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*I.V. Petrenko, L.P. Mironenko**Donetsk National Technical University, Ukraine**Ukraine, 83000, Donetsk, Artema st., 58, mironenko.leon@yandex.ru, eagor0@mail.ru**Parametric Equations of a Plane and the Problem of Description of Plane Curves in Space R^3* *И.В. Петренко, Л.П. Мироненко**Донецкий национальный технический университет, Украина**Украина, 83000, Донецк, ул. Артема, 58, mironenko.leon@yandex.ru, eagor0@mail.ru**Параметрические уравнения плоскости и задача описания плоских кривых в пространстве R^3* *І.В. Петренко, Л.П. Мироненко**Донецький національний технічний університет,**Україна, 83000, м. Донецьк, вул. Артема, 58**Параметричні рівняння площини і завдання опису плоских кривих у просторі R^3*

The aim of this article is the application of the parametric equations of the plane for the description of plane curves in space R^3 . It is found a simple way of parameterization of the plane, which allows you to enter a local coordinate system on the plane in space R^3 . In this local coordinate system of equations of plane curves look easy. In the paper we consider the parameterization of the plane lines on the example of second-order curves. This view may be useful for analytical geometry and mechanics.

Keywords: plane, curve, parametric, normal vector, plane line, the coordinate system.

Целью статьи является применение параметрических уравнений плоскости для описания плоских кривых в пространстве R^3 . Найден простой способ параметризации плоскости, который позволяет ввести локальную систему координат на плоскости. В такой локальной системе координат уравнения плоских кривых выглядят просто. В данной работе рассмотрена параметризация плоских линий на примере кривых второго порядка. Это представление может быть полезным для решения задач аналитической геометрии, механики.

Ключевые слова: плоскость, кривая, параметрический, нормальный вектор, плоская линия, система координат.

Метою статті є застосування параметричних рівнянь площини для опису плоских кривих у просторі R^3 . Знайдено простий спосіб параметризації площини, який дозволяє ввести локальну систему координат на площині. У цій локальній системі координат рівняння плоских кривих мають простий вигляд. У роботі розглянута параметризація плоских ліній на прикладі кривих другого порядку. Це подання може бути корисним для вирішення завдань аналітичної геометрії та механіки.

Ключові слова: площина, крива, параметричний, нормальний вектор, плоска лінія, система координат.

Introduction

The parametric equations of the plane are practically not used in the course of analytic geometry, but are one of the important methods of analytical description of the plane. The matter is that in analytical geometry exists, at least, five various representations of a plane, each of which solves the certain problems [1-4].

Using the equation of a plane passing through three given points make it easy to formulate and solve the problem of the parameterization of arbitrary plane. This problem is a particular case of the parameterization of a smooth surface of arbitrary shape in space R^3 . In general, this problem is discussed in the section of differential geometry involving apparatus of differential calculus [5-6].

In turn, not involving such complex concepts as the first, second quadratic forms, the description of plane curves in Cartesian coordinates can be solved by means of vector algebra. This approach is made available to a wider audience, not only for specialists in differential geometry.

The aim is a simple way of introducing the parametric equations of the plane and the parametric method of describing plane curves in space R^3 . This paper gives examples of second-order curves. Their description in the local coordinate system is very simple. Their equations have the canonical form.

1 Parametric representation of the equation of a plane

The parametric equations of the plane are most easily obtained from the equation of the plane through three given points that do not lie on a straight line

$$M_0(x_0, y_0, z_0), M_1(x_1, y_1, z_1), M_2(x_2, y_2, z_2).$$

Let the point $M(x, y, z)$ be an arbitrary point in the plane. Then three vectors are coplanar (Fig. 1).

$$\begin{aligned} \mathbf{M}_0\mathbf{M} &= \{x - x_0, y - y_0, z - z_0\} \\ \mathbf{M}_0\mathbf{M}_1 &= \{x_1 - x_0, y_1 - y_0, z_1 - z_0\} \\ \mathbf{M}_0\mathbf{M}_2 &= \{x_2 - x_0, y_2 - y_0, z_2 - z_0\} \end{aligned}$$

