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A New Representation of Lagrange's Theorem in Differential Calculus

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Новое представление теоремы Лагранжа в дифференциальном исчислении

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Нове представлення теореми Лагранжа у диференційному численні

It is found a new representation of the mean value Lagrange's theorem in the differential calculus. Any function increment can be expressed through the derivatives in the ending points of a given closed interval. Mean values of the Lagrange derivative and our theory derivative are coincided, but the middle points are different. Our theory allows easily find the middle point and it is not so easy according to Lagrange's theorem. Furthermore, our theory makes it possible to formulate the second mean value theorem in integral calculus, as it is a consequence of differential theorem.
Keywords: technique, theorem, mean value, function, integral, derivative, Lagrange's theorem.

В статье сформулировано новое представление известной теоремы дифференциального исчисления о среднем – теоремы Лагранжа. Приращение функции выражено через производные в концевых точках отрезка. По величине среднее значение производной по Лагранжу и по нашей теории совпадают, однако не совладают средние точки. Наша теория позволяет легко определить среднюю точку, что затруднительно в теореме Лагранжа. Кроме того, наша теория дает возможность сформулировать вторую теорему о среднем в интегральном исчислении, так как она является следствием дифференциальной теоремы.
Ключевые слова: методика, теорема, среднее значение, функция, интеграл, производная, теорема Лагранжа.

У статті сформульовано нове представлення відомої теореми у диференційному численні про середнє – теореми Лагранжа. Прирощення функції представлено через похідні у кінцевих точках відрізку. Середнє значення похідної по Лагранжу і нашої теорії співпадають, але не співпадають середні точки. Наша теорія дозволяє знайти середню точку, що важко зробити на підставі теореми Лагранжа. Крім того, наша теорема дає можливість сформулювати теорему про середнє у інтегральному численні, бо вона просто є наслідком диференціальної теореми.
Ключові слова методика, теорема, середнє значення, функція, інтеграл, похідна, теорема Лагранжа.

Introduction

Lagrange's mean value theorem establishes a relationship between the increment of the function on the interval and its derivative at some midpoint of a given segment. This relationship is given by the formula (2) [1-3]. An important feature of this theorem is an analogue in integral calculus, which is called the first mean value theorem. In the integral calculus there is a second mean value theorem, but there is no such analogue in the differential calculus. This "injustice" can be eliminated, when Lagrange's theorem is formulated in terms of derivatives at the segment endpoints.

1 The Formulation and the Proof of the Main Mean Value Theorem

The Main Mean Value Theorem. Let a function $y = f(x)$ satisfies the following conditions: 1) it is continuous on a segment $[a,b]$, 2) it is differentiable on the interval (a,b) and has a finite one-sided derivatives $f'(a+0)$ and $f'(b-0)$, 3) the derivative $f'(x)$ is monotonic on the interval (a,b). Then there exists a unique point $\xi \in (a,b)$ for which takes place the equality

$$f(b) - f(a) = f'(a)(\xi - a) + f'(b)(b - \xi). \quad (1)$$

Proof. Consider the simplest case, when the derivative does not change its sign on the interval (a,b) (Fig. 1). Figure 1 shows, that by virtue of monotony of the derivative $f'(x)$, the inequality $tgA < tg\alpha < tgB$ takes place. If we multiply it by the difference $(b-a)$ and take into account $tg\alpha(b-a) = f(b) - f(a)$, we will obtain $cg < f(b) - f(a) < cf$. By virtue of a continuity of the function $f(x)$ the sum $fk + eh$ accepts a number of continuous values from cg up to cf , and therefore there is such point ξ, for which the equality $fk + eh = f(b) - f(a)$ takes place. As far as $fk = tgA \cdot (\xi - a) = f'(a)(\xi - a)$ and $eh = tgB \cdot (b - \xi) = f'(b)(b - \xi)$ the formula (1) takes place. The third condition of this theorem guarantees uniqueness of the point ξ .

