REGIMES WITH THE DYNAMIC CHAOS IN ANALISIS OF LINEAR SYSTEMS

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The examples are shown which demonstrate that in the course of analysis of linear systems the modes with dynamic chaos can occur. This can happen when for analysis of linear systems the change of variables is used resulting in the need to investigate either non-linear equations or non-linear functions. In the plasma physics field such situation may arise, e.g., when diagnosing some plasma parameters.

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INTRODUCTION

At present time it is generally accepted that the regimes with dynamic chaos appear only in nonlinear systems. In general, this is correct. However, in some cases, when studying the linear systems one can run into the dynamics similar to the dynamics of modes with dynamic chaos. It may seem that this dynamic is not possible. In reality, in many cases such dynamic is quite natural. It can occur, for example, when for the analysis of linear systems the change of variables is necessary, and thus the initial linear system transforms to the one described by a system of nonlinear equations. The well-known examples are the equations of classical mechanics and the equations of geometrical optics. Original for these equations are linear equations of quantum mechanics and Maxwell equations, respectively.

In such a case, these new variables can lead to appearance of the modes with dynamic chaos. The change of variables modes to those with dynamic chaos can occur in linear systems if the processing of the results of the linear system dynamics is produced by non-linear characteristics. In our previous works (see, for example, [1-3]), we have illustrated the possibility for such regimes to be realized.

This paper presents the results of further studies in this direction. Attention is drawn to the fact that there are two causes of dynamic chaos: (i) an immediate change of variables and (ii) the features of these non-linear equations. The first reason apparently is nonphysical. As an example of the role of the second reason, the dynamics of waves propagating in layered dielectric was considered. The recurrence relations for the coefficients of reflection from and transmission through the layered media weкe found. Thus a non-linear characteristic of the field was inserted. The conditions when this recurrent sequence becomes chaotic have been found too.

1. REPLACEMENT LEADING TO THE CHAOTIC DYNAMICS

A convenient characteristic allowing to determine such changes is a measure of "volume" $\Delta\vec{x}$ in the phase space: $\Delta\mu = p(\vec{x}_i)\cdot\Delta\vec{x}$. Here $p(\vec{x}_i)$ is a probability density for the studied dynamical system to get to point $\vec{x}_i$, that belongs to "volume" $\Delta\vec{x}$. Let's assume that substitution $\vec{z} = f(\vec{x}_i)$ was done in such a way that $\vec{z}$ is an image and $\vec{x}_i$ is an inverse image of the point $\vec{z}$. Principally, there can be many inverse images. The set of vectors $\vec{z} = f(\vec{x}_i)$ do form a new phase space.

We consider in this space the "volume" $\Delta\vec{z}$ ($\vec{z}-\Delta\vec{z}/2$; $\vec{z}+\Delta\vec{z}/2$). By definition, a measure of the magnitude of this volume will be: $\Delta\mu_z = g(\vec{z})\cdot\Delta\vec{z} = \sum\limits_i p(\vec{x}_i)\cdot\Delta\vec{x}_i$. Here the sum is carried out along the number of inverse images. Then the density of probability of the new phase space is determined by the formula:

$$g(\vec{z}) = \sum_i p(\vec{x}_i)\frac{\Delta\vec{x}_i}{\Delta\vec{z}} = \sum_i \frac{p(\vec{x}_i)}{|J|}\ , \qquad (1)$$

where $J = \det(\partial\vec{f}/\partial\vec{x})$ is the Jacobian of transformation.

Equation (1) is practically the Perron - Frobenius formula for transformation of probability density when converting the functions. Let's consider, as an example, the high-usage and most important replacement. Suppose, that we have $\vec{x}_k = A'_k, A''_k$, where $A_k = A'_k + iA''_k$. If one carries out the replacement: $A_k = A'_k + iA''_k = a_k\cdot\exp\ i\varphi_k$, then it is easy to verify that $|J| = 1/|a|$. If in initial variables the motion was regular, then $p(\vec{x}_i) \sim \delta(\vec{x}_i - \vec{x}_i(t))$. The density of probability for new variables become indeterminate ($g(\vec{z})-?$) when $a\to 0$. It is useful to note that this transformation describes the transition from quantum consideration to classical consideration. In this case the condition $a\to 0$ may mean that the velocity of particles are going to zero. This fact corresponds to a well-known result that the transition from quantum to classical consideration for particles with zero speed is not correct. Using this simplest example, we will examine where this might lead to. Let the complex variables satisfy the following equations:

$$\dot{x} = y\ , \quad \dot{y} = -x\ . \qquad (2)$$

Then there are two possible replacements. First $x = x_R + i\cdot x_I$, $y = y_R + i\cdot y_I$. In this case, the new dynamics does not arise. But the new dynamics occurs when another replacement is used, namely: $x = x_0 \cdot \exp\ i \cdot x_2$, $y = x_1 \cdot \exp\ i \cdot x_3$.

