CALCULATION OF GEOMETRIC PARAMETERS OF A MAGNETIC QUADRUPOLE LENS WORKING NEAR TO SATURATION

A.O. Mytsykov

National Science Center "Kharkov Institute of Physics and Technology", Kharkov, Ukraine

The method for defining basic geometric parameters of precision quadrupole lenses is offered. The method allows one to obtain the field of working area differing from required one less than by 10^{-4} compared with the basic component. The lens shape is described by 2 parameters. The method allows taking into account the field distortions caused by saturation of iron. Moreover, it is possible to calculate lenses with a complicated multipolar structure including the forbidden field harmonics. The paper contains the results of calculations for lense shapes used in actual installations.

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INTRODUCTION

The multipolar magnetic lenses are ones from the basic elements in the accelerator engineering. For calculation of lenses the problem is set as follows: according to the specific field distribution it is necessary to obtain geometric parameters of a multipole. Major of these parameters is the pole shape. The basic purpose of this work is deriving of analytical expressions, which connect by a relationship the pole shape with the field multipolar structure.

APPLICATION OF CONFORMAL MAP FOR DESCRIPTION OF MULTIPOLES WITH CURVILINEAR POLES

As is known the magnetic field (hereinafter - field) can be expressed in terms of the complex potential $z(\omega)$ and in terms of the inverse to complex potential function $\omega(z)$ by expression [1]

$$B = B_x + iB_y = \left(i\frac{dz}{d\omega}\right)^* = \left(i\left(\frac{d\omega}{dz}\right)^{-1}\right)^* \qquad (1)$$

The function $\omega(z)$ can be obtained considering a conformal mapping of a rectilinear band 0<Im(z)<H on a "pole band" which is generated by the symmetry lines of a multipole and by the pole profile Fig.1; [2].

$$\omega(z) = \int_{z_0}^{z} \exp[G(z)]dz \qquad (2)$$

where the function $G(z)$ coincides within the accuracy of an integration constants with the Schwarz integral for the band [2]:

$$G(z) = \frac{1}{2H}\left\{\int_{-\infty}^{\infty} v_0(t)\left[\mathrm{cth}\frac{\pi(t-z)}{2H} - \mathrm{th}\frac{\pi t}{H}\right]dt - \int_{-\infty}^{\infty} v_H(t)\left[\mathrm{th}\frac{\pi(t-z)}{2H} - \mathrm{th}\frac{\pi t}{H}\right]dt\right\} \qquad (3)$$

where: H is the height of the rectilinear band on the plane z; $v_0(t)$is the declination angle of the lower shore of the "pole band" (symmetry lines of multipole) as a function of the abscissa t of the z-plane ($z=t+is$) of the rectilinear band; $v_H(t)$is the declination angle of the upper shore of the "pole band" (pole shape) as a function of the abscissa t of the z-plane ($z=t+is$) of the rectilinear band.

At this map (2) the lower shore of the band 0<Im(z)<H of the z-plane is transformed to the symmetry line of a multipole and the upper one to the pole profile. Then the first summand of the integral (3) answers for the problem symmetry. Hereinafter we shall consider only 'pure' multipoles with the rectilinear symmetry lines. Then the following definition is right:

$$v_0(t) = \begin{cases} U_{mult} & t \in [-\infty, 0] \\ 0 & t \in [0, \infty] \end{cases} \qquad (4)$$

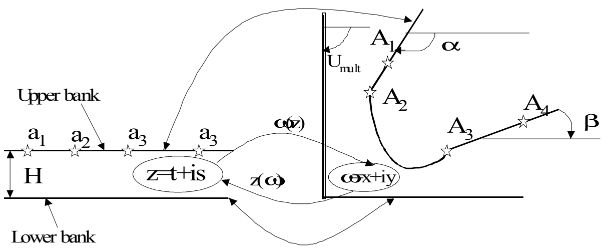

Fig. 1. *The scheme of conformal mappings for a multipole.*

Obviously that for thedipole symmetry U_{mult}=0; for the quadrupole symmetry U_{mult} =-π/2; for the sextupolar symmetry U_{mult} = -π2/3; etc.

