THE KINETIC THEORY OF THE COSMOLOGICAL MODELS

A.I. Akhiezer, S.V. Peletminsky

Institute for Theoretical Physics
National Science Center "Kharkov Institute of Physics and Technology", Kharkov, Ukraine
e-mail: spelet@kipt.kharkov.ua

Theory of Friedman cosmological models is developed. It is based on a common and consistent solution of the kinetic equation for the galaxies distribution function in the approximation of a self-consistent gravitational field and the Einstein equations. Kinetic approach to the relic radiation theory is also under consideration.
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INTRODUCTION

It was Jeans who in 1915 firstly proposed to use a kinetic equation with a self-consistent gravitational field and consider stars as the gas of particles in order to investigate some problems of astrophysics and cosmology. Twenty years later the analogous equation with the self-consistent electromagnetic field was formulated and used by A.A. Vlasov to study oscillating properties of the electron-ionic plasma. In this report we use a kinetic equation with the self-consistent gravitational field for the investigation of different cosmological models. When studying the cosmological models it is typical to use the hydrodynamic approach. It assumes the matter equilibration, when it is possible to speak about its equation of state, but as we show below there is no especial necessity in the hydrodynamic approach and the whole problem can be investigated in a more general kinetic approach. With this consideration there is no need to introduce the assumption about the matter equilibrium and to know its equation of state.

First of all we derive the kinetic equation with the self-consistent gravitational field in the general relativity theory. Further we derive an energy-momentum tensor and vector of the current of particles using a relativistically invariant distribution function. We relate the obtained energy-momentum tensor to the space-time curvature tensor according to the Einstein equations. This allows investigating different cosmological models which represent common and consistent solutions of both the kinetic equation and Einstein equations. The kinetic approach allows to describe the evolution of relict distributions of particles as solutions of the kinetic equations if we assume that the Universe expansion resulted in the separation of the interaction of the particles from the rest of matter in the distant past.

1. RELATIVISTICALY INVARIANT DISTRIBUTION FUNCTION AND KINETIC EQUATION WITH A SELF-CONSISTENT GRAVITATIONAL FIELD

1.1. *Relativistically invariant distribution function and Energy-momentum tensor in the presence of a gravitational field.* Now we focus our attention at the construction of a relativistic kinetic equation without account collisions for the gravitationally interacting particles. Let us denote the four-trajectory of the r-th star (particle) with the mass m_r by $\xi_r(\tau_r)$ (τ_r is the proper time of the r-th star). Then the four-velocity of this star is $\dot{\xi}_r(\tau_r)$. In the special relativity theory a random relativistically invariant distribution function of stars over the positions x, velocities u^μ and masses m is given by

$$\sum_r \delta(x-\xi_r(\tau_r))\delta(u-\dot{\xi}_r(\tau_r))\delta(m-m_r).$$

We want to note that under the general transformations of the space-time coordinates x^μ, $x^\mu \to x'^\mu = x'^\mu(x)$, the coordinates $\xi^\mu(\tau)$ of the particles and their four-velocities $\dot{\xi}^\mu(\tau)$ transform as

$$\xi^\mu \to \xi'^\mu = x'^\mu(\xi), \qquad \dot{\xi}^\mu \to \dot{\xi}'^\mu = \left.\frac{\partial x'^\mu}{\partial x^\nu}\right|_{x=\xi} \dot{\xi}^\nu .$$