Here $x_k(t)$, $k =$ 0,1,2,3, are real functions. After substitution these replacement into (2) we will get the system of equations:

$$\dot{x}_0 = x_1 \cos\Phi \quad \dot{x}_1 = -x_0 \cos\Phi$$

$$\dot{\Phi} = \left(\frac{x_0}{x_1} - \frac{x_1}{x_0}\right) \cdot \sin\Phi \qquad (3)$$

with $\Phi = x_3 - x_2$

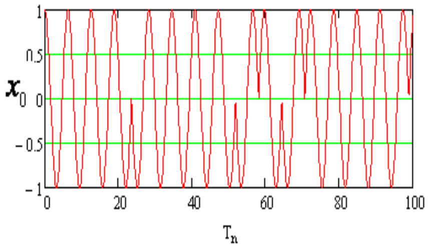

Fig. 1. Time dependence of x_0

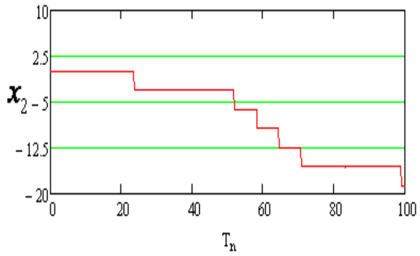

Fig. 2. Time dependence of phase x_2. One can see jumps of phase at the moments when amplitude don't change sign when passing zero point

In addition, it follows from first two equations of the system (6) the existence of the integral: $x_0^2 + x_1^2 = const$. Taking into account this integral, the system of equations (3) has only one degree of freedom. As soon as it gets to the point, for example $x_0 = 0$, then the corresponding phase (x_2) can be undefined. These features of the dynamics of the amplitude x_0 and phase x_2 are shown in Figs. 1-3.

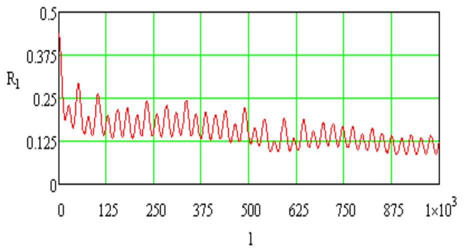

Fig. 3. Autocorrelation function for x_0 *variable*

From Figs. 1, 2 we see that at random time moments the value of x_0 does not change its sign and the value of phase undergoes a jump. Similar random phase jumps for x_3 are observed when $x_1 = 0$.

Above we have considered the transformation occuring most frequently in physics, especially in radiophysics. In addition to this transformation, often is used the transformation that transforms the linear equation of second order to the Riccati equation, which is a first order nonlinear equation. It may be expected that the dynamics of such new variables can also be chaotic. We will show that the exchange, which led to the Riccati equation, is equivalent to the one used above. Indeed, suppose we have the equation $\ddot{x} + \Omega^2 x = 0$. By the use of the replacement $x_0 = \dot{x}/x$ for the new variable this linear equation transforms into nonlinear Riccati equation:

$$\dot{x}_0 = -x_0^2 - \Omega^2 , \qquad (4)$$

with the relation between initial variable and new variable as: $x(t) = a \exp(\int_0^t x_0(t)dt)$

If we present $x_0(t)$ in the form $x_0 = i\Omega + \alpha(t) + i\delta(t)$, $\alpha = \operatorname{Re} x_0$, $\Omega + \delta(t) = \operatorname{Im} x_0$, then the expression for x(t) can be rewritten as: $x(t) = A(t)\exp(i\Omega t + i\beta(t))$, where

$$A(t) = a\exp\left(\int_0^t \alpha dt\right), \quad a = const\,, \quad \beta(t) = \int_0^t \delta(t)dt\,.$$

One can see that this expression is identical to the previous expression.

2. TRANSMISSION OF ELECTROMAGNETIC WAVES THROUGH THE LAYER WITH LAYERED INHOMOGENEITY

Now we shall consider the layered media, with layers arranged perpendicular to the axis z with inhomogeneous medium occupying the region $0 \le z \le L$. For simplicity, it is assumed that there are only two different periodically alternating layers. The dielectric constant of these layers and their thicknesses are equal ε_0, d_0; ε_1, d_1 , respectively.