With such a definition the first summand of the integral (3) takes the form:

$$\frac{\int_{-\infty}^{\infty} v_0(t)\left[\mathrm{cth}\frac{\pi(t-z)}{2H} - \mathrm{th}\frac{\pi t}{H}\right]dt1}{2H} = \frac{U_{mult}}{\pi}\ln\frac{1-\exp\left(\frac{-\pi z}{H}\right)}{\sqrt{2}} = U_{mult}\cdot Y(z). \qquad (5)$$

So, for the multipolar symmetry this factor determines the growth of field from the zero value at the aperture center to the certain value at the aperture edge. The second part of the integral (3) describes the pole shape. Let us assume that the pole has a rectilinear exterior slants and an interior sector. Then the declination angle of the tangent for the upper shore to the abscissa axis of ω-plane varies under the law:

$$v_H(t)=\begin{cases} U_{-\infty} & t\in[-\infty,a_1] \\ Q[t]=\sum_{j=0}^{M} q_j t^j & \\ U_{\infty} & t\in[a_2,\infty] \end{cases} \quad (6)$$

After simple transformations the second part of (3) becomes as

$$\frac{1}{2H}\int_{-\infty}^{\infty} v_H(t)\left(\mathrm{th}\frac{\pi(t-z)}{2H}-\mathrm{th}\frac{\pi t}{H}\right)dt=-T(z)= \quad (7)$$

$$U_{-\infty}\left(S(a_1,z,H,0)-\frac{z}{H}\right)+ \quad (7a)$$

$$\sum_{j=0}^{M} q_j\left(S(a_2,z,H,j)-S(a_1,z,H,j)\right)- \quad (7b)$$

$$U_{\infty}\,S(a_2,z,H,0), \quad (7c)$$

where: the first summand (7a) is the contribution of the left slant from a_1; the second summand (7b) is the contribution of the part between points a_1 and a_2; the third summand (7c) is the contribution of the right slant from a_2.

$$S(t,z,H,j)=\sum_{k=0}^{j} t^{j-k}\frac{H^k}{\pi^{k+1}}\frac{j!}{(j-k)!}\times$$
$$\left(\mathrm{Li}_{k+1}[-\exp\frac{-\pi(t-z)}{2H}]-\frac{1}{2^{k+1}}\mathrm{Li}_{k+1}[-\exp\frac{-\pi t}{H}]\right). \quad (8)$$

Here and further the polylogarithmic function Li_j possessing the following properties is used:

$$\mathrm{Li}_1[-e^x]=-\ln[1+e^x];$$
$$\frac{d}{dx}\mathrm{Li}_{j-1}[-e^x]=\mathrm{Li}_j[j,-e^x]; \quad (9)$$
$$\mathrm{Li}_{j+1}[-e^x]=\int \mathrm{Li}_j[-e^x]dx.$$

DEPENDENCE BETWEEN FIELD HARMONIC STRUCTURE AND POLE SHAPE

The integral (2) representing the function inverse to the complex potential is convenient to be present as:

$$\omega(z)=\int_{z_0}^{z}\exp[U_{mult}Y(z)+T(z)]dz, \quad (10)$$

where: $Y(z)$, $T(z)$ is defined by (5), (9).

Obviously that the field in coordinates of the band $0<\mathrm{Im}(z)<H$ looks like (see (1)):

$$B(z)=\exp[-U_{mult}Y(z)-T(z)]. \quad (11)$$

This expression has as an argument a coordinate of the z-plane instead of physical Cartesian coordinates of the ω-plane containing a pole. Therefore expressions (10,11) are the parametric representation of a field. In practice it is convenient to have expressions describing a field in physical coordinates (i.e. in coordinates of the ω-plane). For deriving such expressions we use the following procedure.

We shall obtain derivatives of the field $dB_i(z)/d\omega(z)^i$ in the ω-plane as functions of z. For this purpose we note that the increment of coordinates on the plane ω is equal to

$$d\omega=\exp[U_{mult}Y(z)+T(z)]dz. \quad (12)$$

Now we can obtain the first derivative of a field in coordinates of a z-plane:

$$\frac{dB}{d\omega}=\frac{\frac{d}{dz}\left(-U_{mult}Y(z)-T(z)\right)}{\exp[2(U_{mult}Y(z)+T(z))]}. \quad (13)$$