That gives us the reason why in the general relativity theory, unlike the special relativity theory, the δ-functions $\delta(x-\xi_r)$ and $\delta(u-\dot{\xi}_r)$ are not invariants but transform obeying the formulae for densities

$$\delta(x-\xi_r) \to \delta(x'-\xi'_r) = \frac{\delta(x-\xi_r)}{|\partial x'/\partial x|}, \tag{1.1}$$

$$\delta\left(u-\dot{\xi}_r\right)\to\delta\left(u'-\dot{\xi}'_r\right)=\frac{\delta\left(u-\dot{\xi}_r\right)}{\left|\partial x'/\partial x\right|},$$

moreover, the quantities x, u must transform in accordance with the formulae

$$x\to x'=x'(x),\quad u^{\mu}(x)\to u'^{\mu}(x')=\frac{\partial x'^{\mu}}{\partial x^{\nu}}u^{\nu}(x),\qquad(1.2)$$

i.e. $u^{\nu}(x)$ must also depend on x and be interpreted as the four-velocity of a particle at the point x ($\left|\partial x'/\partial x\right|$ is the Jacobian of (1.2)).

Bearing in mind that under the space-time transformations the determinant of the metric tensor $g_{\mu\nu}(x)$ ($g\equiv\det g_{\mu\nu}$) transforms as

$$g(x)\to g'(x')=g(x)\left|\frac{\partial x}{\partial x'}\right|^2,$$

we can see that the quantities $\delta\left(x-\xi_r\right)/\sqrt{-g}$, $\delta\left(u-\dot{\xi}_r\right)/\sqrt{-g}$ will be invariants. Therefore, the relativistic invariant distribution function in the general relativity theory can be defined by

$$f(x,u;m)=\qquad(1.3)$$

$$-\frac{1}{g(x)}\left\langle\sum_r\int_{-\infty}^{+\infty}d\tau_r\delta\left(x-\xi_r(\tau_r)\right)\delta\left(u-\dot{\xi}_r(\tau_r)\right)\delta\left(m-m_r\right)\right\rangle,$$

where the brackets $\langle\ldots\rangle$ are used to denote the average over the random coordinates ξ_r, random velocities $\dot{\xi}_r$ and random masses m_r taken at a certain initial moment of the proper time $\tau_r=\tau^0$. The coordinates $\xi_r(\tau)$ satisfy the equations of motion

$$\ddot{\xi}^{\mu}_r=-\Gamma^{\mu}_{\nu\rho}(\xi)\dot{\xi}^{\nu}_r\dot{\xi}^{\rho}_r,\qquad(1.4)$$

where $\Gamma^{\mu}_{\nu\rho}$ are the Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho}(x)\frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\nu}}{\partial x^{\rho}}+\frac{\partial g_{\lambda\rho}}{\partial x^{\nu}}-\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}}\right).\qquad(1.5)$$

Let us explain the integration over τ introduced here. For this purpose we shall consider the integral over τ of the product of one-dimensional δ-functions $\delta\left(x^0-\xi^0(\tau)\right)\delta\left(u^0-\dot{\xi}^0(\tau)\right)$. This integral, evidently, is equal to

$$\int_{-\infty}^{+\infty}d\tau\delta\left(x^0-\xi^0(\tau)\right)\delta\left(u^0-\dot{\xi}^0(\tau)\right)=$$

$$2\frac{u_0}{u^0}\Theta\left(u^0\right)\delta\left(u^{\mu}u_{\mu}-1\right),$$

where Θ is the Heaviside function and u_0, u^0 are the covariant and contravariant time components of the four-velocity. As a result the relativistically invariant distribution function assumes the form

$$f(x,u;m)=2\Theta\left(u^0\right)\delta\left(u^{\mu}u_{\mu}-1\right)\underline{f}(x,\mathbf{u};m),\qquad(1.6)$$

where

$$\underline{f}(x,\mathbf{u};m)=-\frac{1}{g(x)}\frac{u_0}{u^0}\times\qquad(1.7)$$

$$\left\langle\sum_r\delta\left(\mathbf{x}-\boldsymbol{\xi}_r(\tau)\right)\delta\left(\mathbf{u}-\mathbf{u}_r(\tau)\right)\delta\left(m-m_r\right)\right\rangle,\quad\tau=\tau\left(x^0\right)$$

is the usual relativistically noncovariant distribution function satisfying in accordance with the definition the following normalizing condition