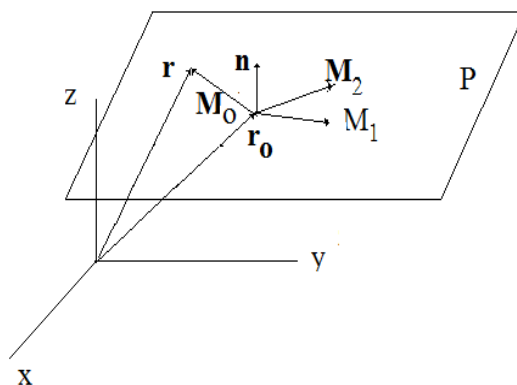


Fig. 1 – Derivation of parametric equations of the plane

The mixed product of any coplanar vectors is zero, i.e. $\mathbf{M}_0\mathbf{M} \cdot \mathbf{M}_0\mathbf{M}_1 \times \mathbf{M}_0\mathbf{M}_2 = 0$.

Using the coordinates this condition can be written as the determinant

$$\begin{vmatrix} x-x_o & y-y_o & z-z_o \\ x_1-x_o & y_1-y_o & z_1-z_o \\ x_2-x_o & y_2-y_o & z_2-z_o \end{vmatrix} = 0.$$

Let us denote $\mathbf{s}_1(l_1, m_1, n_1) = (x_1 - x_o, y_1 - y_o, z_1 - z_o)$, $\mathbf{s}_2(l_2, m_2, n_2) = (x_2 - x_o, y_2 - y_o, z_2 - z_o)$. According to the notation, we write the determinant

$$\begin{vmatrix} x-x_o & y-y_o & z-z_o \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{1}$$

Equality to zero of a determinant means, that its lines (or columns), and in this case lines, are proportional. This fact means, that, for example, the first line of a determinant (1) is a linear combination of the other two with some coefficients α and β

$$\begin{cases} x - x_o = \alpha l_1 + \beta l_2, \\ y - y_o = \alpha m_1 + \beta m_2, \\ z - z_o = \alpha n_1 + \beta n_2. \end{cases} \Rightarrow \begin{cases} x = x_o + \alpha l_1 + \beta l_2, \\ y = y_o + \alpha m_1 + \beta m_2, \\ z = z_o + \alpha n_1 + \beta n_2. \end{cases} \tag{2}$$

In the vector form we have the simple equality

$$\mathbf{r} = \mathbf{r}_o + \alpha \mathbf{s}_1 + \beta \mathbf{s}_2. \tag{3}$$

Here α and β are parameters of the plane, and the equations (2) and (3) are called *the parametric equations of the plane*.

2 The geometric meaning of the parametric equations of the plane of the plane

Explain the geometric meaning of the parametric equations of the plane. For this we construct vectors \mathbf{s}_1 and \mathbf{s}_2 in the plane P and take into account that the point M_o corresponds to the value of parameters $\alpha = \beta = 0$. If $\alpha = \beta = 1$ the point M_o is at the top of the parallelogram formed by the vectors \mathbf{s}_1 and \mathbf{s}_2 (Fig. 2). The further scenario consists in that, at the any fixed values of parameters α and β vectors \mathbf{s}_1 and \mathbf{s}_2 are “extended” accordingly in α and β times (if $\alpha > 1, \beta > 1$) or “shortened” in α and β times (if $\alpha < 1, \beta < 1$), and directed opposite at negative values of parameters $\alpha < 0$ и $\beta < 0$. Since the values of the parameters $-\infty < \alpha < \infty$ and $-\infty < \beta < \infty$ are independent from each other, then the vector \mathbf{r} runs through all points of the plane P .