Similarly it is possible to obtain any derivative, using the recursive ratios:

$$B(z)=\exp[-U_{mult}Y(z)-T(z)];$$
$$\frac{d^iB(z)}{d\omega(z)^i}=\frac{\frac{d}{dz}\left(\frac{d^{i-1}B(z)}{dw^{i-1}}\right)}{\exp[(U_{mult}Y(z)+T(z))]}. \quad (14)$$

Let us relate the point z(0,0) of the z-plane containing the band $0<\mathrm{Im}(z)<H$ and the point $\omega(0,0)$ of the plane containing the pole. Then the factors calculated according to the formulas (14) are the Taylor-series coefficients of the field but now for the ω-plane containing the pole i.e. in physical, Cartesian coordinates. It is very important for the practical applications.

IDEAL QUADRUPOLE SYMMETRY

As an ideal quadrupole we shall mean a case for which $U_{mult}=-\pi/2$ (see (4), Fig.1). Let us consider a map of the rectilinear band $0<\mathrm{Im}(z)<H$ to the "polar band" not laying down conditions to the pole shape. Suppose that there is a certain "shape function" T(z) equal to the integral on the upper shore of the band (see (7)).

The principal performance of a quadrupole lens is the gradient

Having produced all substitutions for the known values U_{mult}, Y(z) we shall obtain:

$$\frac{d\,B(z)}{d\omega(z)}=N[z]=-\frac{-\pi+2\left(-1+\exp[\frac{\pi z}{H}]\right)HT'(z)}{2\sqrt{2}H\exp\left[2T(z)+\frac{\pi z}{H}\right]}. \quad (15)$$

The solution of Eq. (26) is:

$$T_0[z]=\frac{1}{2}\left(\ln[\frac{1-\exp[\frac{-\pi z}{H}]}{2\sqrt{2}\left(C+\int N[z]dz\right)}]\right). \quad (16)$$

Here the label T_0(z) is applied to distinguish the actual "shape function" (7) (in which both angularities and curvature of the medial sector of the pole profile are taken into account) from the ideal one (16). From the conventional definition, that the field at centre of a multipole is equal to zero follows C=0 (constant of the integration in (16). In this case the map of the rectilinear band of the z-plane to the polar band of the ω-plane, looks like (substitution (16) in (10)):

$$\omega(z)=\int_{z_0}^{z}\frac{1}{\sqrt{2\int N[z]dz}}dz; \quad (17)$$

where the following definition is right:

$$N[z] = \sum_{i=0}^{M} N_i z^i = N_0 \left(1 + \sum_{i=1}^{M} \lambda_i z^i \right). \quad (18)$$

In particular, if $N(z) = const = N_0$ the expression (17) becomes simpler to the well-known expression, the inverse of which gives a complex potential for the quadrupole symmetry.

$$\omega(z) = \int_{z_0}^{z} \frac{1}{\sqrt{2N_0 z}} dz = \sqrt{z} \sqrt{\frac{2}{N_0}} \Bigg|_{z_0 = 0}; \quad z = \frac{N_0}{2} \omega^2. \quad (19)$$

The coefficient N_0 has a simple metric sense. It is possible to show that by virtue of properties of conformal representations:

$$N_0 = \frac{1}{H}; \quad H = \frac{R}{\sqrt{2}}. \quad (20)$$

In case of calculation of the lens with a complicated multipolar composition it is necessary to solve the following set of equations:

$$\begin{cases} \operatorname{Im}\left(\int_0^{iH} \frac{dz}{\sqrt{2N_0 \int 1 + \sum_{k=1} \lambda_i H^i dz}} \right) = H; \\ \operatorname{Re}\left(\sum_{i=0} \frac{B^{(i)}_{\omega=0}}{(2+i)!} \left(R_x, R_y \right)^{i+2} \right) = 0; \\ \operatorname{Im}\left(\sum_{i=0} \frac{B^{(i)}_{\omega=0}}{(2+i)!} \left(R_x, R_y \right)^{i+2} \right) = H; \\ \left\| \left(R_x, R_y \right) \right\| = R. \end{cases} \quad (21)$$

