REGULAR AND CHAOTIC DYNAMICS OF DECAYING CASCADE OF WAVES IN PLASMA

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The conditions are formulated, at which realization the dynamics of interaction of finite number of decaying waves becomes chaotic. The estimation of characteristic time of transition from regular dynamics to chaotic is given.
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1. INTRODUCTION

Regular dynamics of three-wave and cascade weak non-linearly processes is investigated quite well [1,2,3]. However at some values for parameters of interacting waves this dynamics can be chaotic. In the previous works [4,5] we formulated the conditions for occurrence of chaotic dynamics for three-wave processes. The main point of considered process is follow. At an initial stage transformation of energy of a wave (the pumping wave) into other waves has a character of instability. Generalizing concepts of nonlinear resonance [6,7] for this case, we suggest that the value of increment of this instability γ is the width of nonlinear resonance. And, as soon as parameter $K \equiv \gamma / \delta\omega$ ($\delta\omega$ - distance between different waves resonances), determining the extent of overlapping for nonlinear resonances of various waves, becomes more than 1 ($K > 1$), one can expect that, dynamics of this waves interaction becomes chaotic. These qualitative considerations were strictly proved by analytical and numerical methods in [4,5].

The occurrence of chaotic dynamics in cascade processes is a subject of the present research. It is necessary to pay attention that the transformation of regular dynamics into chaotic one can occur at enough small intensity of interacting waves fields.

2. STATEMENT OF PROBLEM. BASIC EQUATIONS

In previous investigation we have considered one of the simplest case of interaction - three waves interaction, which is connected among themselves by resonance condition. In common case the number of waves, which take part in interaction, can be rather large. At weak nonlinear interaction of waves the equation for slow changing complex amplitudes of interacting waves can be easily obtained from the Maxwell's equations and equations of medium motion. In an approximation of three-wave interaction we deal with the cascade of interacting waves. Thus, if the conditions of a resonance are fulfilled enough well, then take place the interaction of waves with a fixed phase.

We consider for a simplicity that, many high-frequency (HF) waves and one low-frequency (LF) wave participate in interaction of waves with a fixed phase, waves dumping is neglected, and the matrix elements of waves interaction do not depend on the number of high-frequency wave. At not too large pumping wave amplitudes, when the characteristic times of nonlinear processes is greater than a period of LF wave, the LF wave amplitude is varied slowly during the time, it is possible to average the obtained equations on a period of LF wave. As a result we have a set of equations, which for the first time, apparently, was obtained in [3,8] and was analyzed by many authors:

$$i\frac{da_n}{dt} = b\,a_{n-1}e^{i\delta t} + b^*\,a_{n+1}e^{-i\delta t} \qquad (1a)$$

$$i\frac{db}{dt} = \sum_{n=-N_1}^{N_2} a_{n-1}^* a_n e^{-i\delta t} \qquad (1б)$$

Here detuning $\delta = \omega_n - \omega_{n-1} - \Omega$, ω_n - frequency of HF wave, Ω -frequency of LF wave.

It is important to mark that in real systems the number of high-frequency interacting waves though is large, but finite. Therefore, we shall consider amplitudes of waves $a_n = 0$, if n lies outside the range of values which is determined by condition $-N_1 \le n \le N_2$ (N_1 - number of "red" satellites, N_2 - number of "blue" satellites). The set of equations (1) accurate to terms $\sim 0\,(\Omega/\omega_0, \kappa/k)$ have integrals of motion:

$$\sum_{n=-N_1}^{N2} \omega_n |a_n|^2 + \Omega\,|b|^2 = \varepsilon\,, \qquad (2)$$

$$\sum_{n=-N_1}^{N2} |a_n|^2 = I\,. \qquad (3)$$

The relation (2) represents a full energy of interacting waves, and the relation (3) can be interpreted as number of HF waves. Besides, it is possible to show that

$$\frac{d}{dt}\sum_{n=-N_1}^{N2} a_{n-1}^* a_n = (|a_{-N_1}|^2 - |a_{N_2}|^2)\,. \qquad (4)$$

