GENERATING FUNCTIONAL IN CLASSICAL HAMILTONIAN MECHANICS

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We give a brief survey of a path-integral formulation of classical Hamiltonian dynamics that means a functional-integral representation of classical transition probabilities. This functional exhibits a hidden BRST and anti-BRST invariance. Therefore a simple expression, in terms of superfields, is received for the generating functional. We extend the results for discrete classical systems to continuum mechanics.

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1. INTRODUCTION

The functional-integral approach initially developed for quantum mechanics by R. Feynman [1], is now widely used in classical statistical physics and statistical hydrodynamics. But recently this method is widely spread in classical Hamiltonian mechanics as well. A first attempt to provide a path-integral formulation of classical mechanics was made in [2-4]. There the evolution of dynamical system is considered as a functional-integral representation of classical transition probabilities with such measure as it gives a weight "one" to the classical paths and a weight "zero" to all other, just as functional δ-function for the classical equations of motion. With the help of anticommuting "ghosts" this measure can be rewritten as the exponents of a certain action $\bar{S}$. We provide an interpretation for the "ghosts" fields as being the well-known Jacobi fields (or "geodesic deviations"), which describe behavior of dynamical systems. The remarkable point about the action $\bar{S}$ is that it exhibits an unexpected BRST and anti-BRST type invariant. This invariance allows for simple expressions, in terms of superfields, of this generating functional.

In this article we give a brief survey of applicating the method of generating functional in classical mechanics. The results for Hamiltonian systems with finite degrees of freedom extend to continuum mechanics. We illustrate how the method of the generating functional, so much useful in the quantum theory and statistical physics, is found also in determination of some problems in Hamiltonian mechanics.

2. THE CONSTRUCTION OF AN EFFECTIVE ACTION

Consider the continuous medium and let its mechanic properties are described by the set of functions $\varphi_\alpha(x,t)$, where α =1,2,…n. The equations of classical evolution are presented by a set of functional-form equations:

$$F_\alpha[\varphi(x,t)] = 0, \tag{1}$$

where α is corresponding to different field components. Now accordingly to [2,3] we introduce a classical generating functional Z(J) in the form of a functional integral:

$$Z[J] = \prod_\alpha \int D\varphi_\alpha(\vec{x},t)\delta[\varphi_\alpha(\vec{x},t) - \varphi^{cl}_\alpha(\vec{x},t)]\times \exp\left[\int dtdxJ_\alpha(\vec{x},t)\varphi_\alpha(\vec{x},t)\right], \tag{2}$$

where $\varphi^{cl}_\alpha(x,t)$ is the solution of Eq. (1), $J_\alpha(x,t)$ are the external sources. Then the δ – functional may be written such as

$$\delta[\varphi_\alpha(\vec{x},t) - \varphi^{cl}_\alpha(\vec{x},t)] = \delta[F_\alpha(\varphi(\vec{x},t))]\det\left[\frac{\delta F_\alpha}{\delta\varphi}\right].$$

The calculation of the determinant in last expression is very a difficult problem and depends essentially from the structure of the functional $F_\alpha(\varphi(x,t))$. But we leave these difficulties, reexpressing the determinant over two anticommuting "ghosts" $c_\alpha(x,t), \bar{c}_\alpha(x,t)$.

Then, representing the delta-functional as a functional-integral over auxiliary field $\lambda(x,t)$ one obtains a field theory representation of the generating functional Z(*J*):

$$Z(J) = \prod_\alpha D\varphi_\alpha D\lambda_\alpha Dc_\alpha D\bar{c}_\alpha \exp[iS_{eff}(\varphi,\lambda,c,\bar{c}) + \int J_\alpha(\vec{x},t)\varphi_\alpha(\vec{x},t)d\vec{x}dt], \tag{3}$$

where

$$S_{eff}(\varphi,\lambda,c,\bar{c}) = \int\lambda_\alpha(x,t)F_\alpha[\varphi(x,t)]dxdt - i\int d\vec{x}d\vec{y}dtdt'\bar{c}_\alpha(\vec{x},t)\frac{\delta F_\alpha(\varphi)}{\delta\varphi_\beta}c_\beta(\vec{y},t'). \tag{4}$$

So, we obtain a way to apply the field theory methods to problems of classical mechanics after expressing the generating functional (2) through the effective action (3), or extending the structure of physical field variables with the auxiliary field $\lambda(x,t)$ and two Grassman anticommuting fields, $c(x,t)$ and $\bar{c}(x,t)$, called "ghosts".