$$-\int d^3xd^3u\int_0^{\infty}dmg(x)\frac{u^0}{u_0}\underline{f}(x,\mathbf{u};m)=\qquad(1.8)$$

$$-\int d^4x\int d^4u\int_0^{\infty}dmg(x)u^0f(x,u;m)=N$$

(N is the total number of stars). Therefore, the total mass of stars will be given by

$$M=-\int d^4x\int d^4u\int_0^{\infty}dmmg(x)u^0f(x,\mathbf{u};m).\qquad(1.9)$$

We want to note that the relativistically invariant distribution function does not change under the transformations of the four-coordinates x^{μ} and four-velocities u^{μ}

$$x\to x'^{\mu}=x'^{\mu}(x),\qquad u^{\mu}\to u'^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\nu}}u^{\nu}(x);$$

it means that the function is a scalar

$$f(x,u;m)\to f'(x',u';m)=f(x,u;m).\qquad(1.10)$$

Now we introduce the four-vector $J^{\mu}(x)$ of the mass flux density and the four-tensor $T^{\mu\nu}(x)$ of the energy-momentum. It is known that in the special relativity theory the mass flux vector $J^{\mu}(x)$ and energy-momentum tensor of a point particle are determined by the following formulae

$$J^{\mu}(x)=\int_{-\infty}^{+\infty}d\tau m\dot{\xi}^{\mu}(\tau)\delta\left(x-\xi(\tau)\right),$$

$$T^{\mu\nu}(x)=\int_{-\infty}^{+\infty}d\tau m\dot{\xi}^{\mu}(\tau)\dot{\xi}^{\nu}(\tau)\delta\left(x-\xi(\tau)\right).$$

Thus, in the general relativity theory the quantities, $J^{\mu}(x)$, $T^{\mu\nu}(x)$ referred to a system of particles have in accordance with (1.1) the form

$$J^{\mu}(x)=\sqrt{-g}\int_0^{\infty}dm\int d^4u\times$$

$$\sum_r\int_{-\infty}^{+\infty}d\tau_r mu^{\mu}\frac{\delta\left(x-\xi_r\right)}{\sqrt{-g}}\frac{\delta\left(u-\dot{\xi}_r\right)}{\sqrt{-g}}\delta\left(m-m_r\right),$$

$$T^{\mu\nu}(x)=\sqrt{-g}\int_0^{\infty}dm\int d^4umu^{\mu}u^{\nu}\times$$

$$\sum_r\int_{-\infty}^{+\infty}d\tau_r\frac{\delta\left(x-\xi_r\right)}{\sqrt{-g}}\frac{\delta\left(u-\dot{\xi}_r\right)}{\sqrt{-g}}\delta\left(m-m_r\right)$$

($\sqrt{-g}d^4u$ is an invariant). Averaging $J^{\mu}(x)$ and $T^{\mu\nu}(x)$ over the random coordinates and velocities at the moment τ_0 we get

$$J^{\mu}(x) = \sqrt{-g(x)}\int\limits_0^{\infty} dm m \int d^4 u u^{\mu} f(x,u;m), \qquad (1.11)$$

$$T^{\mu\nu}(x) = \sqrt{-g(x)}\int\limits_0^{\infty} dm m \int d^4 u u^{\mu} u^{\nu} f(x,u;m).$$

We emphasize that here the quantities x^{μ} and u^{μ} are the generalized coordinates and velocities characteristic for the general relativity theory and they may not have the metric meaning.