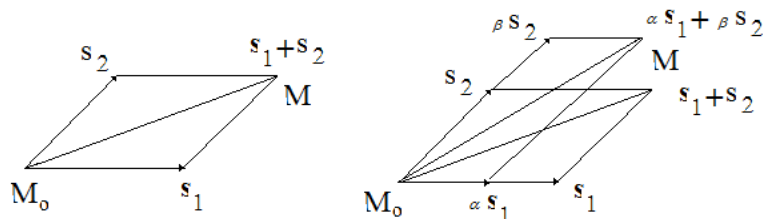


Fig. 2 – Geometrical meaning of the parametrical equations of a plane

From the above arguments follows, if parameters α and β will be connected among themselves by functional dependence the vector \mathbf{r} will describe some flat line L on the plane P .

Let's consider the simplest case when parameters are equal $\alpha = \beta = t$. The equation (3) will become

$$\mathbf{r} = \mathbf{r}_o + t\mathbf{s}, \quad \mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2. \quad (4)$$

These are the parametrical equations of the straight line which is passing through the point M_o with the directed vector $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$. Let's pay attention to the vectors \mathbf{s}_1 and \mathbf{s}_2 , which allow you to orient any straight line in a plane in any desired direction.

3. Parametric representation of plane curves in space R^3

A choice of the vectors \mathbf{s}_1 and \mathbf{s}_2 is simple. They should not be collinear.

It is convenient to choose vectors \mathbf{s}_1 and \mathbf{s}_2 perpendicular each other $\mathbf{s}_1 \perp \mathbf{s}_2$. In that case, for example, the ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will be $\frac{x^2}{|\mathbf{s}_1|^2} + \frac{y^2}{|\mathbf{s}_2|^2} = 1$. Let's consider a parametrical form $x = |\mathbf{s}_1| \cos t$, $y = |\mathbf{s}_1| \sin t$, $0 \leq t < 2\pi$. Using (3), we find representation of an ellipse centered at the point M_o and semi axes $|\mathbf{s}_1| = a$, $|\mathbf{s}_2| = b$ and the normal vector $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$ in the space R^3 .

$$\mathbf{r} = \mathbf{r}_o + \mathbf{e}_1 a \cos t + \mathbf{e}_2 b \sin t, \quad (5)$$

where $\mathbf{e}_1 = \mathbf{s}_1 / |\mathbf{s}_1|$ - $\mathbf{e}_2 = \mathbf{s}_2 / |\mathbf{s}_2|$ are mutual perpendicular unit vectors.

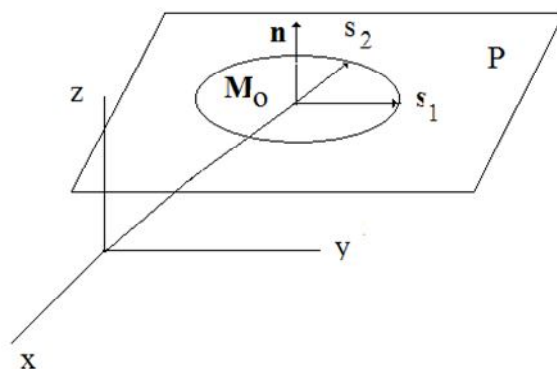


Fig. 3 – Presentation of an ellipse in space

Here are more examples of other second order curves. Parameterize the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ believing $a = |\mathbf{s}_1|$, $b = |\mathbf{s}_2|$ and $x = |\mathbf{s}_1| \cosh t$, $y = |\mathbf{s}_1| \sinh t$, $-\infty < t < \infty$. As a result we have

$$\mathbf{r} = \mathbf{r}_o + \mathbf{e}_1 a \cosh t + \mathbf{e}_2 b \sinh t, \quad (6)$$

In the case of a parabola $y^2 = 2px$ suppose $x = \frac{p}{2} t^2$, $y = pt$.

$$\mathbf{r} = \mathbf{r}_o + \mathbf{e}_1 \frac{p}{2} t^2 + \mathbf{e}_2 pt. \quad (7)$$

