GLOBAL-IN-TIME SOLUTIONS OF 3-DIMENSIONAL VLASOV-POISSON SYSTEM

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In this paper, we consider 3-dimensional Vlasov-Poisson system and study their qualitative properties of global-in-time solutions from the point of view integrability, stability, and optimality using the concept of special constructed integral equations and the Lyapunov direct method. We provide criteria that guarantee the existence of solution for the above mentioned properties By using this approach we reduce the problem of the charged-particle beam transport, in particular, focusing with acceleration, to a problem of optimal control of dynamic system.

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1. INTRODUCTION

It is well known that the Vlasov equation [1] are employed to describe the dynamics of a charged-particle flow under condition that the pair collisions are of little consequence. This equation presents itself a kinematic equation with self-consistent field and free of a collision term. Under absence of the magnetic field and when the particle has charge e and mass M this equation has the form

$$\partial_t + v\partial_x f + \frac{e}{m} E\partial_v f = 0. \tag{1}$$

Here f is a particle density in the phase space, therefore $fdxdv$ is a number of particles inside the volume dx with velocities in the interval dv.

The self-consistent electric field E, appearing in (1), should satisfy Poisson equation (see Vlasov [1])

$$\mathrm{div}E = 4\pi e\int f(t,x,v)dv + 4\pi\rho_{ex}, \tag{2}$$

where ρ_{ex} is an external charge.

Denote by U the potential of electric field E. Then the boundary conditions for the axial-symmetric flow can be presented in the following form:

$$U(0) = 0,\ \partial_r U\big|_{r=0} = 0, \tag{3}$$

where r is a beam radius. First condition determines the potential on the beam axis. Second one can be treated as condition for the absence on the axis of any particle which belongs to the beam moving along the axis.

From the above it appears that the problem at hand consists in constructing solutions to the Vlasov equation (1) with account of the Poisson equation (2) and the boundary conditions (3).

It is customary to seek general solutions to equation (1) on characteristics of the dynamical system

$$\dot{x} = v,\ \dot{v} = \frac{e}{m}E(t,x) \qquad x\in\Omega_1\subset R^3,\ v\in\Omega_2\in R^3, \tag{4}$$

using the Cauchy method.

Thus, the characteristics appear as solutions to the equation, which describes dynamics of an individual particle moving in the force field E. This field presents itself the sum of the exterior and average field`s, the latter characterizing effect of the remaining particles to the analyzed one. Thus, the Vlasov equation is applicable in all cases when the collision forces are negligibly small in comparison with the Coulomb forces, given by equation (2).

Note that equation (2) may have solutions even when the dynamics equation possesses no solutions, especially when we are dealing with generalized solutions.

In the case under study the dynamics equations are essentially nonlinear that presents difficulty in finding the characteristics. In view of this, the idea arose to solve this problem with no use of characteristics. The questions on existence and singularity of solutions of the Vlasov equation were given a wide coverage in many papers (see, e.g., [2-5]). As a rule, in the problems of transport a supplementary condition is imposed. In focusing at a given target the transport should be carried on with acceleration. We assume that this condition is specified by the payoff functional:

$$I = \int_0^T w(t,x,v,E)dt,\ T\le\infty,\ w\ge 0\ \forall\ t>0. \tag{5}$$

In his well-known work [6] Bellman showed that on the optimal motions $(I\to \max/\min)$ the following equation should hold true

$$\partial_t f + v\partial_x f + E\partial_v f = w. \tag{6}$$

Solution of this equation is a some absolutely continuous function defined for each E, $E\subset\overline{E}$, where $\overline{E}$ is a set of admissible values of E, and $w[t]\le 0$ almost everywhere in t, $t\in[0,\infty)$, $w[\infty]=0$.

Denote

$$S[t]\overset{\Delta}{=}S\big(x_t(x_0,v_0),v_t(x_0,v_0)\big),$$

where $\big(x_t(x_0,v_0),v_t(x_0,v_0)\big)$ is a solution to equation (4) under the initial values (x_0,v_0).

Problem of Optimal Stabilization.