3. HIDDEN BRST SYMMETRY

Turn our attention on the wonderful property of the symmetry of effective action (3) that exhibits BRST-like [5,6] invariance of the form:

$$\delta\varphi_\alpha(x,t) = \bar{\varepsilon} c_\alpha(x,t),\ \delta c_\alpha(x,t) = 0,$$
$$\delta\bar{c}_\alpha(x,t) = i\bar{\varepsilon}\lambda_\alpha(x,t),\ \delta\lambda_\alpha(x,t) = 0. \quad (5)$$

where ε and $\bar{\varepsilon}$ are the anticommuting number parameters. It is possible to prove that the invariance of the effective action (3) under transformations (5) does not depend on the shape and structure original functional. Therefore arises a question about a character of symmetry (3) under anti-BRST transformations:

$$\delta c_\alpha(x,t) = -i\varepsilon\lambda_\alpha(x,t),$$
$$\delta\bar{c}_\alpha(x,t) = 0,$$
$$\delta\lambda_\alpha(x,t) = 0,$$

$$\delta\varphi_\alpha(x,t) = -\varepsilon\bar{c}_\alpha(x,t). \quad (6)$$

It is possible to prove [7] that after transformation (6) we have the following expression for the variation of effective action:

$$\delta S_{eff} = -\int dxdydtdt'\lambda_\alpha(\vec{x},t)c_\beta(\vec{y},t')\left[\frac{\delta F_\alpha(\varphi)}{\delta\varphi_\beta} - \frac{\delta F_\beta(\varphi)}{\delta\varphi_\alpha}\right].$$

In general it is nonzero. But this variance of the effective action vanisches only if the functional F[$\varphi(x,t)$] satisfies the condition of "potentiality", so if it may be present as:

$$F_\alpha[\varphi(\vec{x},t)] = \frac{\delta S(\varphi)}{\delta\varphi_\alpha(x,t)}.$$

It is evident, that the functional $S(\varphi)$ means a classical action, and the Eq. (1) describes a condition of it minimum. So, the classical dynamics equations obtained from the minimum action principle can be described by means of generating functional (3) with the effective action

$$S_{eff}(\varphi,\lambda,c,\bar{c}) = \int\lambda_\alpha(\vec{x},t)\frac{\delta S}{\delta\varphi_\alpha(\vec{x},t)}dxdt - i\int\bar{c}_\alpha(\vec{x},t)\frac{\delta^2 S}{\delta\varphi_\alpha(\vec{x},t)\delta\varphi_\beta(\vec{y},t)}c_\beta(\vec{y},t')dxdydtdt', \quad (7)$$

which has property of invariance under the BRST and anti-BRST transformations.

4. HAMILTONIAN DYNAMICS, CONSTRUCTION OF "SUPERHAMILTONIAN"

So far, our study of classical motion was founded by the functional of action without using Hamilton's properties of its equations. Now, let us consider the consequence, if our system is described by Hamilton equations. The equations of motion for the continuum system with Hamiltonian H(φ) has the form:

$$\dot{\varphi}^b(\vec{x},t) = \omega^{ab}(\vec{x})\frac{\delta H(\varphi)}{\delta\varphi_\alpha(x,t)}, \quad (8)$$

where $\omega(x)$ is the local symplectic two-form and the corresponding symplectic matrix has the property:

$$\omega^{ab}(x)\omega_{bc}(x) = \delta^a_c.$$

Then we construct the corresponding generating functional for equations (8) using the above-mentioned method. Now from (4) and (7) we obtain the expression for the generating functional of Hamilton mechanics:

$$Z(J,\eta,\psi,\bar{\psi}) = \prod_a \int D\varphi^\alpha D\lambda_\alpha Dc^\alpha D\bar{c}_\alpha \exp\left[\int \bar{L}dt\right]\times$$
$$\exp\int[J_\alpha(\vec{x},t)\varphi^\alpha(\vec{x},t) + \eta^\alpha(\vec{x},t)\lambda_\alpha(\vec{x},t) + \bar{\psi}_\alpha(\vec{x},t)c^\alpha(\vec{x},t) + \bar{c}_\alpha(\vec{x},t)\psi^\alpha(\vec{x},t)]dxdt, \quad (9)$$

where the Lagrange's function ("super-Lagrangian") has the form:

$$\bar{L}(\varphi,\lambda,c,\bar{c}) = \int[\lambda_a(\vec{x},t)[\dot{\varphi}^a(x,t) - \omega^{ab}(\vec{x})\varphi_b)] + i\bar{c}_a(\vec{x},t)[\partial_t\delta^a_b + \omega^{ac}\delta_c\delta_b H(\varphi)]c^b(\vec{x},t)]dx. \quad (10)$$

Here we introduce the new external sources, $\eta(x,t)$ and $\psi(x,t)$ corresponding to the additional fields $\lambda(x,t)$ and $c(x,t)$ respectively. From the effective Lagrangian (10) we obtain as usually the general Hamilton's function ("super-Hamiltonian"):

$$\bar{H}(\varphi,\lambda,c,\bar{c}) = \int dxdt[\lambda_\alpha(\vec{x},t)\omega^{ab}\delta_b H(\varphi) + i\bar{c}_\alpha(\vec{x},t)\omega^{ac}\delta_c\delta_b H(\varphi)c^b(\vec{x},t)]. \quad (11)$$