1.2. Kinetic equation with a self-consistent gravitational field. Now we derive a kinetic equation for the relativistically invariant distribution function f . To this end it should be noted that the following evident relation is valid

$$\left\langle \sum_r \int\limits_{-\infty}^{+\infty} d\tau_r \frac{d}{d\tau_r} \delta(x - \xi_r(\tau_r)) \times \right. \qquad (1.12)$$

$$\left. \delta(u - \dot{\xi}_r(\tau_r))\delta(m - m_r) \right\rangle = 0.$$

Performing in this formula the differentiation with respect to τ_r and using the δ -functions which enter the equation (1.12) we can obtain according to (1.4)

$$\left\{ u^{\mu}\left(\frac{\partial}{\partial x^{\mu}} - \frac{\partial u^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} \right) + \frac{\partial}{\partial u^{\mu}} F^{\mu}(u,x) \right\} gf = 0,$$

where

$$F^{\mu}(u,x) = -\Gamma^{\mu}_{\nu\rho}(x) u^{\nu} u^{\rho} \qquad (1.13)$$

(the derivatives $\partial/\partial x^{\mu}$, $\partial/\partial u^{\nu}$, $\partial/\partial u^{\mu}$ act also on the function gf ; we assume that the four-velocity components $u^{\mu}(x)$ depend on x). Noting that

$$\frac{\partial F^{\mu}}{\partial u^{\mu}} = -2\Gamma^{\nu}_{\mu\nu} u^{\mu}, \qquad \frac{1}{g}\frac{\partial g}{\partial x^{\mu}} = 2\Gamma^{\nu}_{\mu\nu}, \qquad (1.14)$$

we have

$$\left\{ u^{\mu}\frac{\partial}{\partial x^{\mu}} + \left(F^{\mu} - u^{\nu}\frac{\partial u^{\mu}}{\partial x^{\nu}} \right)\frac{\partial}{\partial u^{\mu}} \right\} f = 0.$$

Recalling the definition of the covariant derivative of the contravariant four-vector

$$D_{\nu} u^{\mu} = \frac{\partial u^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho} u^{\rho} \qquad (1.15)$$

and using the formula (1.13) we finally obtain

$$u^{\mu}\left\{ \frac{\partial}{\partial x^{\mu}} - D_{\mu} u^{\nu}\frac{\partial}{\partial u^{\nu}} \right\} f = 0. \qquad (1.16)$$

If one considers the four-velocity u^{μ} being independent of x , then the equation (1.16) will lose its explicit covariance but take a more evident form

$$\left\{ u^{\mu}\frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\nu\rho}(x) u^{\nu} u^{\rho}\frac{\partial}{\partial u^{\mu}} \right\} f = 0. \qquad (1.17)$$

We have hitherto assumed that the gravitational field is in no way related to the distribution function f . Further we shall be interested in the self-consistent problem, where the field $g_{\mu\nu}(x)$ itself is determined by the distribution function. With this purpose the Einstein equations for the metric tensor $g_{\mu\nu}(x)$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}, \qquad (1.18)$$

where $R^{\mu\nu}$ is a contracted curvature tensor and $T^{\mu\nu}$ is an energy-momentum tensor governed by (1.11) (G is a gravitational constant), must be attached to (1.17). Thus, the full system of equations for the distribution function f and field $g_{\mu\nu}$ is determined by (1.17), (1.18).

We have seen that the relativistically invariant distribution functioncontains as a multiplier the quantity $\Theta(u^0)\delta(u^{\mu} u_{\mu} - 1)$, which indicates that this function is non-zero only when $u^{\mu} u_{\mu} = 1$ and $u^0 > 0$ (the first relation is connected with the fact that τ is a proper time of a particle and the second, that u^0 is a time component of the four-velocity). Such a structure of the relativistically invariant distribution function must not contradict the kinetic equation (1.17).

