RELATIVISTIC EFFECTS IN ELECTRON NEOCLASSICAL TRANSPORT

I. Marushchenko, N.A. Azarenkov
V.N. Karazin Kharkiv National University, Kharkiv, Ukraine

The parallel momentum correction technique is generalized for relativistic approach. It is required for proper calculation of the parallel neoclassical flows and, in particular, for the bootstrap current at fusion temperatures. It is shown that the obtained system of linear algebraic equations for parallel fluxes can be solved directly without calculation of the distribution function if the relativistic monoenergetic transport coefficients are already known, while the latter can be calculated by any non-relativistic solver.

PACS: 52.55.Dy, 52.25.Fi, 52.27.Ny

INTRODUCTION

Since the fusion reactor scenarios require high temperatures, the role of relativistic effects for electrons in toroidal devices also becomes an actual question. In particular, the relativistic effects in the transport processes need to be examined for the next generation of such devices as ITER and DEMO [1, 2], where the expected electron temperatures are sufficiently high, T_e > 25 keV. Apart from this, the aneutronic fusion reactors with $D-{}^3He$, $D-D$ and $p-{}^{11}B$ reactions require actually relativistic temperatures of up to T_e ~ 100 keV [3, 4]. It was already shown [5, 6] that relativistic effects in electron transport can be non-negligible in the range of electron temperatures typical for fusion. In the previous publications [6, 11, 12] the relativistic approach has been developed mainly for radial neoclassical transport. However, the account of relativistic effects is much more important for parallel neoclassical fluxes since there is no anomalous transport along the magnetic surfaces and neoclassical fluxes are in good agreement with experiment.

The standard technique for solving drift kinetic equation (DKE) includes the expansion of the distribution function in Sonine (associated Laguerre) polynomials $L_n^{(3/2)}(x)$ [7-10]. In relativistic approach, due to the additional relativistic term in right-hand-side (RHS) of DKE, the standard technique with Sonine polynomials $L_n^{(3/2)}(x)$ becomes inefficient, and the generalization of this technique for arbitrary temperatures is proposed in this paper.

1. RELATIVISTIC DRIFT KINETIC EQUATION

Generally, relativistic consideration at fusion temperatures only for electrons is required. However, for convenience, below we consider all species and distinguish between electrons and ions only when it is necessary.

It was shown before [6, 11, 12] that the relativistic DKE (rDKE) for neoclassical electron transport can be written as:

$$V(f_e) - C_e^{lin}(f_e) = -\dot{\rho}\left(A_1 + \left(\kappa - \frac{5}{2} - \mathcal{R}\right)A_2\right)F_{eMJ} + +bv_{\parallel}A_3F_{eMJ}, \quad (1)$$

where f_e is the disturbance of the electron distribution function driven by the thermodynamic forces A_i, V is the relativistic Vlasov operator, $C^e(f_e) = \sum_b\left(C^{eb}[f_e, F_{bMJ}] + \delta_{eb}C^{eb}[F_{bMJ}, f_e]\right)$ is the relativistic linearized Coulomb operator [13] (here, $\delta_{aa} = 1$ and $\delta_{ab} = 0$ if $b \neq a$), $\dot{\rho} = V_{dr} \cdot \nabla\rho$ where ρ is the flux-surface label, $\kappa = m_e c^2(\gamma - 1)/T_e$ is the normalized kinetic energy of the electrons with $\mu_r = m_e c^2/T_e$, $v_{\parallel} = u_{\parallel}/\gamma$ with $u = p/m_e$, and $b = B/B_0$ is the magnetic field normalized by its reference value. The thermodynamic forces in RHS of Eq. (1) are defined as

$$A_1(\rho) = \frac{p_e'}{p_e} - \frac{e\Phi_e'}{T_e},$$
$$A_2(\rho) = \frac{T_e'}{T_e},\ A_3(\rho) = -\frac{e}{T_e}\frac{\langle E \cdot B\rangle}{\langle B^2\rangle}, \quad (2)$$

where $p_e = n_e T_e$ is the thermodynamic pressure, $\boldsymbol{E}$ is the inductive electric field, and prime means $d/d\rho$. Note that in contrast to definitions used in [6, 11, 12], the term $\mathcal{R}$ is not included in the thermodynamic force. Instead, in Eq. (1) only the standard definition is applied.

The additional term in the RHS of Eq. (1) with

$$\mathcal{R}(\mu_r) = \mu_r\left(\frac{K_3(\mu_r)}{K_2(\mu_r)} - 1\right) - \frac{5}{2} = \frac{15}{8\mu_r} + \cdots (\mu_r \gg 1) \quad (3)$$

appears due to specific features of F_{eMJ}, and $K_n(x)$ is the modified Bessel function of the second kind of *n*-th order.

