SUPERCONDUCTIVITY AND INTEGRABILITY

E.D. Belokolos

Institute of Magnetism, National Academy of Sciences, Kiev, Ukraine
e-mail: bel@imag.kiev.ua

The paper is a review of studies of integrability of the BCS Hamiltonian with discussion of some its integrable generalization which present an interest for a number of physical problems.

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1. INTRODUCTION

In 1957 J. Bardeen, L.N. Cooper and J.R. Schrieffer have introduced the BCS Hamiltonian which was very successful in description of the superconductivity. In 1958 N.N. Bogoljubov et al. proved an equivalence of the BCS Hamiltonian to the quadratic one in the thermodynamic limit. At a finite number of particles R.W. Richardson (1965) proved an integrability of the BCS Hamiltonian [1] and M. Gaudin (1976) built an appropriate mathematical theory [2,3]. Recently an interest to the integrability of BCS Hamiltonian was renewed in connection with different applications.

2. INTEGRABILITY OF THE BCS HAMILTONIAN

The BCS Hamiltonian is

$$H_{BCS} = \sum_{i,\sigma} \varepsilon_i c^+_{i,\sigma} c_{i,\sigma} - g\sum_{i,j} c^+_{i\uparrow} c^+_{i\downarrow} c_{j\downarrow} c_{j\uparrow},$$

where are the annihilation and creation operators of electrons. In this Hamiltonian the pairing interaction does not act on singly occupied levels. As a result we may study these levels separately.

By means of the operators

$$S_j^- := c_{j\downarrow} c_{j\uparrow}, S_j^+ := (S_j^-)^+ = c^+_{j\uparrow} c^+_{j\downarrow},$$

$$S_j^z := \frac{1}{2}(c^+_{j\uparrow} c_{j\uparrow} + c^+_{j\downarrow} c_{j\downarrow} - 1),$$

which obey to the commutation relations we can present the BCS Hamiltonian in a form

$$H_{BCS} = \sum_{j=1}^{L} 2\varepsilon_j (S_j^z + 1/2) - g\sum_{j=1}^{L}\sum_{k=1}^{L} S_j^+ S_k^-.$$

The BCS Hamiltonian has the integrals of motion [4]

$$R_{i=} = S_i^z - gH_i, H_i = \sum_{kj=1,k\neq i}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_k}{\varepsilon_i - \varepsilon_k},$$

which commute with each other. The number of pairs N and the Hamiltonian H_{BCS} are linear and quadratic forms of these integrals of motion respectively,

$$N = \sum_{i=1}^{L} (R_i + 1/2),$$

$$H_{BCS} = \sum_{i=1}^{L} 2\varepsilon_i (R_i + 1/2) + g(\sum_{i=1}^{L} R_i)^2 - g\sum_{i=1}^{L} \mathbf{S}_i^2.$$

3. THE GAUDIN ALGEBRA, THE RICHARDSON EQUATIONS, EIGENSTATES AND EIGENVALUES OF THE BCS HAMILTONIAN

1. The Gaudin algebra. Given a set of complex numbers $\{\varepsilon_j, j = 1,...,L\}$ and a set of independent spin operators $\{S_j^+, S_j^-, S_j^z, j = 1,...,L\}$, satisfying the commutation relations

$$[S_j^z, S_k^\pm] = \pm\delta_{jk} S_k^\pm, [S_j^+, S_k^-] = 2\delta_{jk} S_k^z,$$

we define the operator rational functions

$$S^\alpha(\omega) = \sum_{j=1}^{L} \frac{S_j^\alpha}{\varepsilon_j - \omega}, \alpha = z, \pm.$$

The operators $S^\alpha(\omega)$ are the generators of the Gaudin algebra and obey to the commutation relations

$$\left[S^z(\omega), S^z(\omega')\right] = 0, \left[S^\pm(\omega), S^\pm(\omega')\right] = 0,$$

$$\left[S^z(\omega), S^\pm(\omega')\right] = \pm\frac{S^\pm(\omega) - S^\pm(\omega')}{\omega - \omega'},$$

$$\left[S^+(\omega), S^-(\omega')\right] = 2\frac{S^z(\omega) - S^z(\omega')}{\omega - \omega'}.$$

The Gaudin algebra is an infinitedimensional extension of the $sl(2)$ algebra.

