local regular solution of Eq. (1) if $f \in W^{1,1}_{\rm loc}$, $J_f(z) \neq 0$, and f satisfies (1) a. e. in $B(z_0,\varepsilon_0)$. The local regular solutions $f\colon B(z_0,\varepsilon_0) \to \mathbb{C}$ and $f_*\colon B(z_*,\varepsilon_*) \to \mathbb{C}$ of Eq. (1) will be called the extensions of each to other if there is a finite chain of solutions $f_i\colon B(z_i,\varepsilon_i) \to \mathbb{C}$, $i = 1,\ldots,m$, such that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) \equiv f_{i+1}(z)$ for $z \in E_i := B(z_i,\varepsilon_i) \bigcap B(z_{i+1},\varepsilon_{i+1}) \neq \varnothing$, $i = 1,\ldots,m-1$. A collection of local regular solutions $f_j\colon B(z_j,\varepsilon_j) \to \mathbb{C}$, $j \in J$, will be called a multivalent solution of Eq. (1) in D if the disks $B(z_j,\varepsilon_j)$ cover the whole domain D, and if f_j are extensions of one to another through the collection, and the collection is maximal by inclusion. A multivalent solution of Eq. (1) will be called a multivalent solution of the Dirichlet problem for a prescribed continuous function $\varphi\colon E_D \to \mathbb{R}$ if $u(z) = \operatorname{Re} f(z) = \operatorname{Re} f_j(z)$, $z \in B(z_j,\varepsilon_j)$, $j \in J$, is a single-valued function in D satisfying the condition $\lim\limits_{z\to P} u(z) = \varphi(P)$ along any ways in D going to $P \in E_D$.

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До теорії задачі Діріхле для рівнянь Бельтрамі

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Вивчається задача Діріхле для вироджених рівнянь Бельтрамі в довільних скінченнозв'язних областях. У термінах дотичних дилатацій сформульовано цілий ряд критеріїв існування регулярних розв'язків цієї проблеми в довільних обмежених однозв'язних областях, а також псевдорегулярних і багатозначних розв'язків в довільних обмежених скінченнозв'язних областях без вироджених граничних компонентів.

Ключові слова: рівняння Бельтрамі, задача Діріхле, прості кінці, регулярні розв'язки, однозв'язні області, скінченнозв'язні області, псевдорегулярні та багатозначні розв'язки.

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К теории задачи Дирихле для уравнений Бельтрами

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Изучается задача Дирихле для вырожденных уравнений Бельтрами в произвольных конечносвязных областях. В терминах касательных дилатаций сформулирован целый ряд критериев существования регулярных решений этой проблемы в произвольных ограниченных односвязных областях, а также псевдорегулярных и многозначных решений в произвольных ограниченных конечносвязных областях без вырожденных граничных компонент.

Ключевые слова: уравнения Бельтрами, задача Дирихле, простые концы, регулярные решения, односвязные области, конечносвязные области, псевдорегулярные и многозначные решения.