The electron equilibrium is given by the Juttner-Maxwellian distribution function F_{eMJ} [13, 14]

$$F_{eMJ}(u) = C_{MJ}(\mu_r)\frac{n_e}{\pi^{3/2}u_{th}^3}e^{-\mu_r(\gamma-1)}, \quad (4)$$

where $\gamma = (1 + u^2/c^2)^{1/2}$ is the Lorentz factor and $u_{th} = p_{th}/m_e$ is the thermal momentum per unit mass with $p_{th} = (2m_e T_e)^{1/2}$. The Maxwellian is normalized by density, $\int d^3u\, F_{eMJ} = n_e$, and

$$C_{MJ} = \sqrt{\frac{\pi}{2\mu_r}}\frac{e^{-\mu_r}}{K_2(\mu_r)} = 1 - \frac{15}{8\mu_r} + \cdots\ (\mu_r \gg 1). \quad (5)$$

The ion distribution function, F_{iM}, is considered as non-relativistic Maxwellian one.

2. MOMENT EQUATIONS FOR PARALLEL FLUXES CALCULATION

In the moment-equation method for the parallel neoclassical fluxes calculation [8-10], the non-relativistic distribution function was expanded by

Sonine polynomials $L_n^{(3/2)}(x)$, and its zeroth and first moments gave the parallel fluxes of particles, $\Gamma_{e\parallel}$, and heat, $q_{e\parallel}$, respectively. However, the relation between fluxes of the heat, the energy and particles in the relativistic approach differs from the classical one [7] by the additional relativistic term [6, 11], and

$$q_a = Q_a - \left(\frac{5}{2} + \mathcal{R}\right) T_a \Gamma_a, \tag{6}$$

where $\Gamma_a = \int d^3u\, v f_a$ and $Q_a = \int d^3u\, v\, m_a c^2(\gamma - -1) f_a$ are the fluxes of particles and energy, respectively. Since the expression for the relativistic heat flux was derived in [6, 11] only phenomenologically, the rigorous derivation of Eq. (6) is given in the Appendix.

From Eq. (1) with $A_3 = 0$ one can easily recognize that, in contrast to the non-relativistic approach ($c \to \infty$ and $\mathcal{R} = 0$), the expansion of RHS with the Sonine polynomials $L_n^{(3/2)}(\kappa)$ cannot be represented by only zeroth and first terms. However, if the generalized Laguerre polynomials $L_n^{(\alpha)}(\kappa)$ with $\alpha = 3/2 + \mathcal{R}$ are chosen, the standard procedure can be applied. Here, $L_0^{(\alpha)}(x) = 1$, $L_1^{(\alpha)}(x) = \alpha + 1 - x$, etc. [15]. The use of $L_n^{(\alpha)}(\kappa)$ is perfectly applicable also for the representation of the heat flux, given by Eq. (6).

Following [10], let us introduce the parallel-flow-velocity moments,

$$n_a V_{\parallel i}^a = \int d^3u\, v_\parallel L_i^{(\alpha)}(\kappa) f_a. \tag{7}$$

Then the moment with $i = 0$ can be identified with the parallel flow velocity or flux of particles, $n_a V_0^a = \int d^3u\, v_\parallel f_a = \Gamma_\parallel^a$, while the moment with $i = 1$ is related to the parallel heat flux, $n_a V_1^a = \int d^3u\, v_\parallel (5/2 + \mathcal{R} - \kappa) = -q_\parallel^a / T_a$.

Representing f_a as the Legendre polynomials series and taking into account that only the first Legendre harmonic contributes in parallel fluxes, one can replace f_a with ξf_{a1}, where $\xi = \frac{u_\parallel}{u}$ and $f_{a1} = (3/2)\int_{-1}^{+1} d\xi\, \xi f_a$. Solution of Eq. (1) can be sought as $f_a = \xi f_{a1}$ with the series

$$f_{a1} = \frac{m_a u}{T_a} w(\kappa) F_{aMJ}(\kappa) \sum_i a_i V_{\parallel i}^a L_i^{(\alpha)}(\kappa), \tag{8}$$

where $w(\kappa)$ is the weight function. Direct substitution of Eq. (8) into Eq. (7) gives

$$w(\kappa) = C_{aMJ}^{-1} \kappa^{\mathcal{R}} \left(\frac{2}{\gamma+1}\right)^{3/2}, \tag{9}$$

($w = 1$ when $c \to \infty$) and

$$a_i = a_i^0 \frac{\Gamma\left(i + \frac{5}{2}\right)}{\Gamma\left(i + \frac{5}{2} + \mathcal{R}\right)}, \quad a_i^0 = \frac{3\,(2i)!!}{(2i+3)!!}. \tag{10}$$

Here it is taken into account that

$$\int_0^\infty d\kappa\, e^{-\kappa} \kappa^\alpha L_i^{(\alpha)}(\kappa) L_j^{(\alpha)}(\kappa) = \frac{\Gamma(i+\alpha+1)}{i!} \delta_{ij}. \tag{11}$$

Find that in the case $c \to \infty$, i.e. with $w = 1$ and $\mathcal{R} = 0$, the expressions (8-10) perfectly coincide with the non-relativistic formulas given in [10].