And in case, when the number of "red" and "blue" satellites is equal ($N_1 = N_2$) take place one more integral of motion

$$\sum_{n=-N_1}^{N_2} a_{n-1}^* a_n = const\,. \qquad (5)$$

Really, if at initial time it is given HF wave ($a_0^0 = a_0(t=0)$) and small impurity of LF wave and the number of excited satellites is large ($n \to \infty$) it is possible (see [2, 3]) to present solution of the equation (1a) as:

$$a_n(t) = a_0^0 i^n e^{in\beta} J_n(|\mathrm{B}|)\,, \qquad (6)$$

$$\mathrm{B}(t) = 2\int_0^t b(t')\exp(-i\delta t')dt' \;;\; \beta = \arg \mathrm{B}\,.$$

One can easily verify this, by direct substitution (6) in (1a) and using a recursive relation $2J_n' = J_{n-1} - J_{n+1}$. From (6) follows, that the process of satellites excitation is symmetric with respect to changing $n \to -n$. Therefore

$|a_n| = |a_{-n}|$, if the initial conditions are also symmetric, that confirms existence of the integral of motion (5).

The equations of system (1) at this are decoupled. From (5) follows, that if at initial time only one HF wave (pumping wave with amplitude a_0^0) and only one LF wave are given, the process of interaction of HF waves occurs at constant amplitude of LF wave and is described by a system of the linear differential equations (1a).

As can be see from (6), more and more high harmonics a_n become excited during the time due to interaction with LF wave. From properties of Bessel's functions $J_n(x)$ follows that amplitude of excited HF harmonics a_n will be small, if the index of a Bessel's function exceeds the values of argument (n> x), and also at large values of argument (x >> 1, x > n). Thus, if argument of Bessel's function ($|B(t)|$) is a limited value, the number of excited HF waves is limited, the maximum number of excited HF harmonics is determined by a maximum B: $n_{\max} \sim |B_{\max}|$. At large values of argument ($x >> 1, x > n$), amplitude $a_n \sim const/\sqrt{x}$. If, $B(t)$ is growing with t function, the number of excited HF waves will be determined from condition $N_1 \le n \le N_2$. Thus, the dynamics of HF wave amplitude will be determined by dynamics of LF wave, to consideration of which we shall pass.

From equations (1) it is possible to receive the equation for amplitude of low-frequency wave as:

$$\ddot{b}_1 + i\delta \dot{b}_1 + \omega_{NL}^2 b_1 = 0, \quad (7)$$

where $\omega_{NL}^2 = |a_{N_1}|^2 - |a_{N_2}|^2$.

Let's analyze the solutions of this equation in the most simply cases. If the number of «red» satellites is equal to number of «blue» satellites $(N_1 = N_2)$, then $\omega_{NL}^2 = 0$. The solution of the equation (7) becomes:

$$b(t) = b_0 + \frac{1}{i\delta}\dot{b}(0)(e^{i\delta t} - 1) \quad (8)$$

From (8) follows, that value of amplitude of LF wave remains constant if, at $t = 0$ it is given HF and LF waves ($b(0) = b_0$, $a_0^0 = a_0(0)$) and $\dot{b}(0) = 0$.

When two HF waves are given (beat-wave case) the amplitude of LF wave oscillates with the frequency δ. When $\delta \to 0$ the amplitude $b(t)$ increase linearly with time.

3. NUMERICAL ANALYSIS

These analytical results relate to case $n \to \infty$ and do not give a possibility to investigate the dynamics of finite number of interacting waves. For finite case n the set of equations (1) is solved numerically at various initial conditions for fields and detuning parameters δ. We have investigated: temporal dynamics of waves interaction, power spectra (S_ω) of realizations and their autocorrelation functions (C_f), there were selected the real initial values of the fields: $\mathrm{Re}a_0(t=0)=a_0^0$; $\mathrm{Re}a_{-1}(t=0)=a_{-1}^0$; $b(t=0)=b_0$. The imaginary parts of the fields and amplitude of other waves at $t = 0$ are equal zero.

Accuracy of solution was evaluated with the help of integrals of motion. By a choice of an integration step the absolute accuracy of integrals was conserved not more then $10^{-5} \div 10^{-6}$, when the integrals values are about unit.