The first equation of this system expresses the fact that with a conformal map the value of scalar potential (imaginary part complex) on equipotential lines which are mapped each other is intact. The second equation expresses the circumstance that a certain point (R_x,R_y) of the ω-plane is mapped to the point (0,H) of z-plane. The last equation defines the position of the point (R_x,R_y) on the ω plane by setting the aperture radius. The system of four equations (21) contains 4 unknowns R_x, R_y, N_0, H and can by solved if coefficients λ_i are known quantities. The coefficients λ_i can be obtained from consideration of field derivatives on the rules (14) with allowance for (18):

$$\begin{aligned} & B^{(1)}(0) = N_0; \\ & B^{(5)}(0) = 6N_0^3 \lambda_2; \\ & B^{(9)}(0) = 21N_0^5 \left(16\lambda_2^2 + 120\lambda_4 \right); \\ & B^{(13)}(0) = 693N_0^7 \left(112\lambda_2^3 + 3360\lambda_2\lambda_4 + 10800\lambda_6 \right) \\ & B^{(17)}(0) = 693N_0^9 \left(73472\lambda_2^4 + 5698560\lambda_2^2\lambda_4 + 49766400\lambda_2\lambda_6 + \right. \\ & \qquad \left. 325\left(32256\lambda_4^2 + 362880\lambda_8 \right) \right). \end{aligned} \quad (22)$$

The process can be prolonged as much as long. However in practice the low numbers of harmonics are most significant. The system (22) can be solved concerning the even expansion coefficients λ_i. Here is the solution for the first terms of expansion:

$$\begin{aligned} & \lambda_2 = \frac{B^{(5)}(0)}{6N_0^3}; \\ & \lambda_4 = \frac{-28B^{(5)}(0)^2 + 3N_0 B^{(9)}(0)}{7560N_0^6}; \\ & \lambda_6 = \frac{N_0^2 B^{(13)}(0) + 1078B^{(5)}(0)^3 - 154N_0 B^{(5)}(0) B^{(9)}(0)}{7484400N_0^9}; \\ & \lambda_8 = \frac{N_0^3 B^{(17)}(0) - 768N_0^2 B^{(5)}(0) B^{(13)}(0) - 560560B^{(5)}(0)^4}{81729648000N_0^{12}} + \\ & \frac{96096N_0 B^{(5)}(0)^2 B^{(9)}(0) - 1144N_0^2 B^{(9)}(0)^2}{81729648000N_0^{12}}; \\ & \ldots; \lambda_1, \lambda_3, \lambda_5 \ldots = 0. \end{aligned} \quad (23)$$

Thus set of equations (21), (23) completely define the ideal shape function according to the given multipolar structure (coefficients B_i).

ACTUAL PROFILES OF QUADRUPOLE LENSES

Now, when the forms of ideal shape functions (16) and actual one (7) are known, we shall require their coincidence in M points on a segment [0,H] which corresponds to the radius of the aperture on the ω-plane. (see (19,20)). The better is the coincidence of these functions in a working area, the less is the deviation of the field in actual geometry from the required field. Therefore requirement of equality of ideal and actual functions of the shape in M points of the working area generates a system M of equations, which can be used for determination of conformal map parameters.

The reasons of symmetry reduce in the requirements:

$$\begin{aligned} & a_1 = -a_2 = a; \quad U_{-\infty} = -\frac{\pi}{2} - \alpha; \quad U_{\infty} = \alpha; \\ & Q[t] = -\frac{\pi}{4} + q_1 t + q_3 t^3 + q_5 t^5 + \ldots. \end{aligned} \quad (24)$$

Let us require that in points of conjugation of various profile sectors the function Q(t) was continuous. Add to this the requirements (24) and we shall obtain a set of equations for definition of another parameters.

$$\begin{cases} Q[-a] = \sum_{i=0}^{M} q_i t^i = U_{-\infty} = -\frac{\pi}{2} - \alpha; \\ Q[a] = \sum_{i=0}^{M} q_i t^i = U_{\infty} = \alpha; \\ a_2 = a = -a_1; \\ q_0 = -\frac{\pi}{4}, \quad q_{2i+1} = 0, \quad i = 1,2,3\ldots \end{cases} \quad (25)$$

Thus if, for example, we tried to specify $Q(t)$ by the polynomial of 7-th degree, then only 2 parameters will be by free. Hereinafter we shall use the q_i parameters at the lowest degrees of a polynomial as free ones. The coefficients at the higher degrees will answer for conjugation of the pole surface.