We assume that from the left a homogeneous half-space ($z < 0$) a flat monochromatic electromagnetic wave falls on a non-uniform layer. For simplicity, the wave is incident at right angle to the interface. The components of electric and magnetic fields of the wave obey the following equations:

$$\frac{\partial E_x}{\partial z} = ikH_y\,, \quad \frac{\partial H_y}{\partial z} = ik\varepsilon(z)E_x\,, \quad \frac{\partial^2 E_x}{\partial z^2} + k^2\varepsilon(z)E_x = 0\,. \quad (5)$$

We are looking for reflectance and transmittance of the wave through an inhomogeneous layer with the fields in an arbitrary homogeneous layer having the form:

$$E_x = A_n exp(ik_n z) + B_n exp(-ik_n z)\,, \qquad (6)$$

$$H_y = \sqrt{\varepsilon_n}\ \ A_n exp(ik_n z) - B_n exp(-ik_n z) \ ,$$

where $n_k \equiv \sqrt{\varepsilon_k}$ - the refractive index of the layer.

We use the matrix method (see, e.g., [4, 5]) for determination of the transmission and reflection coefficients. Under this method, the fields at the left and right edges of the layer are connected with each other by the following matrix:

$$\begin{pmatrix}E_x^-\\ H_y^-\end{pmatrix}=M_n\begin{pmatrix}E_x^+\\ H_y^+\end{pmatrix},\quad M_n=\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix},\qquad (7)$$

where $a_{11}=\cos k_n d_n=a_{22}$; $a_{12}=-i\sin\ k_n d_n\ /\sqrt{\varepsilon_n}$; $a_{21}=-i\sin\ k_n d_n\ \cdot\sqrt{\varepsilon_n}$;

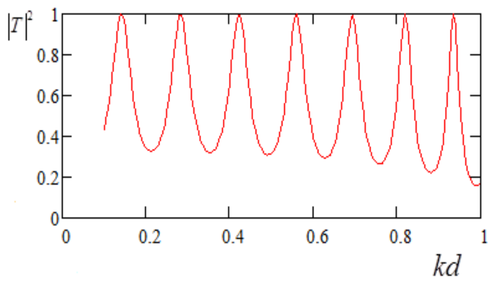

Fig .4. Energetic coefficient of the transmission (number of layers is 10). $n_0=5,\ n_1=2$

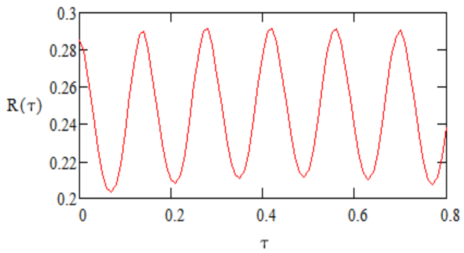

Fig. 5. Autocorrelation function (number of layers is 10). $n_0=5,\ n_1=2$

The connection between fields at the left and right boundaries of the double layer will be determined by the following relation:

$$\begin{pmatrix}E_x^-\\ H_y^-\end{pmatrix}=M_1M_0\begin{pmatrix}E_x^+\\ H_y^+\end{pmatrix}\ .\qquad (8)$$

The elements of matrix $M\equiv M_0M_1=\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix}$ are easy to determine:

$$A_{11}=\cos k_0d_0\cos k_1d_1-\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_0}}\sin k_0d_0\sin k_1d_1\,,$$

$$A_{22}=\cos k_0d_0\cos k_1d_1-\frac{\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_1}}\sin k_0d_0\sin k_1d_1\,,$$

$$A_{12}=\frac{-i}{\sqrt{\varepsilon_1}}\cos k_0d_0\sin k_1d_1-\frac{i}{\sqrt{\varepsilon_0}}\sin k_0d_0\cos k_1d_1\,,$$

$$A_{12}=-i\sqrt{\varepsilon_1}\cos k_0d_0\sin k_1d_1-i\sqrt{\varepsilon_0}\sin k_0d_0\cos k_1d_1\,.$$

If there are N such double layers, the relation between fields at the left ($z=0$) and right ($z=L$) boundaries will be expressed by the formula:

$$\begin{pmatrix}E_x^-\\ H_y^-\end{pmatrix}=M^N\begin{pmatrix}E_x^+\\ H_y^+\end{pmatrix}.\qquad (9)$$

In the general case the matrix $L\equiv M^N$ looks like a quite complicated one, and its elements could be found only by calculus of approximations. However, in some special cases the elements of the matrix can be strongly simplified, in particular, the elements become much simpler if the layers with identical optical thickness $k_0d_0=k_1d_1=kd$, are under consideration.