For calculation of neoclassical parallel fluxes, collision operator with parallel momentum conservation can be approximated as [8]

$$C_{ab}(f_a) = \nu_D^{ab}(u) L(f_a) + \xi\left(C_1^{ab}(f_{a1}) + \nu_D^{ab}(u) f_{a1}\right) \tag{12}$$

and $C_1^a(f_{a1}) = \sum_b C_1^{ab}(f_{a1})$ can also be sought as series

$$C_1^a(f_{a1}) = \frac{m_a u}{T_a} w(\kappa) F_{aMJ}(\kappa) \sum_j a_j F_{\parallel j}^a L_j^{(\alpha)}(\kappa), \tag{13}$$

with the parallel collisional friction forces

$$n_a F_{\parallel j}^a = \int d^3u\, v_\parallel L_j^{(\alpha)}(\kappa) \times \times \sum_b \left(C^{ab}[f_{a1}, F_{bMJ}] + \delta_{ab} C^{ab}[F_{bMJ}, f_{a1}]\right). \tag{14}$$

Taking Eq. (8) into account, one can see that the integrals in Eq. (13) are well defined and can be calculated directly through the parallel fluxes as $F_{\parallel j}^a = \sum_{l,b} c_{jl}^b V_{\parallel l}^b$ [8 – 10]. The coefficients c_{jl}^b can be easily calculated and they are not shown here due to its cumbersomeness.

Now, following [8], let us introduce the adjoint monoenergetic rDKE,

$$V(g_a) + \nu_D^a \mathcal{L}(g_a) = \frac{1}{R_0} b v_\parallel F_{aMJ}, \tag{15}$$

with the solution which gives the mono-energetic transport coefficients. Here, $\mathcal{L} = \frac{1}{2}\frac{\partial}{\partial \xi}(1 - \xi^2)\frac{\partial}{\partial \xi}$ is the Lorentz operator, $\nu_D^a(u) = \sum_b \nu_D^{ab}(u)$ is the deflection frequency and R_0 is the characteristic spatial scale. Then, applying the operation $\langle \int d^3u\, \hat{f}_a L_i^{(\alpha)} \ldots \rangle$ to Eq. (15), where $\hat{f}_a = f_a / F_{aMJ}$ and $\langle \ldots \rangle$ is the averaging over the magnetic flux surface, and using then the adjoint properties of the Vlasov and collision operators, one can obtain the following:

$$\sum_j \left\{ \langle b V_{\parallel j}^b \rangle \left(\delta_{ij} - \frac{2a_j}{\langle b^2 \rangle u_{th}^2} \times \left[[\![\gamma w \nu_D^a L_i^{(\alpha)} D_{33}^a]\!]_i + \sum_{l,b} c_{jl}^b [\![\gamma w \nu_D^a L_i^{(\alpha)} D_{33}^a]\!]_i \right] \right) \right\} = -[\![D_{31}^a]\!]_i \cdot A_1^a + [\![L_i^{(\alpha)}(\kappa) D_{33}^a]\!]_i \cdot A_2^a - [\![D_{31}^a]\!]_i \cdot A_3^a. \tag{16}$$

Here, the Onsager symmetry, $D_{31} = -D_{13}$, is used, and D_{nm} are the relativistic mono-energetic coefficients [12]. The operation of the energy convolution with the relativistic Maxwellian is defined as

$$[\![\varphi(\kappa)]\!]_i = \frac{2C_{MJ}}{\sqrt{\pi}} \int d\kappa\, \kappa^{1/2} e^{-\kappa} \gamma \sqrt{\frac{\gamma+1}{2}} L_i^{(\alpha)} \varphi(\kappa). \tag{17}$$

One can see that this system of algebraic equations with respect to the parallel fluxes can be easily solved if the mono-energetic transport coefficients are already calculated. If the series in Eq. (8) and Eq. (13) are cut off with only first two terms taken into account (i.e.

with $i = 0, 1$), then the heat and particles fluxes are directly defined.

SUMMARY

In this paper, the moment-equation technique, previously developed for non-relativistic plasmas [8-10], is generalized to use in the relativistic approach. It opens a possibility to apply the neoclassical transport theory at arbitrary temperatures. It is shown that the obtained system of linear algebraic equations for parallel fluxes, $\langle bV_{\parallel 0}^e \rangle = \frac{\Gamma_{\parallel}^e}{n_e}$ and $\langle bV_{\parallel 1}^e \rangle = -q_{\parallel}^e / n_e T_e$, can be solved directly without calculation of the distribution function if the monoenergetic transport coefficients are already known. Note, that the relativistic monoenergetic transport coefficients can be calculated by any non-relativistic solver [12].