We construct the representation of the Gaudin algebra, fixing the highest weight vector $|0\rangle$ by means of the following relations,

$$S^+(\omega)|0\rangle = 0,$$

and define the representation space as a linear hull of vectors

$$|N\rangle = |\omega_1,...,\omega_N\rangle = S^-(\omega_1)...S^-(\omega_N)|0\rangle$$

with arbitrary N complex numbers $\omega_1,...,\omega_N$.

2. The generating function of the integrals of motion.

The operator rational function

$$F(\omega) = S^z(\omega) + (g/2)\mathbf{S}^2(\omega) =$$

$$= \sum_j \frac{R_j}{\varepsilon_j - \omega} + \frac{g}{2}\sum_j \frac{S_j^2}{(\varepsilon_j - \omega)^2}$$

is a generating function of the integrals of motion of the BCS Hamiltonian since

$$R_j = -res_{\omega=\varepsilon_j} F(\omega).$$

It is easy to prove that values of the operator $F(\omega)$ at a different values of ω commute with each other,

$$[F(\omega), F(\omega')] = 0.$$

We can prove that the $|N\rangle = |\omega_1,...,\omega_N\rangle$ are eigenstates of the operator $F(\omega)$ if the quantities $\omega_1,...,\omega_N$ satisfy the Richardson equations

$$\sum_{j=1}^{L} \frac{s_j}{\varepsilon_j - \omega_\alpha} - \sum_{\beta,\beta\neq\alpha}^{L} \frac{1}{\omega_\beta - \omega_\alpha} = \frac{1}{g}, \alpha = 1,...,N.$$

Eigenvalues of the $F(\omega)$ are of the following form

$$F(\omega) = \sum_{j=1}^{L} \frac{s_j}{\varepsilon_j - \omega} + \sum_{\alpha=1}^{N} \frac{1}{\omega - \omega_\alpha} +$$

$$\frac{g}{2}\left(\sum_{j=1}^{L} \frac{s_j}{\varepsilon_j - \omega} + \sum_{\alpha=1}^{N} \frac{1}{\omega - \omega_\alpha}\right)^2 +$$

$$\frac{g}{2}\left(\sum_{j=1}^{L} \frac{s_j}{(\varepsilon_j - \omega)^2} - \sum_{\alpha=1}^{N} \frac{1}{(\omega - \omega_\alpha)^2}\right).$$

3. Eigenstates and eigenvalues of the H_{BCS}.

The function $|N\rangle = |\omega_1,...,\omega_N\rangle$ with parameters $\omega_1,...,\omega_N$, satisfying the Richardson equation, is an eigenstate of integrals of motion R_j with the eigenvalue

$$R_j = s_j + gs_j\left(\sum_{k=1,k\neq j}^{L} \frac{s_k}{\varepsilon_k - \varepsilon_j} + \sum_{\alpha=1}^{N} \frac{1}{\varepsilon_j - \omega_\alpha}\right).$$

The same function $|N\rangle = |\omega_1,...,\omega_N\rangle$ with the same parameters $\omega_1,...,\omega_N$ is an eigenstate of the Hamiltonian H_{BCS} which is a quadratic form of the integrals of motion R_j. In order to calculate the eigenvalues of the H_{BCS} we ought to put the expressions for eigenvalues of the integrals of motion R_j in the formula

$$H_{BCS} = \sum_{i=1}^{L} 2\varepsilon_i(R_i + 1/2) +$$

$$g\left(\sum_{i=1}^{L} R_i\right)^2 - g\sum_{i=1}^{L} \mathbf{S}_i^2.$$

We can calculate the eigenvalues of the H_{BCS} also by means of the asymptotic expansion of the generating function $F(\omega)$ at $\omega \to \infty$,

$$F(\omega) = \sum_{m=1}^{\infty} F^{(m)}\omega^{-m} = -\frac{1}{\omega}\left(\sum_{j=1}^{L} R_j\right) -$$

$$\frac{1}{\omega^2}\left(\sum_{j=1}^{L} \varepsilon_j R_j - \frac{g}{2}\sum_{j=1}^{L} S_j^2\right) + O\left(\frac{1}{\omega^3}\right).$$