We should pay attention that the matrix M_n is a 2x2 matrix, which diagonal elements are real functions and nondiagonal elements are imaginary functions. The matrix M is of a similar structure. It is easy to show that similar structurewill also the transfer matrix, which links the fields at the outer boundaries of the inhomogeneous layer:

$$L\equiv M^N=\begin{pmatrix}l_{11}&il_{12}\\ il_{21}&l_{22}\end{pmatrix}.\qquad (10)$$

These elements, after some transformations, can be expressed in terms of Chebyshev polynomials of the second kind:

$$l_{11}=A_{11}U_{N-1}(s)-U_{N-2}(s)\,,$$

$$l_{22}=A_{22}U_{N-1}(s)-U_{N-2}(s)\,,\quad l_{12}=A_{12}U_{N-1}(s)\,.$$

Here $s=\ A_{11}+A_{22}\ /2$; $U_N(s)$ - Chebyshev polynomials .

Using elements of matrix L it is easy to express transmission and reflection coefficients:

$$T=\frac{2}{\left[l_{11}+l_{22}\sqrt{\varepsilon_N}+i(l_{12}\sqrt{\varepsilon_N}+l_{21})\right]}\ ,\qquad (11)$$

$$R=\frac{\left[\ l_{11}-l_{22}\sqrt{\varepsilon_N}\ \ +i(l_{12}\sqrt{\varepsilon_N}-l_{21})\right]}{\left[\ l_{11}+l_{22}\sqrt{\varepsilon_N}\ \ +i(l_{12}\sqrt{\varepsilon_N}+l_{21})\right]}.\qquad (12)$$

Expressions (11) and (12) can be analyzed by numerical methods. The most informative are energetic reflection coefficients and energetic transmission coefficient, which have been studied.

It was shown that there is a range of parameters, where the dependence of these characteristics on the frequency of the incident radiation is irregular. Figs. 4-7 show two typical cases. In the first case (see Figs. 4, 5) parameters of the inhomogeneous layers are chosen so that the dependence of the transmission coefficient on frequency is a regular function.

In this case the correlation function oscillates without decreasing the amplitude of the oscillations. However, with increasing the number of layers, this dependence becomes irregular (see Figs. 6, 7), and the amplitude of the correlation function decreases rapidly.

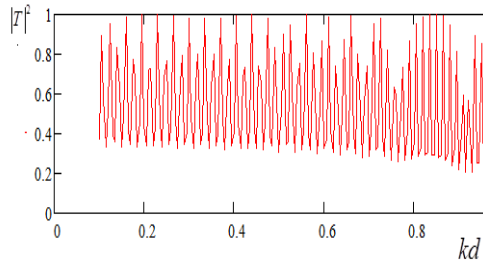

Fig. 6. Energetic coefficient of the transmission (number of layers is 80). $n_0 = 5,\ n_1 = 2$

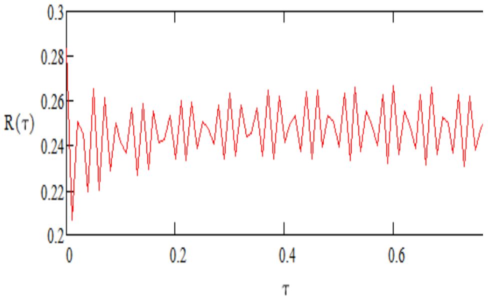

Fig. 7. Correlation function (number of layers is 80). $n_0 = 5,\ n_1 = 2$

CONCLUSIONS

Thus, the above presented results do additionally confirm the following main conclusions: when providing the study of linear systems, there is a high probability to encounter the dependences that are inherent in systems with dynamic chaos. This situation arises when for analysis of the dynamics of linear systems variables that are either themselves nonlinear, or obey to non-linear equations are used. In plasma physics, such a situation can be met, for example, in the diagnosis of plasma parameters.

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РЕЖИМЫ С ДИНАМИЧЕСКИМ ХАОСОМ ПРИ АНАЛИЗЕ ДИНАМИКИ ЛИНЕЙНЫХ СИСТЕМ

В.С. Антипов, В.А. Буц

Приведены примеры, которые показывают, что при анализе динамики линейных систем могут возникать режимы с динамическим хаосом. Это случается, когда для анализа линейных систем используются замены переменных, которые приводят к необходимости исследовать либо нелинейные уравнения, либо нелинейные функции. Для физики плазмы такая ситуация может возникнуть, например, при диагностике параметров плазмы.

РЕЖИМИ З ДИНАМІЧНИМ ХАОСОМ ПРИ АНАЛІЗІ ДИНАМІКИ ЛІНІЙНИХ СИСТЕМ

В.С. Антіпов, В.О. Буц

Наведено приклади, які показують, що при аналізі динаміки лінійних систем можуть виникати режими з динамічним хаосом. Це трапляється, коли для аналізу лінійних систем використовуються заміни змінних, які призводять до необхідності досліджувати, або нелінійні рівняння, або нелінійні функції. Для фізики плазми така ситуація може виникнути, наприклад, при діагностиці параметрів плазми.