Here an essential moment is that expression (11) can be interpreted as a Hamiltonian's function to be correct, we have to make sure that the fields $\lambda(x,t)$ and $\varphi(x,t)$, as well as $c(x,t)$ and $\bar{c}(x,t)$ are pairs of canonically conjugate variables with the following (graded) Poisson structure:

$$\{\varphi_a(x,t),\lambda_b(x',t)\} = \delta_{ab}\delta(x-x'),$$
$$\{c^b(x,t),c^a(x,t)\} = -i\delta^{ab}\delta(x-x').$$

The formulation of classical mechanics on the basis of introducing the generating functional with an effective Lagrangian (10) gives a possibility to describe from the unified point of view such problems as obtaining the equations of dynamics, problems of ergodicity and stochastization of Hamilton system. We note here that after field's variation in generating functional (9) with the Lagrange's function (10) and by using the minimum action principle we obtain the following equations:

$$\dot{\varphi}^{a}(x,t) - \omega^{ab}\delta_{b}H(\varphi) = 0, \qquad (12.a)$$

They, obviously, form the initial Hamilton's equations.

The equations of "motion" of "ghosts" $c(x,t)$ and $\bar{c}(x,t)$ are defined by the form of the second variation of the effective action (7) and we find:

$$\left[\partial_{t}\delta^{a}_{b} - \omega^{ac}\delta_{c}\delta_{b}H(\varphi)\right]c^{b}(x,t) = 0, \qquad (12.b)$$

$$\left[\partial_{t}\delta^{a}_{b} + \omega^{ac}\delta_{c}\delta_{b}H(\varphi)\right]\bar{c}_{a}(x,t) = 0. \qquad (12.c)$$

It was shown in [4] that the "ghosts" $c(x,t)$ and $\bar{c}(x,t)$ can be interpreted as Jacobi fields $\delta\varphi(x,t)$, i.e. the infinitesimal displacement between two classical trajectories (or "geodesic deviations") [9]. Such correspondence between the "ghosts" and fields of Jacobi explains that there is a possibility to apply them in such problems as beginning of a dynamical chaos in the Hamilton's system with the exponential running of trajectories.

5. SUPERFIELDS IN CLASSIC MECHANICS

Now note the importance meaning of the invariance of effective action (7) under simultaneous BRST and anti-BRST transformations. Just this extended symmetry of the action gives a superfield description of classical mechanics. Let us define the following superfield $\phi(x,t,\theta)$ on the space of initial fields as their combination:

$$\phi^{a}(x,t,\underset{\rightarrow}{\theta},\bar{\theta}) = \varphi^{a}(x,t) + \theta\underset{\rightarrow}{c}^{a}(x,t) + \bar{\theta}\omega^{ab}\bar{c}_{b}(x,t) + i\bar{\theta}\theta\omega^{ab}\lambda_{b}(x,t), \qquad (13)$$

where θ are the Grassmann variables. In terms of superfield Lagrangian (10) and also Hamiltonian (11) assume a very simple form:

$$\bar{L}(\phi,\dot{\phi}) = i\int d\theta\, d\bar{\theta}\left[\frac{1}{2}\phi^{a}\omega_{ab}\dot{\phi}^{b} - H(\phi^{a})\right], \qquad (14)$$

$$\bar{H}(\phi) = i\int d\theta\, d\bar{\theta}\, H(\phi). \qquad (15)$$

Note, that the functional dependence of "super-Hamiltonian" (15) from the superfield has such structure as an initial Hamiltonian in dynamics equations (8). The generating functional (9) in terms of superfield obtaines a form:

$$Z(\Gamma_{\alpha}) = \prod_{\alpha}\int D\phi_{\alpha}\exp[i\int \bar{L}(\phi)d\theta\, d\bar{\theta}\, dt + \int\Gamma_{\alpha}\phi_{\alpha}\, d\theta\, d\bar{\theta}\, dt].$$

Now we see the similarities between this functional integral and the quantum mechanical one. Obviously, the analysis of phase space structure of classical dynamics reduces to the study of the superfield theory.

6. CONCLUSION

In this paper we have reviewed the general framework of a path-integral approach to classical Hamiltonian dynamics. We confirm that our path-integral method is the correct counterpart of the operatorial approach to classical mechanics. The crucial elements of the operatorial formalism mentioned above are the "classical commutation relations" which follow from the classical path- integral.

In this approach we had a lot of extra variables (auxiliary fields) besides those labelling the phase-space of the original mechanical system. It was discovered that these extra variables had a beautiful geometric meaning and therefore unexpected universal symmetries generated by the functional technique were received.

We call the reader's attention to a very important point. The invariance that we have found does not rely on any particular form of the Lagrangian. It is valid for any action, so its consequences must be very general independently on the initial construction of the phase space of the classical system.

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