Further the lens with the aperture 35 mm and with the slant angle 30° was considered, Fig. 2. This lens is planed to be used in the installation [4]. The parameters a, q_1, q_3 were defined. The pole profile constructed on these parameters (2) was substituted in the program MERMEID [3] as an input. The multipolar composition of the field was determined at various currents, Fig. 3. **The harmonics presented in the expansion** are caused by the influence of the excitation coil and saturation effects. Therefore calculated multipolar composition was substituted in (23) with the opposite sign for definition of preliminary changed parameters of conformal transformation. As a result a new profile of the pole was obtained. The allocation of the gradient for the lens with such a pole is shown in Fig. 4.

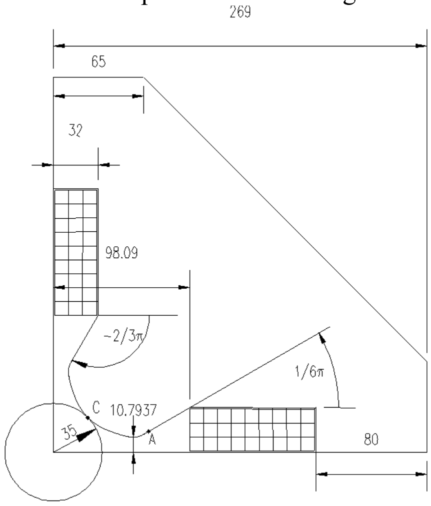

Fig. 2. *Cross-section of quadrupole lens.*

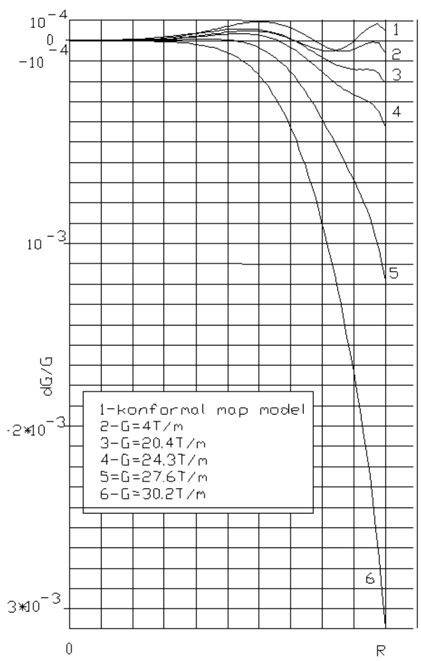

Fig. 3. *The radial dependence of the gradient in the lens (R=0.035 m) with the pole shape obtained by the model based on the conformal map for various gradient values.*

CONCLUSION

The analytical expressions, obtained in this paper**,** at shared use with programs of **numerical simulation** allow to obtain by 1-2 iterations the design solution for a quadrupole magnet, having eliminated from operation a phase "of creative searching" of designer. The calculation results are directly applicable for synthesis of profiles of quadrupole lenses "burdened" by even harmonics. The possibilities of a method do not eliminate operation with forbidden, odd harmonics too. For this purpose it is necessary to take into account the curvature of lines corresponding to lines of symmetry of an ideal quadrupole (i.e. lower coast of the band) and to conduct evaluations for each quadrant.

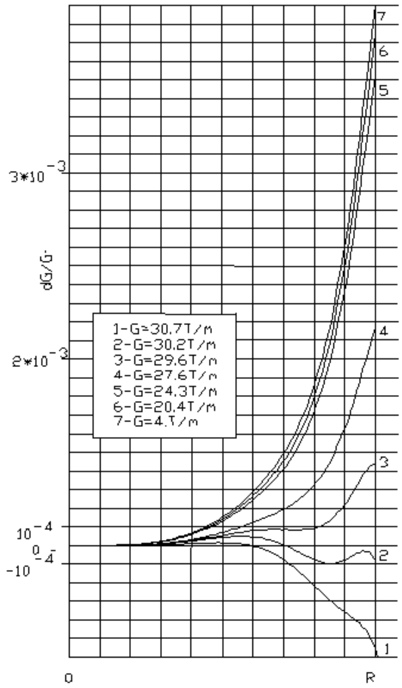

Fig. 4. *The radial dependence of the gradient in the lens (R=0.035 m) with the preliminary changed pole for various gradient values.*

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