2. THE EINSTEIN EQUATIONS AND KINETIC EQUATION WITH A SELF-CONSISTENT GRAVITATIONAL FIELD

2.1. *Covariant Conservation Laws.* Since we are interested in the self-consistent problem of a common and consistent solution of the Einstein equations (1.18) and kinetic equation (1.17) and since for an arbitrary $g_{\mu\nu}$ the following relation holds

$$D_{\nu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0,$$

we must make sure that owing to the kinetic equation (1.17) the relation

$$D_{\nu} T^{\mu\nu} = 0 \qquad (2.1)$$

is valid. Using the definition (1.11) of the energy-momentum tensor we get

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\nu}} T^{\mu\nu} +$$

$$\sqrt{-g}\int\limits_0^{\infty} dm m \int d^4 u u^{\mu} u^{\nu}\frac{\partial f}{\partial x^{\nu}}.$$

Taking into account, further, the kinetic equation (1.17) and integrating by parts over the variable u^{μ} we obtain

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = T^{\mu\nu}\frac{\partial \ln\sqrt{-g}}{\partial x^{\nu}} - 2\sqrt{-g}\Gamma^{\rho}_{\nu\rho}\int\limits_0^{\infty} dm m \int d^4 u u^{\nu} u^{\mu} f -$$

$$\sqrt{-g}\Gamma^{\mu}_{\nu\rho}\int\limits_0^{\infty} dm m \int d^4 u u^{\nu} u^{\rho} f.$$

Making use again of the definition of the tensor $T^{\mu\nu}$ and noting that

$$\frac{\partial \ln\sqrt{-g}}{\partial x^\nu} = \Gamma^\rho_{\nu\rho}$$

we have

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\lambda_{\nu\lambda} T^{\mu\nu} + \Gamma^\mu_{\nu\rho} T^{\nu\rho} = 0.$$

Remembering the definition of the covariant derivative of the tensor $A^{\mu\nu}$

$$D_\rho A^{\mu\nu} = \frac{\partial A^{\mu\nu}}{\partial x^\rho} + \Gamma^\mu_{\lambda\rho} A^{\lambda\nu} + \Gamma^\nu_{\lambda\rho} A^{\mu\lambda}$$

we come to the relation (2.1). It expresses the covariant conservation law of energy-momentum in the general relativity theory (to obtain the ordinary conservation laws we must, as it is known, add an energy-momentum pseudotensor of the gravitational field to the quantity $T^{\mu\nu}$).

In a similar manner the covariant conservation law of the four-vector of mass flux density can be obtained

$$D_\mu J^\mu = 0. \tag{2.2}$$

This law, unlike the conservation law of energy-momentum, can be easily put in a form of an ordinary conservation law. Indeed, using the definition of the covariant derivative (1.15) and the relation (1.14) we get

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0, \qquad j^\mu(x) = \sqrt{-g}J^\mu(x), \tag{2.3}$$

where the quantity $J^\mu(x)$ is related to the distribution function f by (1.11). This relation ensures that $\int d^3x\, j^0(x)$ is independent of x^0. This quantity in accordance with (1.11) defines a total mass of particles (stars; see. (1.9))

$$M = \int d^3x\, j^0(x). \tag{2.4}$$

2.2. Solution of the Kinetic Equation for the Spatially-Homogeneous and Isotropic Metric. We want to find a certain class of solutions for the kinetic equation in the simplest case of spatially-homogeneous and isotropic metric. In a synchronous reference system, where $g_{00} = 1$, $g_{0i} = 0$, ($i = 1,2,3$) this metric is determined by

$$ds^2 = dt^2 - dl^2, \qquad dl^2 = -g_{ik}(x)dx^i dx^k,$$

where

$$g_{ik}(x) = a^2(x^0)g^0_{ik}(\mathbf{x}), \tag{2.5}$$

$a(x^0)$ is a certain function of time and g^0_{ik} is a function of space coordinates alone. The time $t \equiv x^0$ is interpreted as the unique common time shown by a clock at rest with respect to this reference system and dl^2 defines a space metric at the moment t. Note that the synchronous reference system represents a system moving together with matter if the forces between particles of matter are only of gravitational nature.