Since the numbers of pairs N and the Hamiltonian H_{BCS} are expressed in terms $F^{(1)}, F^{(2)}$,

$$N = -F^{(1)} + (1/2)L,$$

$$H_{BCS} = -2F^{(2)} + g\left(F^{(1)}\right)^2 + \sum_{j=1}^{L} \varepsilon_j,$$

and since we know the eigenvalues of the operator $F(\omega)$, we can obtain the following expression for the eigenvalues E_N of the BCS Hamiltonian,

$$E_N = -2\sum_{\alpha=1}^{N} (\omega_\alpha - (g/2)) + E_0,$$

$$E_0 = \sum_{j=1}^{L} ((2s_j + 1)\varepsilon_j - s_j g).$$

4. SOLUTIONS OF THE RICHARDSON EQUATIONS. CLASSIFICATION OF EIGENSTATTES

The Richardson equations

$$\sum_{\beta,\beta\neq\alpha}^{N} \frac{1}{\omega_\alpha - \omega_\beta} - \sum_{j=1}^{L} \frac{s_j}{\omega_\alpha - \varepsilon_j} = \frac{1}{g}, \alpha = 1,...,N,$$

admit different interpretations.

We can interpret the Richardson equations as conditions of local equilibrium for a set of charges on a plane (actually lines of charge perpendicular to the plane) which interact with each other by means of a logarithmic potential and with a uniform external field. Indeed if we assume that there are the N free charges of unit strength at points $z_1,...,z_N$ and the L fixed charges with a charge of strength b_j located at a point a_j of the real axis where $j = 1,...,L$ and a uniform external field $-1/g$ then the energy of such a system of charges is

$$W(z_1,\dots,z_N)=\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\ln|z_\alpha-z_\beta|+\sum_{\alpha=1}^{N}\sum_{j=1}^{L}b_j\ln|z_\alpha-a_j|+\frac{1}{g}\sum_{\alpha=1}^{N}z_\alpha.$$

These charges are in equilibrium if their energy is stationary with respect to coordinates of free charges, i.e.

$$\frac{\partial}{\partial z_\alpha}W(z_1,\dots,z_N)=0,\alpha=1,\dots,N.$$

These equations coincide with the Richardson equations if we put $z_\alpha=\omega_\alpha,\alpha=1,\dots,N$ and $a_j=\varepsilon_j,b_j=-(1/2)s_j,j=1,\dots,L.$

The energy $W(z_1,\dots,z_N)$ of the *N* free charges on a plane does not have a global extremum since it is not bounded from above and below but it has a number of local extrema described by solutions of the Richardson equations.These solutions of the Richardson equations correspond different quantum states of the BCS Hamiltonian. Since

$$\lim_{g\to+0}\omega_\alpha(g)=\varepsilon_\alpha,\alpha=1,\dots,N,$$

we may label these quantum states by the quantum numbers of the free Hamiltonian corresponding to $g=0.$ In such a way we come to conclusion that there are C_L^N states with *N* pairs for the BCS Hamiltonian.

We can interpret the Richardson equations also as the equations for zeros of a polynomial satisfying a special ordinary differential equation of the second order with polynomial coefficients. To this end let us consider the polynomial

$$f(z)=\prod_{\beta=1}^{N}(z-z_\alpha)$$

with zeros $z_\beta,\beta=1,\dots,N.$ It is easy to show that

$$\frac{1}{2}\frac{f''(z_\alpha)}{f'(z_\alpha)}=\sum_{\beta=1,\beta\neq\alpha}^{N}\frac{1}{(z_\alpha-z_\beta)}.$$

If we insert this expression into the Richardson equations we obtain

$$f''(z_\alpha)-2\left(\sum_{j=1}^{L}\frac{s_j}{z_\partial-\varepsilon_j}+\frac{1}{g}\right)f'(z_\alpha)=0.$$

Since the polynomial $f(z)$ of the order *N* and the polynomial

$$\prod_{k=1}^{L}(z-\varepsilon_k)\left[f''(z)-2\left(\sum_{j=1}^{L}\frac{s_j}{z-\varepsilon_j}+\frac{1}{g}\right)f'(z)\right]$$

of the order *N+L-1* have the same zeros $z_\alpha,\alpha=1,\dots,N$ there must exist such a polynomial $C(z)$ of the order *L-1* that

$$\prod_{k=1}^{L}(z-\varepsilon_k)\left[f''(z)-2\left(\sum_{j=1}^{L}\frac{s_j}{z-\varepsilon_j}+\frac{1}{g}\right)f'(z)\right]+C(z)f(z)=0.$$

Therefore the polynomial $f(z)$ with zeros $z_\alpha,\alpha=1,\dots,N$ must satisfy the written above differential equation of the second order

$$A(z)f''(z)+B(z)f'(z)+C(z)f(z)=0$$

with polynomial coefficients $A(z),B(z),C(z).$ There are several polynomials $C(z)$ with this property and their number is equal to the number of solutions of the Richardson equations.