Let the payoff functional of the transport flow (5) is chosen. It is required to find a self-consistent electrostatic field $\{E_s(t,x,v)\}_{s=1}^3$, providing asymptotic stability of the stationary (non-disturbed) motion by virtue of (1). Herewith, the inequality should be fulfilled

$$\int_0^\infty w\left(t, E, x^0, v^0\right) dt \le \int_0^\infty w(t, E, x, v) dt$$

for all initial values $\{t_0, x_0, v_0\}$ from some domain $\Omega_x \times \Omega_v$.

2. STATIONARY SOLUTIONS

To begin with, we find stationary solutions, appearing in the problem of optimal stabilization. For this purpose we shall analyze

$$Lf = gw \text{ a.e.}, \tag{7}$$

where operator L is introduced for the convenience sake:

$$Lf \equiv v\partial_x f + E\partial_v f,$$

and a $g(\cdot)$ is an arbitrary function from the space $L^2(\overline{\Omega})$, $\overline{\Omega}$ is a closure of the union $\Omega_1 \cup \Omega_2$. Thus, by L is denoted the derivative of function f by virtue of (3):

$$\frac{df[t]}{dt} = Lf.$$

From the conditions imposed above on function w there follows

$$f(x,v) = \int_0^\infty g[t]w[t]dt. \tag{8}$$

Function $f(x,v)$ is absolutely continuous of its initial values $\{x,v\}$ belonging to domain $\overline{\Omega}$. For the convenience sake index 0 is omitted.

Function w under the integral in (7) is fixed while as g may stand any function from $L^2(\overline{\Omega})$. This situation can be defined as follows

$$f = Rg \text{ a.e.} \tag{9}$$

We see that formula (7) with the help of function w specifies the operator R acting on $L^2(\overline{\Omega})$. Let $\{\varphi_k\}_0^\infty$ be the orthonormal system in $L^2(\overline{\Omega})$, the inequality

$$|Rg| \le \left| \sum |c_k|^2 \left| \int_0^\infty \varphi_k w dt \right|^2 \right|^{1/2} \le \left| \sum |c_k|^2 \right|^{1/2} \cdot v(x)$$

is satisfied almost everywhere, where $c_k = \int_\Omega g\varphi_k dxdv$ is the Fourier coefficient of function g,

$$\sum |c_k|^2 = \|g\|_{L^2}^2,$$

$$V(x,v) = -\int_0^\infty w(x[t], v[t])dt.$$

Thus, the inequality

$$|Rg| \le \|g\|_{L^2} V(x,v) \text{ a.e.} \tag{10}$$

is true for all $g \in L^2(\overline{\Omega})$ and consequently R is the Hilbert-Schmidt operator ([7], p.136). That is why, the operator R has a full set of eigen-functions and its spectrum contains only eigen-values. Operator R is predetermined by some symmetric kernel $k(x,v;y,u)$, $\{x,y;v,u\} \in \overline{\Omega}$.

Now we find solutions to the equation

$$Lf_0 = f_0 w. \tag{11}$$

It follows from the above considerations that

$$f_0 = \int_{\overline{\Omega}} k(x,v;y,u) f_0(y,u) dydu.. \tag{12}$$

Let us act by the operator L upon both the left-hand and the right-hand sides of equation (12). Since $f_0 w$ can be uniquely presented in domain $\overline{\Omega}$ as the orthogonal sum

$$f_0 w = \lambda f_0 + g, \int_{\overline{\Omega}} f_0 g dxdv = 0,$$

we come to the Fredholm equation of second kind

$$\lambda f_0 + g = \int \tilde{k}(x,v;y,u) f_0 dydu. \tag{13}$$

where $\tilde{k} = Lk(x,v;y,u)$.

The homogeneous part of the equation has non-trivial solutions which are orthogonal to the free term g, when λ belongs to the spectrum of operator $LR \equiv \tilde{R}$.

In can easily be shown, using relationship (9), that $\tilde{R}$ is the Toeplitz operator. It is generated by function w, that is spectrum $\sigma(\tilde{R})$ completely coincides with the values of function $w(x,v)$ when $\{x,v\} \in \overline{\Omega}$.

The following conclusion can be reached. If the spectrum is concentrated at the zero value of function w and the solution f_0 of equation (11) is absolutely continuous function that is f_0 is the first integral, then $\lambda = 0$ is the eigen-value of operator $\tilde{R}$.