In the synchronous reference system the quantity u^0 entering the kinetic equation (1.17), is of the form

$$u^0 = \sqrt{1 - u^i u^k g_{ik}(x)}. \tag{2.6}$$

For the case of the spatially-homogeneous and isotropic distribution of particles, that we are interested in, the distribution function $\underline{f}$ depends only on the time t and space invariant of the squared velocity $u^i u^k g_{ik}(x)$ (only the quantity $g_{ik}(x)$ entering this expression contains the dependence on space coordinates; in the Euclidean case homogeneity and isotropy correspond to the fact that the distribution function depends only on t and $|\mathbf{u}|$, but it is independent of $\mathbf{x}$). Thus, in the case under consideration the distribution function $\underline{f}$ is dependent only on two arguments, t and u^0

$$\underline{f}(x, u^k) = f(t, u^0). \tag{2.7}$$

Using the definition of the Christoffel symbols (1.5) and remembering that $g^{00} = 1$, $g^{0i} = 0$, the kinetic equation (1.17) can be represented as

$$\frac{\partial f}{\partial t} + \frac{\dot a}{a}\frac{1 - u^{0\,2}}{u^0}\frac{\partial f}{\partial u^0} = 0.$$

We see that in the case of the homogeneous and isotropic metric the solution of the kinetic equation can be found in the form (2.7). Evidently, the general solution of this equation has the form

$$f = f\left(a(t)\sqrt{u^{0\,2} - 1}\right),$$

where f is an arbitrary function of one argument. Using the definition of u^0 (see (2.6)) we get

$$\underline{f}(x, u^k) = f\left(a(t)\sqrt{-g_{ik}u^i u^k}\right). \tag{2.8}$$

3. RELATIVISTIC COSMOLOGICAL MODELS

3.1. Spatially-homogeneous and isotropic solutions of the Einstein equations and kinetic equation. The obtained distribution function of particles in the spatially-homogeneous and isotropic case allows to determine according to (1.11) the energy-momentum tensor $T^{\mu\nu}(x)$ (with the unknown quantity $a(t)$). Making the substitution of the found expression for $T^{\mu\nu}(x)$ in the Einstein equation we can get an equation for the determination of $a(t)$.

Using the spherical coordinates we have the following relations:

-in the case of the closed cosmological model

$$dl^2 = a^2(t)\left(d\chi^2 + \sin^2\chi\, d\vartheta^2 + \sin^2\chi \sin^2\vartheta\, d\varphi^2\right), \tag{3.1}$$

$$0 < \chi < \pi, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi,$$

-in the case of the open cosmological model

$$dl^2 = a^2(t)\left(d\chi^2 + \mathrm{sh}^2\chi\, d\vartheta^2 + \mathrm{sh}^2\chi \sin^2\vartheta\, d\varphi^2\right), \tag{3.2}$$

$$0 < \chi < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi,$$

-and in the case of the Euclidean model

$$dl^2 = a^2(t)\left(d\chi^2 + \chi^2 d\vartheta^2 + \chi^2\sin^2\vartheta\, d\varphi^2\right), \tag{3.3}$$

$$0 < \chi < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi.$$

The quantity $a(t)$ entering the first two formulae for the closed and open models has the meaning of the curvature radius of the three-dimensional space (the curvature tensor P_{ik} of the three-dimensional space equals $P_{ik} = \pm\left(1/a^2\right)g_{ik}$). Here we shall consider in greater detail the case of the closed model and give only final results referring to the open and Euclidean models.

In the case of the closed model non-zero components of the metric tensor $g_{\mu\nu}$ are

$$g_{00} = 1, \quad g_{11} = -a^2, \tag{3.4}$$

$$g_{22} = -a^2\sin^2\chi, \quad g_{33} = -a^2\sin^2\chi\sin^2\vartheta$$

(index 1 corresponds to the variable χ , index 2 to ϑ and index 3 to φ). Non-zero components of the tensor $R_{\mu\nu} - (1/2)g_{\mu\nu}R$, entering the Einstein equation, in the form

$$R_{00} - \frac{1}{2}R = 3\frac{1 + \dot{a}^2}{a^2}, \tag{3.5}$$

$$R_{ik} - \frac{1}{2}g_{ik}R = -g_{ik}\,\frac{1 + 2a\ddot{a} + \dot{a}^2}{a^2}.$$