APPENDIX: DERIVATION OF THE RELATIVISTIC HEAT FLUX IN THE COVARIANT FORMULATION

It is shown that the definition of the relativistic heat flux introduced in Eq. (6) and [6, 11] conforms well with the definition accepted in the relativistic kinetics, based on the covariant formalism [14].

In the relativistic covariant formalism, all values required for transport equations (in particular, the energy, as well as fluxes of particles, the energy and the heat) are usually defined with the help of the four-vector of the particle flow,

$$N^{\alpha} = \int \frac{d^3p}{p^0} p^{\alpha} f(\boldsymbol{x}, \boldsymbol{p}), \qquad \text{(A1)}$$

and the four-tensor of the energy-momentum,

$$T^{\alpha\beta} = c \int \frac{d^3p}{p^0} p^{\alpha} p^{\beta} f(\boldsymbol{x}, \boldsymbol{p}), \qquad \text{(A2)}$$

where $p^0 = (m^2c^2 + p^2)^{1/2}$ and $\alpha, \beta = 0, 1, 2, 3$ (we do not consider the moments of higher order). Note that $T^{\alpha\beta}$ naturally contains the rest energy and its flow, while the transport equations do not require these values. Since $v = cp/p^0$, the spatial flow of particles is $\Gamma^i \equiv N^i$ with $i = 1, 2, 3, 4$, while the time-like flow is related to the density as $N^0 = cn$. The internal energy enclosed in the electron distribution function can be expressed from the energy-momentum tensor as $W_e = T^{00} - n_e m_e c^2$, which for the relativistic Maxwellian is

$$T^{00} = n_e T_e \left(\mu_r \frac{K_3}{K_2} - 1 \right).. \qquad \text{(A3)}$$

Now, one can easily prove that $W_e = n_e T_e \left(\frac{3}{2} + \mathcal{R} \right)$ with $\mathcal{R}$ given by Eq. (3).

In a similar way, the spatial energy flux, Q^i, and the heat flux, q^i, can be calculated [14],

$$Q^i = cT^{0i} - m_e c^2 \Gamma^i, \qquad \text{(A4a)}$$

$$q^i = cT^{0i} - h_e \Gamma^i,, \qquad \text{(A4b)}$$

where cT^{0i} is the total energy flow and $h_e = \frac{T^{00}}{n_e} + T_e$ is the enthalpy of electrons, that for the relativistic Maxwellian is $h_e = m_e c^2 K_3 / K_2$ [15]. The heat flux, Eq. (A4b), can be written as

$$q^i = Q^i - \mu_r \left(\frac{K_3}{K_2} - 1 \right) T_i \Gamma_i, \qquad \text{(A5)}$$

and, finally, using the definition of $\mathcal{R}$ given by Eq. (3), one can easily obtain Eq. (6).

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Article received 22.12.2014

РЕЛЯТИВИСТСКИЕ ЭФФЕКТЫ В НЕОКЛАССИЧЕСКОМ ПЕРЕНОСЕ ЭЛЕКТРОНОВ

И. Марущенко, Н.А. Азаренков

Предложено обобщение метода сохранения параллельных импульсов на случай релятивистского приближения. Это необходимо для корректного вычисления параллельных неоклассических потоков и, в частности, бутстреп-тока при температурах, которые требуются для осуществления термоядерного синтеза. Показано, что полученная система линейных алгебраических уравнений может быть решена непосредственно, без вычисления функции распределения, если известны релятивистские моноэнергетические коэффициенты, причем последние могут быть вычислены любым нерелятивистским солвером путём переобозначения величин, входящих в дрейфово-кинетическое уравнение.

РЕЛЯТИВІСТСЬКІ ЕФЕКТИ В НЕОКЛАСИЧНОМУ ПЕРЕНЕСЕННІ ЕЛЕКТРОНІВ

І. Марущенко, М.О. Азарєнков

Запропоновано узагальнення методу збереження паралельних імпульсів на випадок релятивістського наближення. Це необхідно для коректного обчислення паралельних неокласичних потоків і, зокрема, бутстреп-струму за температур, які потрібні для здійснення термоядерного синтезу. Показано, що здобуту систему лінійних алгебраїчних рівнянь можна розв’язати безпосередньо, без обчислення функції розподілу, якщо відомі релятивістські моноенергетичні коефіцієнти, причому останні можуть бути обчислені будь-яким нерелятивістським солвером шляхом перепозначення величин, що входять до дрейфово-кінетичного рівняння.