On Fig.1-2 represents results of calculation of fields amplitudes dynamics for symmetrical case (equal number of «red» and «blue» satellites - $N_1 = N_2 = N = 11$) with $\delta = 0$ for two case of initial values of fields: $a_0^0 = 1$ and $b_0 = 0.2$, and beat-wave case $a_0^0 = 1$, $a_{-1}^0=0.04$.

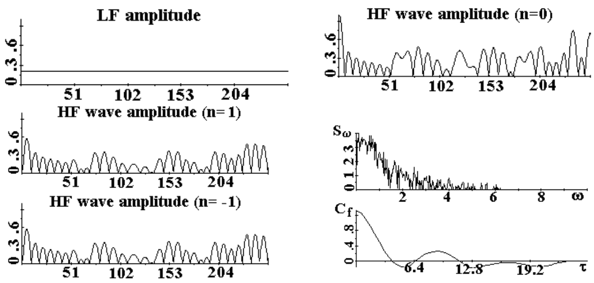

Fig.1. Amplitude of LF wave and HF harmonics with $n = 0, \pm 1$ versus time, S_ω and C_f for HF wave $n = 0$, $a_0^0=1$, $b_0=0.2$, $\delta = 0$. Symmetrical case $N_1 = N_2$.

As expected, at initial value of pumping wave amplitude $a_0^0=1$ and LF wave amplitude $b_0 = 0.2$ amplitude of LF wave remains constant (see Fig.1).

Other character of LF wave dynamics is observed in beat-wave case with $a_0^0=1$ and $a_{-1}^0=0.04$ (see Fig.2). The

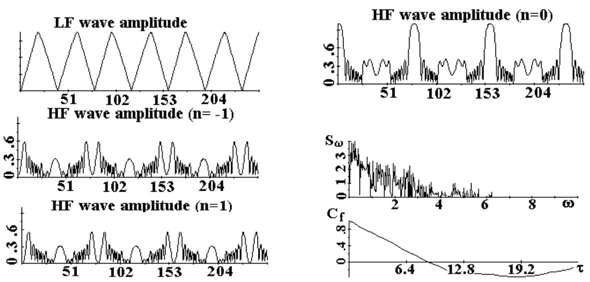

Fig.2 Amplitude of LF wave and HF harmonics with $n = 0, \pm 1$ versus time, S_ω and C_f for HF wave $n = 0$, $a_0^0=a_0(0)$, $a_{-1}^0=0.04$. Symmetrical case $N_1 = N_2$.

amplitude of LF wave at the beginning linearly increases up to value $b_{\max}$ with a velocity V_1 during time $t^{(1)}$. Then the linear growth is replaced by a linear decreasing with a velocity V_2 ($V_2 < V_1$) to zero during $t^{(2)}$. Further follows the linear growth of LF wave amplitude with velocity V_2 up to $b_{\max}$ during $t^{(2)}$ which is replaced by linear decreasing with velocity V_1 to zero during $t^{(1)}$. After that the dynamics of low-frequency wave is repeated with a period equal $T = 2(t^{(1)} + t^{(2)})$. In the both case excitation process of HF harmonics has symmetrical character $|a_{-n}| = |a_n|$. The HF waves amplitudes exactly follows to Bessel's functions with index «n» ($|a_0(t)|=|J_0(|b|t)|$, $|a_1(t)|=|J_1(|b|t)|$ etc.) so long as the inequality $t \le t^*$ ($t^* = N/|b|$ for case $a_0^0=1$, $b_0 = 0.2$ and $t^* = t^{(1)}$ in beat-wave case) is approximately fulfilled.

When $t > t^* = N/b$ (see Fig.1) the dynamics of HF waves amplitudes ceases to follow Bessel's functions becomes more complicated, remaining symmetrical and, apparently, regular.

In beat-wave case (see Fig.2) at times $t \approx t^{(1)}$ the transient period occurs. When $t \ge t^{(1)} + t^{(2)}$ the dynamics of amplitude of HF harmonics has character of a mirror reflection relatively point $t_{ref} = t^{(1)} + t^{(2)}$ and periodically change during the time with period T=2 ($t^{(1)} + t^{(2)}$).

A behavior of spectra and correlation functions correspond a character of realizations. The maximum of power spectra of HF waves is located in low-frequency region and broaden, that agreed with slow change of amplitudes and their enrichment by higher frequencies.