Let us turn now to calculation of the energy-momentum tensor. According to the formulae (1.11) its non-zero components are equal to

$$T_{00} = \sqrt{-g}\int_0^\infty dmm\int d^3u u_0 f\left(a\sqrt{-g_{ik}u^iu^k}\right),$$

$$T_{ik} = \frac{1}{3}g_{ik}\sqrt{-g}\int_0^\infty dmm\int d^3u\frac{u_iu^i}{u_0}f\left(a\sqrt{-g_{ik}u^iu^k}\right).$$

Going over from the variables $u^1,\ u^2,\ u^3$ to $x,\ y,\ z$

$$x = a^2u^1, \quad y = a^2\sin\chi\, u^2, \quad z = a^2\sin\chi\sin\vartheta\, u^3$$

and taking into account that

$$\sqrt{-g}d^3u = \frac{dxdydz}{a^3}$$

we obtain

$$T_{00} = \frac{1}{a^3}\int_0^\infty dmm\int_{-\infty}^{+\infty} dxdydz\left(1 + \frac{x^2+y^2+z^2}{a^2}\right)^{1/2} \times f\left(\sqrt{x^2+y^2+z^2}\right),$$

$$T_{ik} = \frac{1}{3a^3}g_{ik}\int_0^\infty dmm\int_{-\infty}^{+\infty} dxdydz\,\frac{x^2+y^2+z^2}{a^2}\times\left(1 + \frac{x^2+y^2+z^2}{a^2}\right)f\left(\sqrt{x^2+y^2+z^2}\right)$$

or introducing the notation $x^2 + y^2 + z^2 = r^2$

$$T_{00} = \frac{4\pi}{a^3}\int_0^\infty dmm\int_0^\infty drr^2\left(1 + \frac{r^2}{a^2}\right)^{1/2} f(r), \tag{3.6}$$

$$T_{ik} = -\frac{4\pi}{3a^5}g_{ik}\int_0^\infty dmm\int_0^\infty drr^4\left(1 + \frac{r^2}{a^2}\right)^{-1/2} f(r).$$

In a similar way one can find formulae for the mass flux density. Using (1.11) and (2.8) we have

$$J^0 = \frac{4\pi}{a^3}\int_0^\infty dmm\int_0^\infty drr^2 f(r), \quad J^k = 0. \tag{3.7}$$

To derive the formulae (3.6), (3.7) we used the solution of the kinetic equation (2.8) in the case of the homogeneous and isotropic metric, found in the previous Section. It should be stressed that this solution, as well as the formulae (3.6) and (3.7), hold for all three cosmological models.

One can easily see that Eqs. (3.6), (3.7) yield in the limit of $a \to \infty$

$$T^{00} \to J^0, \quad a \to \infty \tag{3.8}$$

and in the limit of $a \to 0$

$$p \to \frac{1}{3}T^{00}, \tag{3.9}$$

where the pressure p is given by $T_{ik} = -pg_{ik}$.

In the case of the closed cosmological model the Universe mass is determined by (see (2.4))

$$M = \int d^3x\sqrt{-g}J^0 = 2\pi^2 a^3 J^0. \qquad (3.10)$$

We want to note that the quantities T_{00}, $T_{ik}g^{ik}$, as it must be in the case of the space-homogeneous metric, do not depend on x^k and in virtue of the metric isotropy the quantity T_{ik} is proportional to g_{ik}.