A dependence of the quantities $\omega_\alpha,\alpha=1,\dots,N$ on the interaction constant $g,0\leq g<\infty$ has the following properties:

A. If there exist such a $g_0\neq 0$ that $\omega_1(g_0)=\dots=\omega_K(g_0)=a$ then we have 1) a=ε_j; 2) $K=s_j+1$; 3) $\omega_p(g)=C(g-g_0)^{1/K},p=1,\dots,K$ in small neighborhood $|g-g_0|$; 4) g_0 is a solution of the algebraic equation of the *K*-th degree.

B. Ata $g\to\infty$ we have $\omega_\beta(g)\to\varepsilon_\beta,\beta=1,\dots,P$ or $\omega_\gamma(g)\to\varepsilon_\gamma+i\Delta_\gamma,\gamma=1,\dots,Q,$ where $P+Q=N.$ It means that $\omega_\alpha(g),\alpha=1,\dots,N$ are N branches of the algebraic function $\omega(g).$

C. At $g\to+0$ we have

$$\lim_{g\to+0}\omega_\alpha(g)=\varepsilon_\alpha,\ \lim_{g\to+0}\frac{d}{dg}\omega_\alpha(g)=-s_\alpha,$$

$$\alpha=1,\dots,N$$

5. THERMODYNAMIC LIMIT FOR THE BCS HAMILTONIAN

Now let us consider according to the paper [5] the thermodynamic limit

$$L\to\infty,N\to\infty,\lim_{L\to\infty}\frac{N}{L}=const,\lim_{L\to\infty}\frac{g}{L}=G.$$

Let us assume that there exist the density of states $\rho(\varepsilon)$ and the density of pairs $r(\xi)$ satisfying conditions

$$\int_\Omega\rho(\varepsilon)d\varepsilon=L/2,\int_\Gamma r(\xi)d\xi=N.$$

Here Ω is a support of unperturbed spectrum and $\Gamma=\prod_{k=1}^{K}\Gamma_k$ is a support of spectrum of pairs and they are symmetrical with respect of the real axis.

In the thermodynamic limit the Richardson equations

$$\frac{1}{2}\sum_{j=1}^{L}\frac{1}{\varepsilon_j-\omega_\alpha}-\sum_{\beta=1;\beta\neq\alpha}^{N}\frac{1}{\omega_\beta-\omega_\alpha}=\frac{1}{2g},\alpha=1,\dots,L,$$

are transformed to the singular integral equation

$$\int_\Omega\frac{\rho(\varepsilon)d\varepsilon}{\varepsilon-\xi}-P\int_\Gamma\frac{r(\xi')|d\xi'|}{\xi'-\xi}=\frac{1}{2G},\xi\in\Gamma.$$

According to a theory of singular integral equations this equation has a solution

$$r(\xi)=\frac{1}{\pi}|h(\xi)|,h(\xi)=R(\xi)\int_\Omega\frac{\rho(\varepsilon)d\varepsilon}{R(\varepsilon)(\varepsilon-\xi)},$$

where a_k,b_k are initial and final points of the line Γ_k. The following conditions must be satisfied

$$\int_\Omega\frac{\rho(\varepsilon)}{R(\alpha)}\varepsilon^{K-1}d\varepsilon=\frac{1}{2G},$$

$$\int_\Omega\frac{\rho(\varepsilon)}{R(\varepsilon)}\varepsilon^{k}d\varepsilon=0,0\le k<K.$$

The support $\Gamma=\prod_{k=1}^{K}\Gamma_k$ is defined by the equation

$$\Re\int_{a_k}^{\xi}h(\xi')d\xi'=0,\xi\in\Gamma_k.$$

Example. Let us consider a simple example when Γ consists of one segment with limit points

$$a=\varepsilon_0-i\Delta,b=\varepsilon_0+i\Delta.$$

Applying the theory presented above we obtain the following results:

(1) the density of spectrum for pairs

$$r(\xi)=\frac{1}{\pi}|h(\xi)|=\frac{1}{\pi}\left|\sqrt{(\xi-\varepsilon_0)^2+\Delta^2}\right|\times$$
$$\left|\int_\Omega\frac{\rho(\varepsilon)d\varepsilon}{(\varepsilon-\xi)\sqrt{(\xi-\varepsilon_0)^2+\Delta^2}}\right|,$$

(2) the gap equation

$$\int_\Omega\frac{\rho(\varepsilon)d\varepsilon}{\sqrt{(\varepsilon-\varepsilon_0)^2+\Delta^2}}=\frac{1}{2G},$$

(3) the Fermi energy equation

(4) the following expression:

$$N=\frac{1}{2\pi i}\int_\Gamma h(\xi)d\xi=$$
$$\int_\Omega\left(1-\frac{\varepsilon-\varepsilon_0}{\sqrt{(\varepsilon-\varepsilon_0)^2+\Delta^2}}\right)\rho(\varepsilon)d\varepsilon,$$

(4) the ground state energy

$$E=\frac{1}{2\pi i}\int_\Gamma\xi h(\xi)d\xi=-\frac{\Delta^2}{2G}+$$
$$\int_\Omega\left(1-\frac{\varepsilon-\varepsilon_0}{\sqrt{(\varepsilon-\varepsilon_0)^2+\Delta^2}}\right)\varepsilon\rho(\varepsilon)d\varepsilon.$$

6. NORMS OF THE EIGENSTATES CORRELATTION FUNCTIONS

The normalization factor of eigenstate $|N\rangle$ is expressed in terms of the Jacobi matrix Δ of the Richardson equations:

$$R(\varepsilon)=\left[\prod_{k=1}^{K}(\xi-a_k)(\xi-b_k)\right]^{1/2}.$$

$$\langle N|N\rangle=N!|\Delta|,\Delta_{\alpha\beta}=\frac{1}{(\omega_\alpha-\omega_\beta)^2}+$$
$$\delta_{\alpha\beta}\left(\sum_j\frac{s_j}{(\varepsilon_j-\omega_\alpha)^2}-\sum_\nu\frac{1}{(\omega_\nu-\omega_\alpha)^2}\right).$$

The correlation functions of variables S_j are

$$\langle\mathbf{S}_j\cdot\mathbf{S}_k\rangle=s_js_k-s_j\sum_{\alpha=1}^{N}\left(\frac{\varepsilon_j-\varepsilon_k}{\varepsilon_j-\omega_\alpha}\right)^2\frac{\partial\omega_\alpha}{\partial\varepsilon_k}.$$

Sklyanin has developed mathematical means to calculate different other correlation functions [6]

7. THE INTEGRABLE GENERALIZATIONS OF THE BCS HAMILTONIAN

The generalized BCS Hamiltonian

$$H=\sum_i\varepsilon_in_{i\sigma}-\sum_{i,j}g_{ij}c_{i\uparrow}^{+}c_{i\downarrow}^{+}c_{i\downarrow}c_{i\uparrow}+\sum_{i,j}U_{ij}n_{i\sigma}n_{i\sigma'},$$

is integrable at a special form of the interaction functions g_{ij} and U_{ij}.

Let us consider an integrable Hamiltonian

$$H_N=\sum_i\varepsilon_i\tau_i+A\sum_{i,k}\tau_i\tau_k+\sum_i\beta_i\mathbf{S}_i^2,$$

with the integrals of motion τ_j of the form

$$\tau_j=S_j^z+\Xi_j,\Xi_j=\sum_{k=1,k\neq j}^{\Omega}w_{jk}^{\alpha}S_j^{\alpha}S_k^{\alpha}.$$

The operators τ_j are called isotropic when $w_{jk}^x=w_{jk}^y=w_{jk}^z$ otherwise we call them anisotropic. The operators τ_j commute with each other if

$$[\Xi_j,\Xi_k]=0,[S_i^z,\Xi_j]+[\Xi_i,S_j^z]=0,$$

or, in other words, if

$$w_{ij}^{\alpha}w_{jk}^{\gamma}+w_{ji}^{\beta}w_{ik}^{\gamma}=w_{ik}^{\alpha}w_{jk}^{\beta},w_{ij}^{x}=-w_{ji}^{y}.$$