In the polar coordinate system $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, therefore

$$\mathbf{r} = \mathbf{r}_o + \rho(\varphi)(\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi). \quad (8)$$

Put the point M_o in the right focus curve of the second order (ellipse, hyperbole or parabola). Then the equations of curves of the second order in the polar coordinate system will be of the form

$$\mathbf{r} = \mathbf{r}_o + \frac{p}{1 - \varepsilon \cos \varphi} (\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi), \quad (9)$$

where ε is the eccentricity of a second order curve. At $\varepsilon < 1$ we have an ellipse, at $\varepsilon > 1$ - the right branch a hyperbole (for the left branch of a hyperbole in the general equation (8) should be put $\rho = \frac{-p}{1 + \varepsilon \cos \varphi}$), at $\varepsilon = 1$ we have a parabola, p is the parameter of the curve of the second order (9). In particular, the equation of a circle of radius R is obtained from the general equation (8) if we set $\rho = R$.

$$\mathbf{r} = \mathbf{r}_o + R(\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi).$$

4 Other representations of parametric equations of the plane

Let's build a local basis of mutually perpendicular unit vectors \mathbf{e}_1 and \mathbf{e}_2 on a given plane. Then equation (3) takes the form

$$\mathbf{r} = \mathbf{r}_o + \alpha \mathbf{e}_1 + \beta \mathbf{e}_2. \quad (10)$$

The scalar multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ is satisfied to the equalities $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1$. Then the scalar multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ by the equation (10) gives the system of the equations

$$\begin{cases} \mathbf{e}_1 \cdot (\mathbf{r} - \mathbf{r}_o) = \alpha \\ \mathbf{e}_2 \cdot (\mathbf{r} - \mathbf{r}_o) = \beta. \end{cases}$$

We summarize both equalities and denote $\alpha + \beta = t$. Then we get a one-parameter vector equation of a plane $(\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{r} - \mathbf{r}_o) = t$, which corresponds to a family of the parametrical equations of straight lines in the perpendicular direction to the vector $\mathbf{e}_1 + \mathbf{e}_2$.

Vector multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ is satisfied to the equalities $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{n}$ and $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = 0$. Then the vector multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ by the equation (10) gives the next vector system of the equations

$$\begin{cases} \mathbf{e}_1 \times (\mathbf{r} - \mathbf{r}_o) = -\alpha \mathbf{n} \\ \mathbf{e}_2 \times (\mathbf{r} - \mathbf{r}_o) = \beta \mathbf{n} \end{cases}$$

We summarize both the last equalities and denote $-\alpha + \beta = t$. Then we get another one-parameter vector equation of a plane $(\mathbf{e}_1 + \mathbf{e}_2) \times (\mathbf{r} - \mathbf{r}_o) = t \mathbf{n}$.

We can exclude the parameter t using the triple vector multiplication of the vectors $\mathbf{n} \times (\mathbf{e}_1 + \mathbf{e}_2) \times (\mathbf{r} - \mathbf{r}_o)$. According to the formula $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ we obtain $\mathbf{n} \times (\mathbf{e}_1 + \mathbf{e}_2) \times (\mathbf{r} - \mathbf{r}_o) = \mathbf{n} \times \mathbf{n} = \mathbf{0} \Rightarrow (\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o)) - (\mathbf{r} - \mathbf{r}_o)(\mathbf{n} \cdot (\mathbf{e}_1 + \mathbf{e}_2)) = \mathbf{0}$.

Now $\mathbf{n} \perp (\mathbf{e}_1 + \mathbf{e}_2)$, therefore $\mathbf{n} \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 0$ and $\mathbf{e}_1 + \mathbf{e}_2 \neq \mathbf{0}$. As a result we have the equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o) = 0$. This is well-known equation of a plane. In the coordinate representation the equality looks like

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0.$$

Findings

1. The simple way of introduction of the parametrical equations of a plane is offered.
2. The effective way of the description of plane curves in space is offered.
3. Results of the work are approved on curves of the second order.

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