The availability of asymmetry in number of excited HF waves (for example, $N_1 = N_2 - 1$) essentially changes the dynamics of amplitudes as HF and LF.

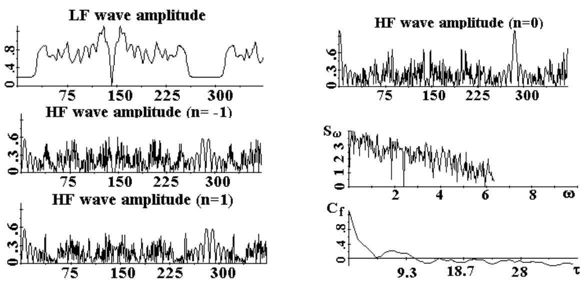

Fig.3. Amplitude of LF wave and HF harmonics with $n = 0, \pm 1$ versus time, S_ω and Cf for HF wave $n = 0$, $a_0^0=1$, $b_0=0.2$, $\delta = 0$. Nonsymmetrical case $N_1 \ne N_2$.

On Fig.3 the results of calculation of dynamics of fields amplitudes for nonsymmetrical case are shown. As it is visible in the beginning the integral (5) is saved and LF wave amplitude is constant. Then the amplitude of LF wave becomes complicated function of time. The dynamics of LF amplitude periodically changes with time. At that the periodic recovery of integral (5) takes place, accompanying by the constancy of amplitude of LF wave. In dynamics of HF waves the obvious periodicity is not observed.

Spectrum of HF power is wide with a slow decreasing to the area of high frequencies. The correlation function fast decreases up to zero and has irregular oscillations near to a zero level that is characteristically for chaotic processes.

For the initial conditions of the beatwave type- $a_0^0=1$, $a_{-1}^0=0.04$ at $\delta = 0$ the dynamics of fields is qualitatively similar to case of the set a_0^0 and b_0. The dynamics of HF waves becomes even more complicated.

One more parameter, which essentially influence on the process of waves transformation, is the availability of detuning $\delta \ne 0$. As follows from the previous analytical considering at small detuning values ($\delta << 1$) the dynamics of LF wave insignificantly differs from the case with $\delta = 0$ on times $t < 1/\delta$. At large times the dynamics of nonresonance interaction is investigated numerically. The availability of detuning parameter leads to additional complicating of the dynamics of waves interaction. Just in case with a broken symmetry ($N_1 \ne N_2$) and small values of detuning the dynamics of waves have chaotic character.

On Fig.4 the results of numerical calculations for asymmetrical case ($N_1 \ne N_2$) are shown with the initial conditions beat-wave $a_0^0=1$ and $a_{-1}^0=0.04$, at detuning parameters $\delta = 0.05$. From these plots it is visible, that

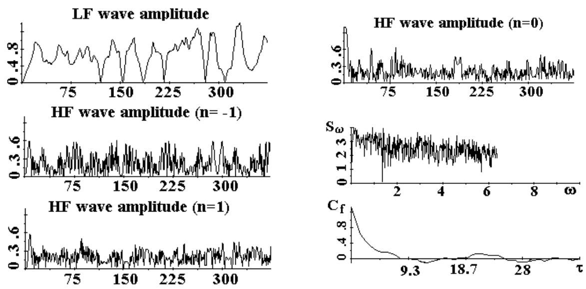

Fig.4. Amplitude of LF wave and HF harmonics with $n = 0, \pm 1$ versus time, S_ω and Cf for HF wave $n = 0$, $a_0^0=1$, $a_{-1}^0=0.04$, $\delta = 0.05$. Nonsymmetrical case $N_1 \ne N_2$.

the amplitudes of high-frequency oscillations look like the form of irregular oscillations with various intensity, which frequency also varies irregular. Spectra of these waves acquire the form like «desktop», the correlation functions fast decrease during time. Therefore, one can speak about random amplitude and frequency modulation of HF waves.

4. CONCLUSION

Thus it is possible to formulate the conditions of dynamic chaos arising at decaying interaction of the finite number of waves. The dynamics of decaying cascade becomes chaotic at small values of detuning parameter, when the finite number of waves with number of «red» satellites which is not equal to number of «blue» satellites participates in interaction.

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