Thus, the Einstein equation for the time component is of the form [1,2]

$$\dot{a}^2 + 1 = p(a), \qquad (3.11)$$

$$p(a) = \frac{32\pi^2 G}{3}\frac{1}{a}\int_0^\infty dmm\int_0^\infty drr^2\sqrt{1+\frac{r^2}{a^2}}f(r). \qquad (3.12)$$

For the open model the analogous equation is

$$\dot{a}^2 - 1 = p(a), \qquad (3.13)$$

and for the Euclidean model it takes the form

$$\dot{a}^2 = p(a), \qquad (3.14)$$

where the function $p(a)$ is governed, as before, by (3.12). Formulae (3.11), (3.13), (3.14) along with (3.12) describe a space-time structure of the Universe for the three Friedman cosmological models and in certain limiting cases (we do not dwell on these cases) lead to well-known results.

3.2. Relic Distributions of Particles. The kinetic equation (1.17) can be used in order to describe the evolution of the stars distribution function as well as the distribution functions of elementary particles such as photons, neutrino, electrons and others, which originated after the Big Bang. Notice that one can use the kinetic equation after a certain lapse of time, when as a result of the Universe expansion we can neglect the collisions between particles.

Recalling reasoning given in Sections 1.1 and 1.2 we come to the kinetic equation for massless particles

$$\left(k^\mu\frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\rho}k^\nu k^\rho\frac{\partial}{\partial k^\mu}\right)f = 0, \qquad (3.15)$$

which is analogous to the kinetic equation (1.17) for particles possessing mass (k^μ is the wave vector of photon); the difference between them is that the relativistically invariant distribution function f, determined by the formula

$$f(x,k) = 2\vartheta(k_0)\delta\left(k^\mu k_\mu\right)\underline{f}(x,\mathbf{k}) \qquad (3.16)$$

is other than zero not at the mass surface $p^\mu p_\mu = m^2$ but only when $k^\mu k_\mu = 0$. The kinetic equation for $\underline{f}(x,\mathbf{k})$, obviously, takes the form

$$\left(k^0\frac{\partial}{\partial t} + k^l\frac{\partial}{\partial x^l} - \Gamma^k_{\nu\rho}k^\nu k^\rho\frac{\partial}{\partial k^k}\right)\underline{f} = 0, \qquad (3.17)$$

$$\omega \equiv k^0 = \sqrt{-g_{ik}k^i k^k}$$

(as previously we use the synchronous reference system). The general isotropic solution of this equation is

$$\underline{f}(x,\mathbf{k}) = f(a(t)\omega). \qquad (3.18)$$

The expressions for the energy-momentum tensor and flux density vector for massless particles have the form

$$T^{\mu\nu}(x) = \sqrt{-g}\int d^4kk^\mu k^\nu f(x,k)\square, \qquad (3.19)$$

$$J^\mu(x) = \sqrt{-g}\int d^4kk^\mu f(x,k).$$

As it is mentioned above the distribution function of photons in the moment when radiation separates from matter is determined by the formula

$$\underline{f}(t_0,\mathbf{k}) = \frac{1}{(2\pi)^3}\left\{\exp\left(\frac{\hbar k^0}{T(t_0)}\right) - 1\right\}^{-1}.$$

This function is normalized so that the quantity $\int d^3k\underline{f}(t_0,\mathbf{k})$ represents the density of photons. Bearing in mind (3.18) we obtain the distribution function at the moment $t > t_0$ in the form

$$\underline{f}(t,\mathbf{k}) = \frac{1}{(2\pi)^3}\left\{\exp\left(\frac{\hbar\omega}{T(t)}\right) - 1\right\}^{-1}, \qquad (3.20)$$

where

$$T(t) = T(t_0)\frac{a(t_0)}{a(t)}.$$

We see that when the distribution function evolution takes place the form of distribution remains unaltered while the temperature, which occurs to be inversely proportional to the curvature radius, is altered.

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REFERENCES

1. J. Ehlers, in: R.K. Sachs (Ed.). *General Relativity and Kinetic Theory, General Relativity and Cosmology.* New York: "Academic Press", 1971.
2. A.S. Monin, P.Ya. Polubariniva-Kochina, V.I. Khlebnikov. *Cosmology, Hydrodynamics, Turbulence.* Moscow: "Nauka", 1989.