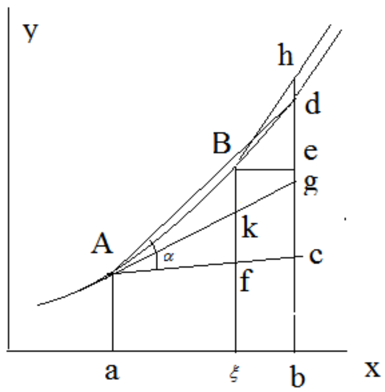

Figure 1 – A geometric method for proving the theorem, $A = \angle gAc,\ B = \angle eBh$

The second geometrical way of the proof of the theorem is easier than the first one. We shall divide both parts of equality (1) into the difference $(b-a)$ and we shall enter a parameter $t = \dfrac{\xi - a}{b - a}$, where $0 < t < 1$: $\dfrac{f(b) - f(a)}{b - a} = f'(a)t + f'(b)(1-t),\ 0 < t < 1.$

Taking into account the geometrical meaning of a derivative, we obtain the equality $tg\alpha = tgA \cdot t + tgB \cdot (1-t)$. It is necessary to consider an inequality $tgA < tg\alpha < tgB$, which is provided with monotony of the derivative $f'(x)$. Existence of the point ξ follows from Weierstrass theorem [2] according to which continuous function on a segment has a continuous number of values between $f(a)$ and $f(b)$.

2 The Formulation of Lagrange's Theorem and Compare it With the Main Mean Value Theorem

Lagrange's Theorem. Let a function $y = f(x)$ satisfies the following conditions: 1) it is continuous on a segment $[a,b]$, 2) it is differentiable on the interval (a,b) . Then there is a point $\xi \in (a,b)$, for which there is the following equality

$$f(b) - f(a) = f'(\xi)(b-a). \quad (2)$$

When comparing equations (1) and (2) we obtain

$$f'(a)(\xi - a) + f'(b)(b - \xi) = f'(\mu)(b-a). \quad (3)$$

Note that these two theorems midpoints are different, i.e. $\xi \neq \mu$. Let's find a function for which the points ξ and μ will coincide. If in the formula (3) to take $\xi = \mu$, you will get a simple differential equation

$$f'(a)(\xi - a) + f'(b)(b - \xi) = f'(\xi)(b-a)$$

After integrating it with the variable ξ we obtain

$$f(\xi) = \frac{f'(a) - f'(b)}{b-a}\frac{\xi^2}{2} - \frac{f'(a)a - f'(b)b}{b-a}\xi + C.$$

This is parabola, whose derivative is a linear function. From our theorem easily there is the point ξ:

$$\xi = \frac{f(b) - f(a) + f'(a)a - f'(b)b}{f'(a) - f'(b)}. \quad (4)$$

Note that by Lagrange's theorem it to make much harder.

From here the important application of our theorem follows. In the discrete mathematics nodes, values of functions and its derivatives are set discretely. By Lagrange's theorem discrete function needs to be approximated by a smooth function, as a rule, a polynomial [4]. According to our theorem midpoints are calculated by the formula (4) and it does not run any questionable approximations in the form of polynomials.

Example: $y = 4x^2 + 3x - 5$, $x \in (1,3)$.

Find the point μ by Lagrange's theorem: $y(1) = 2$, $y(3) = 40$, $y' = 8x + 3$,

$y(3) - y(1) = 38$, $\frac{y(3) - y(1)}{3-1} = 19$, $y'(\mu) = 8\mu + 3$, $8\mu + 3 = 19 \Rightarrow \mu = 2$.

By our theorem it is calculated by formula (4): $y'(1) = 11$, $y'(3) = 27$.

$$\xi = \frac{f(3) - f(1) + f'(1) - f'(3)3}{f'(1) - f'(3)} = \frac{40 - 2 + 11 - 81}{11 - 27} = 2.$$

This example demonstrates the coincidence of midpoints for the parabola by Lagrange and by our theory.

3 Roll's And Cauchy's Theorems in the New Edition

Roll's Theorem. Let a function $y = f(x)$ satisfies the following conditions: 1) it is continuous on a segment $[a,b]$, 2) it is differentiable on the interval (a,b), 3) the derivative $f'(x)$ is monotonic on the interval (a,b), 4) $f(b) = f(a)$. Then there exists a unique point $\xi \in (a,b)$ for which takes place the equality

$$f'(a)(\xi - a) + f'(b)(b - \xi) = 0. \quad (5)$$

If to take into account $f(b)=f(a)$, then the formula (5) follows from formula (1). The expression for the point $\xi=\dfrac{f'(a)a-f'(b)b}{f'(a)-f'(b)}$ is followed from the formula (5). The same expression can be obtained from the formula (4), provided that $f(b)=f(a)$.