Furthermore we impose an additional condition

$$\left[\sum_{i=1}^{\Omega} S_i^z, \Xi_j\right] = 0, j = \{1,...,\Omega\},$$

which is equivalent to the equations

$$w_{ij} v_{jk} + w_{ji} v_{ik} = w_{ik} w_{jk},$$

$$w_{ij} := w_{ij}^x = -w_{ji}^x = w_{ij}^y = -w_{ji}^y, v_{ij} := w_{ij}^z.$$

These equations for the quantities w_{ij} and v_{ij} have solutions

$$w_{jk} = \frac{qK}{\sinh q(u_j - u_k)},$$

$$v_{jk} = qK \coth q(u_j - u_k),$$

where u_j are arbitrary complex parameters such that the quantities v_{jk}, w_{jk} are real. The parameter q can be real or imaginary. If q is real then we have hyperbolic functions, if $q = i, G = iK$, and K, u_j are real then we have trigonometric functions.

The eigenfunctions of integral of motions τ_j are of the form

$$|\Psi_j\rangle = \sum\nolimits'_{j_1...j_N} c(j_1...j_N) S_{j_1}^+ ... S_{j_N}^+ |0\rangle +$$

$$\sum\nolimits'_{j_1...j_{N-1}} c(j_1...j_{N-1}) S_{j_1}^+ ... S_{j_{N-1}}^+ S_j^+ |0\rangle.$$

Here $|0\rangle = |\downarrow,...,\downarrow\rangle$ is the vacuum, the primes at the sums mean that the indices run in the range $\{1,...,\Omega\} \setminus \{j\}$. The eigenvalues of τ_j are defined by the equalities

$$\tau_j |\Psi_j\rangle = (1/2)(h_j - 1)|\Psi_j\rangle,$$

where h_j are solutions of the equations

$$h_j / K = \sum_{l=1}^{\Omega} q \coth q(u_j - u_l) -$$

$$2\sum_{\alpha=1}^{L} q \coth q(u_j - \omega_\alpha),$$

and ω_α fulfill the equations

$$1/K = \sum_{l=1}^{\Omega} q \coth q(\omega_\alpha - u_l) -$$

$$2\sum_{\beta=1, \beta\neq\alpha}^{L} q \coth q(\omega_\alpha - \omega_\beta).$$

Thus we obtain the integrable BCS Hamiltonian

$$H_N = \sum_j 2\varepsilon_j S_j^z - \sum_{jk} g_{jk} S_j^+ S_k^- + \sum_{jk} U_{jk} S_j^z S_k^z,$$

with the following interaction functions

$$g_{jk} = -qK(\varepsilon_j - \varepsilon_k) / \sinh q(u_j - u_k), j \neq k,$$

$$U_{jk} = A + qK(\varepsilon_j - \varepsilon_k) \coth q(u_j - u_k), j \neq k,$$

$$g_{jj} = -\beta_j, U_{jj} = A + \beta_j.$$

Here parameters A, β_j, K are arbitrary real constants, while q can be real or imaginary. The eigenfunctions Ψ_N and eigenvalues E_N of the Hamiltonian H are

$$\Psi_N = \prod_{\alpha=1}^{N} \sum_{j=1}^{\Omega} \frac{c_{j\uparrow}^+ c_{j\downarrow}^+}{\sinh(\omega_\alpha - u_j)} |0\rangle,$$

$$E_N = \sum_{j=1}^{\Omega} \varepsilon_j + \sum_{j,k=1}^{\Omega} U_{jk} + 4AKN(KN - \Omega) +$$

$$2K\sum_{j=1}^{\Omega} \sum_{\alpha=1}^{N} \varepsilon_j \coth(\omega_\alpha - u_j).$$

At limit case $q \to 0$ we come to the isotropic case, i.e. to the BCS Hamiltonian with the following constants

$$K = g / E_D, \beta_j = -g, A = g,$$

$$u_j = \varepsilon_j / (E_D \theta(\varepsilon_j - E_D)).$$

Diagonal elements g_{jj}, U_{jj} are arbitrary since they renormalize ε_j.

There are generalizations to integrable quantum models with arbitrary Lie algebras, in particular, with $O(N)$ and $Sp(2k)$.

8. CONCLUSION

We have presented above a review of the integrability of the BCS Hamiltonian. Further studies show that it has deep connections to integrable vertex models, conformal field theory, Chern-Simons theory, Bethe ansatz and quantum groups (see e.g. [7]). Since the integrability of the BCS Hamiltonian have been used essentially in description of superconductivity of nuclei, ultrasmall metallic grains and quantum dots at low temperatures it presents an interest also from point of view of applications.

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