Well known, the symmetric kernel function k may be written down as $\sum \mu_k \psi_k(x,v)\psi_k^*(y,u)$, where μ_k are eigenvalues and $\psi_k(x,v)$ are eigenfunctions for the operator R. This reasoning yields the equation (13) to simple integral equation:

$$\lambda f_p = \int \sum \mu_k L\psi_k(x,v)\psi_k^*(y,u) f_p(y,u) dydu. \tag{14}$$

It is easy to verify that the equation (14) reduces to an algebraic equation

$$\lambda h_p = \sum k_{pq} h_q,$$

where

$$h_p = \int \psi_p^*(y,u) f_o(y,u) dydu,$$

$$k_{pq} = \int L\psi_p(y,u)\psi_m^*(y,u) f_o(y,u) dydu.$$

Then there exists a set $\{f_p\}$ of solutions the equation (14) such that

$$S = \sum_1^6 \alpha_p e^{i\lambda_p t} f_p(x,v) + A \tag{15}$$

is a complete integral of the equation (1) if $\{\alpha_k\}_{k=1}^6$ and A is some set of constants. The approach adopted also permits us to investigate an integrability of Vlasov-Poisson system.

3. COMPLETE INTEGRABILITY

Definition. The system (1) is said to be the completely integrable, if the Hamilton function $H=\sum_{s=1}^{3}\theta_s v_s+\sum_{s=1}^{3}\theta_{3+s}E_s$ may be yields to a function $H'=H'(J_1,...J_6)$.

As well known, if now $\{f_s\}$ in (15) is replaced with new impulse $\{J_s\}$, then we obtain $H'=H+\partial_t S$ and can find the following differential equations

$$\frac{dJ_s}{dt} = 0, \quad s=1,2,...,6,$$

$$\frac{d\theta_s}{dt} = \lambda_s, \; s=1,2,...,6.$$

Here $\{\theta_s, J_s\}_{s=1}^{6}$ are angle - action coordinates.

We considered the situation in which the set $\{f_p\}$ contain the independent first integrals $\{f_p\}_1^5$ only, i.e. the Poisson brackets

$$[H,f_p]=0\ .$$

and $f_s=c_{s1}f_1+c_{s2}f+...+c_{s6}f_6)$, $s\geq 7$, where $\{c_{si}\}_{i=1}^{6}$ are arbitrary constants.

Here the statement $[H,f_p]=0$ is true so that f_p is independent of variables $(\psi_1,...,\psi_6)$, and $Lf_p=0$.

But if all set $\{f_p\}_1^\infty$ is linear independent set of the first integrals then we may construct a soliton-like solution of Vlasov-Poisson systems.

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ГЛОБАЛЬНЫЕ ВО ВРЕМЕНИ РЕШЕНИЯ ТРЕХМЕРНОЙ СИСТЕМЫ УРАВНЕНИЙ ВЛАСОВА-ПУАССОНА

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В этой работе мы рассматриваем 3-х мерную систему уравнений Власова-Пуассона и изучаем качественные свойства их глобальных во времени решений с точки зрения интегрируемости, устойчивости и оптимальности использования концепции специально сконструированных интегральных уравнений и прямого метода Ляпунова. Мы приводим критерии, которые гарантируют существование решения для вышеупомянутых свойств. Благодаря использованию этого подхода мы свели проблему транспорта пучка заряженных частиц, в частности, фокусировку с ускорением к задаче оптимального управления динамической системой.

ГЛОБАЛЬНІ ЗА ЧАСОМ РОЗВ'ЯЗКИ ТРИВИМІРНОЇ СИСТЕМИ РІВНЯНЬ ВЛАСОВА-ПУАССОНА

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В цій роботі ми розглядаємо 3-и вимірну систему рівнянь Власова-Пуассона та вивчаємо якісні властивості їх глобальних за часом розв'язків з точки зору інтегрованості, стійкості і оптимальності використання концепції спеціально сконструйованих інтегральних рівнянь та прямого методу Ляпунова. Ми приводимо критерії, котрі гарантують існування розв'язків для вищезазначених властивостей. Завдяки застосуванню цього підходу ми звели проблему транспорту пучка заряджених частинок, зокрема, фокусування з прискоренням до задачі оптимального керування динамічною системою.