Cauchy's Theorem. Let functions $y=f(x)$ and $g(x)$ satisfy the following conditions: 1) they are continuous on a segment $[a,b]$, 2) they are differentiable on the interval (a,b), 3) the derivatives $f'(x)$ and $g'(x)\neq 0$ are monotonic on the interval (a,b). Then there exists a unique point $\xi\in(a,b)$ for which takes place the equality

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(a)(\xi-a)+f'(b)(b-\xi)}{g'(a)(\xi-a)+g'(b)(b-\xi)}. \quad (6)$$

Proof. In the proof we use the method of undetermined Lagrange's multiplier [1]. Let us introduce an auxiliary function $F(x)=f(x)-\lambda g(x)$ and find the uncertain factor λ to run the condition of Roll's theorem: $F(b)=F(a)$. We will get

$$f(b)-\lambda g(b)=f(a)-\lambda g(a)\Rightarrow \lambda=\frac{f(b)-f(a)}{g(b)-g(a)}. \quad (7)$$

According to Roll's theorem there is a point ξ, for which takes place

$$F'(a)(\xi-a)+F'(b)(b-\xi)=0$$

or

$$\left(f'(a)-\lambda g'(a)\right)(\xi-a)+\left(f'(b)-\lambda g'(b)\right)(b-\xi)=0.$$

Taking into account the expression λ (7), after simple transformations we obtain the formula (6).

4 Simple Application of our Theory is the Second Integral Mean Value Theorem

If we apply the formula (1) to the case when the function $f(x)$ acts as its antiderivative, i.e. $F'(x)=f(x)$, then we immediately get the integral analogue of our theorem

$$\int_a^b f(x)dx=f(a)(\xi-a)+f(b)(b-\xi).$$

This is so-called the second mean value theorem of integral calculus. Its generic equivalent is obtained by presenting this formula in the form

$$\int_a^b f(x)dx=f(a)\int_a^\xi dx+f(b)\int_\xi^b dx$$

and replacing the integral measure $dx\to dG(x)=g(x)dx$ [5]. Thus, finally, we obtain

$$\int_a^b f(x)g(x)dx=f(a)\int_a^\xi g(x)dx+f(b)\int_\xi^b g(x)dx.$$

Conclusions

The formula (2) is not the unique form of representation of Lagrange's theorem. The variant of representation of an increment of a function through its derivatives at the endpoints of a segment in the form of the formula (1) is possible still. Such formulation of Lagrange's theorem does not replace it, but even more expanding its capabilities. For example, the article provides an example of application of the formula (1) in the integral calculus - it is a relatively simple formulation of the second mean value theorem.

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RESUME

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A New Representation of Lagrange's Theorem in Differential Calculus

Background: in the integral calculus there is a second mean value theorem, but there is no such analogue in the differential calculus. This "injustice" can be eliminated, when Lagrange's theorem is formulated in terms of derivatives at the segments endpoints.

Materials and methods: we used methods of differential calculus, derivative geometric meaning, the notion of arithmetic mean value and weighted mean value, uncertain Lagrange's multipliers.

Results: the main result of the paper is the wording of the new mean value theorem in differential calculus, which in turn gives rise to a number of theorems analogues of Rolle and Cauchy. Classical Rolle's, Lagrange's and Cauchy's theorems are based on the concept of a usual mean and arithmetic mean values. The new theorems are based on the concept of the weighted mean value.

Conclusion: it is found a new representation of the mean value Lagrange's theorem in the differential calculus. The change of any functions is expressed in the terms of the derivatives at the end points of a given closed interval. Mean values of the function are coincided for both theories, but the midpoints are different. Our theory allows find the midpoint easily. Such possibility is absent in Lagrange's theorem. Besides our theory is fundamental for the second mean value integral theorem.

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